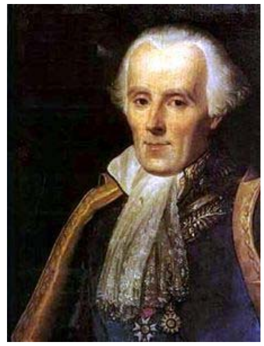
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

One of the most useful differential equations for engineers is **Laplace's equation**.

Pierre-Simon, marquis de Laplace (March 23, 1749 – March 5, 1827) was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics.



He formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming.

The Laplacian differential operator, widely used in applied mathematics, is also named after him.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Laplace's equation can describe torsion in solids, flow in porous media, steady state heat transfer, incompressible flow of inviscid fluids, electrostatic problems, and magneto-statics.

The general form of Laplace's equation in Cartesian coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The partial differential operator, ∇^2, or Δ, (which may be defined in any number of dimensions) is called the Laplace operator, or just the Laplacian.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

The nonhomogeneous form of Laplace's equation is called the **Poisson equation:**

$$\nabla^2 u + f = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f = 0$$

Siméon-Denis Poisson (June 21, 1781 – April 25, 1840), was a French mathematician, geometer, and physicist.

Poisson's well-known correction of Laplace's second order partial differential equation for potential was first published in the Bulletin de la société philomatique (1813).

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Each of the physical problems mentioned above involve either equilibrium or time independent states.

This type of problem is called an **elliptic boundary value problem**.

In general, a two-dimensional elliptic boundary value problem has the form:

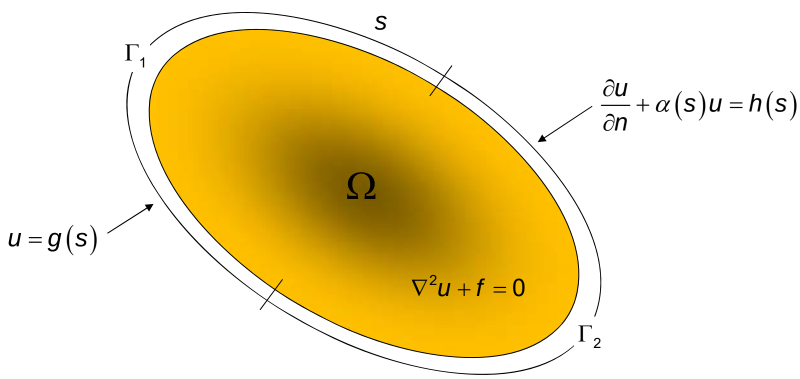
$$\nabla^2 u(x,y) + f(x,y) = 0 \qquad \text{in } \Omega$$

$$u = g(s) \qquad \text{on } \Gamma_1$$

$$\frac{\partial u}{\partial n} + \alpha(s)u = h(s) \qquad \text{on } \Gamma_2$$

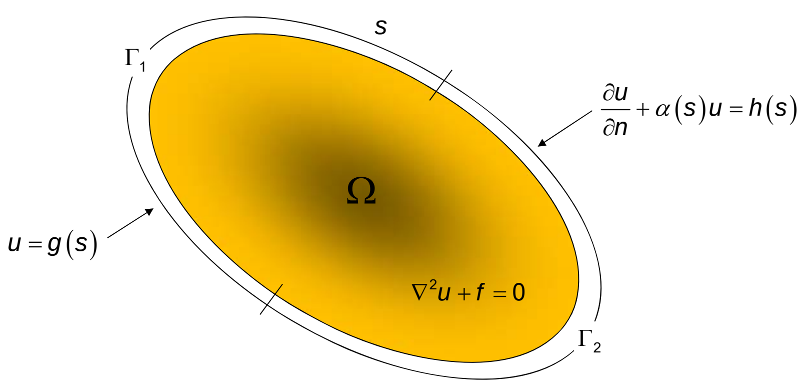
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Where Ω is the interior domain, and Γ_1 and Γ_2 form the boundary of the domain.



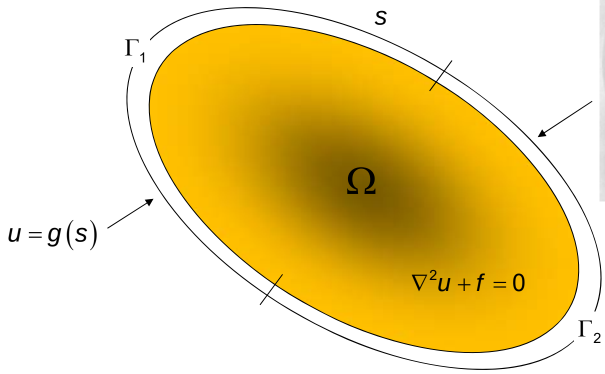
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

A boundary condition that specifies the value of the function *u* on the surface Γ_1 is called a **Dirichlet** (dee ree KLAY) **boundary condition** or type one condition.



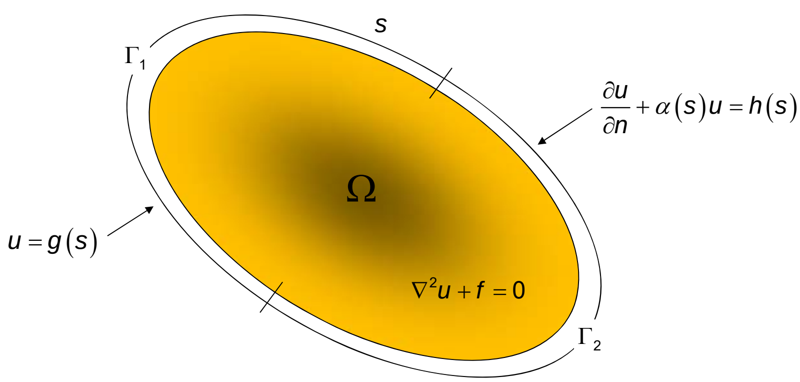
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Johann Peter Gustav Lejeune Dirichlet (February 13, 1805 – May 5, 1859) was a German mathematician credited with the modern formal definition of a function



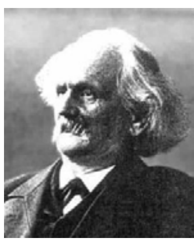
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

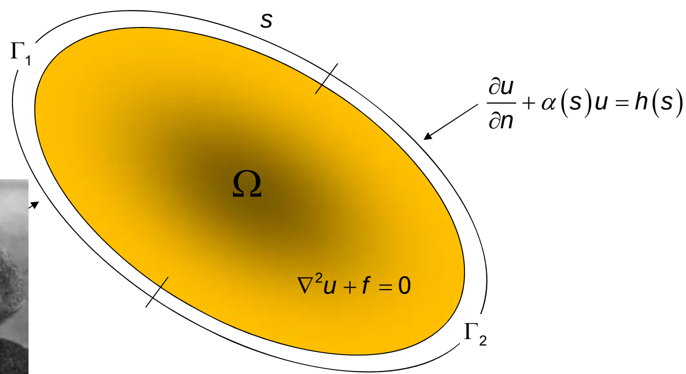
A boundary condition prescribed in the form of a derivative of the function *u* on the surface Γ_2 is called a type two or a **Neumann boundary condition**.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

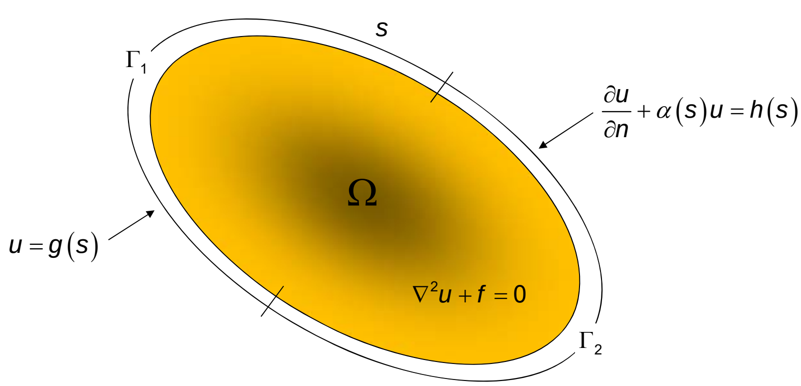
Carl Gottfried Neumann (May 7, 1832 - March 27, 1925) was a German mathematician. Neumann worked on the Dirichlet principle, and can be considered one of the initiators of the theory of integral equations.





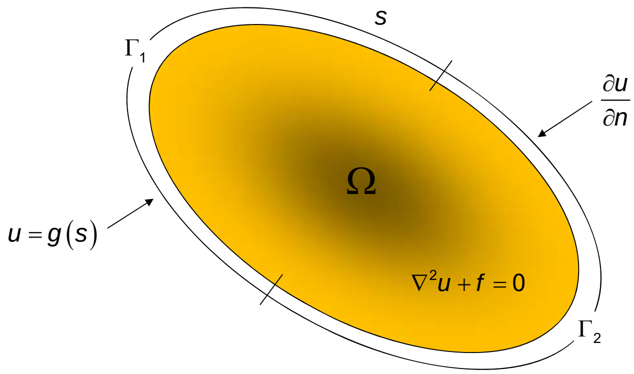
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

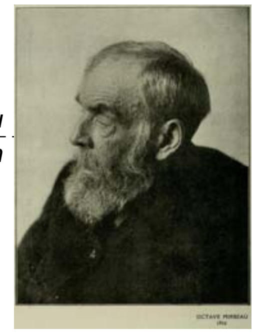
If the value of α is not zero then the condition is called a **Robin boundary condition**.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

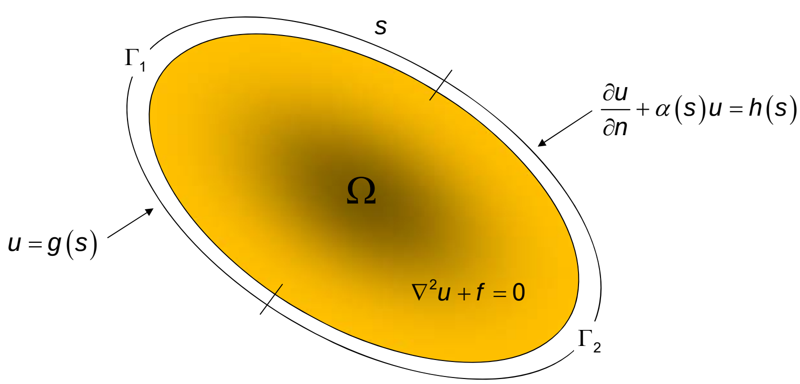
Victor Gustave Robin (1855-1897) was a French mathematical analyst and applied mathematician who lectured in mathematical physics at the Sorbonne in Paris and also worked in the area of thermodynamics.





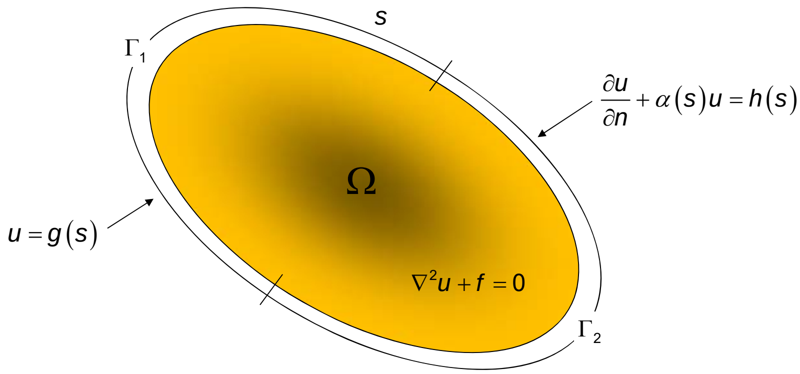
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

If the entire boundary is a Type 1 boundary condition, then the boundary value problem is called a **Dirichlet Problem**.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

If the entire boundary is a Type 2 with α = 0 then the boundary value problem is called a **Neumann Problem**.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

The Ritz finite element formulation is based on the functional for the Poisson equation.

If you are interested in how the energy functional is developed refer to your textbook.

The Poisson functional is:

$$Z(u) = \frac{1}{2}\iint_{\Omega}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]d\Omega - \iint_{\Omega} uf\,d\Omega$$

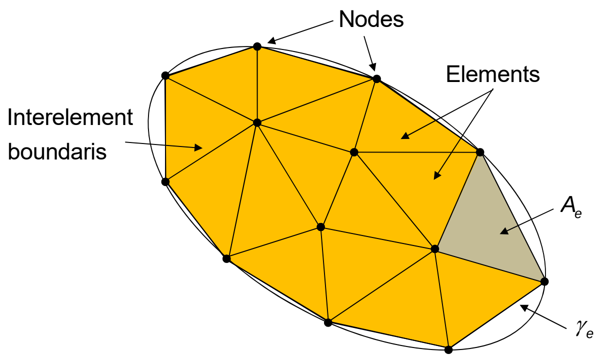
$$+\frac{1}{2}\int_{\Gamma_2}\alpha u^2\,ds + \int_{\Gamma_2} uh\,ds = 0$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Discretization - The first step in developing a finite element model, just as in one-dimensional analysis, is discretization.

The first type of two-dimensional discretization we will discuss utilizes straight-sided or linear triangular elements.

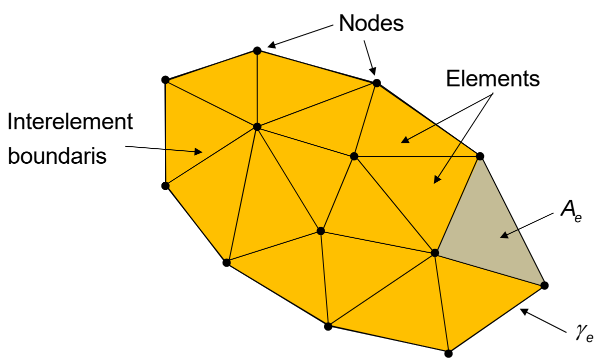


TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Discretization - The linear triangular elements are not capable of describing curved geometries with much accuracy.

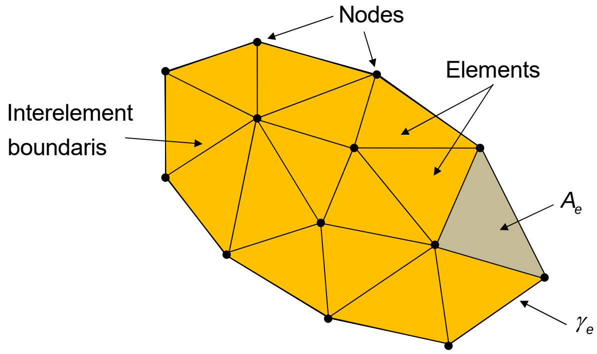
The error between the elemental model and the actual domain may be improved by using more elements.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

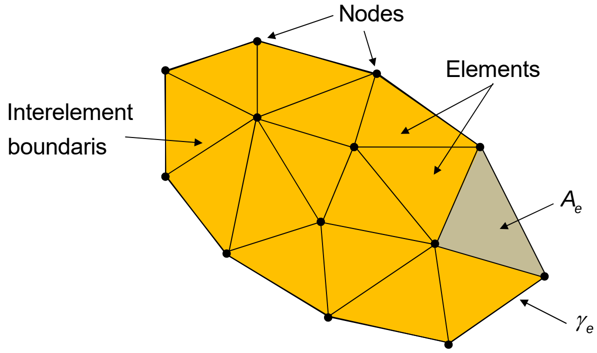
Discretization - Another difficulty is that the boundary conditions which are generally described as continuous function over the boundary are distributed over a piece-wise linear representation of the surface.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Discretization - Both of these difficulties may be modeled more accurately by using more sophisticated elements, for example a triangular element with curved sides.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Discretization - In terms of the discretization, the functional ***Z*** may be represented by a sum of the integrals over each element area A_e and each elemental surface γ_e as:

$$Z(u) \approx \frac{1}{2}\sum_e \iint_{A_e}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dA - \sum_e \iint_{A_e} uf\, dA$$

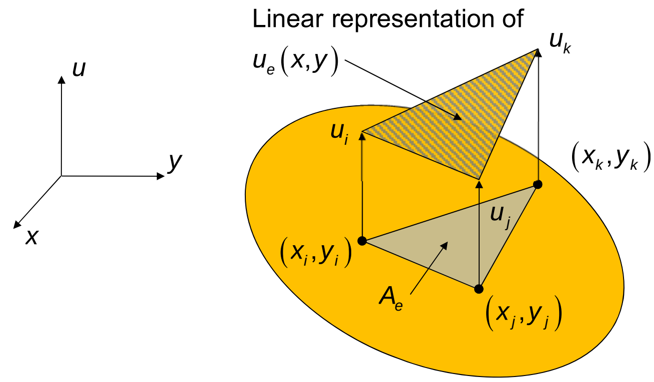
$$+\frac{1}{2}\sum_e{}' \int_{\gamma_{2e}} \alpha u^2\, ds - \sum_e{}' \int_{\gamma_{2e}} uh\, ds = 0$$

where the sum is over all the elements, and Σ' is the sum over each elemental segment γ_{2e} of the Γ_2 portion of the surface.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - The simplest interpolation over a straight-sided three node triangular element is to assume the function $u(x, y)$ is represented by a linear plane.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - In a manner similar to that used to develop the linear, quadratic, and cubic shape functions for one-dimensional problems, we may describe the variation of *u* over the element as:

$$u_e(x,y) = \alpha + \beta x + \gamma y$$

where α, β, and γ are constants determined by matching the function u_e with the nodal values of the element:

$$u_e(x_i,y_i) = \alpha + \beta x_i + \gamma y_i = u_i$$

$$u_e(x_j,y_j) = \alpha + \beta x_j + \gamma y_j = u_j$$

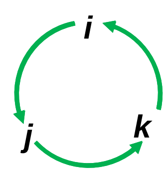
$$u_e(x_k,y_k) = \alpha + \beta x_k + \gamma y_k = u_k$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - Solving the three equations for α, β, and γ and substituting back into the expression representing the variation of *u* over the element results in:

$$u_e(x,y) = N_i u_i + N_j u_j + N_k u_k$$



where: $N_i = \dfrac{a_i + b_i x + c_i y}{2A_e}$

with $a_i = x_j y_k - x_k y_j \qquad b_i = y_j - y_k \qquad c_i = x_k - x_j$

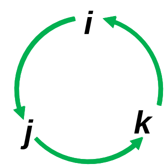
where *i*, *j*, and *k* are permuted cyclically

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - As before the functions N are called the ***shape functions***. The determinant of the coefficients is:

$$2A_e = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$



where A_e is the area of the element.

Any numbering scheme that proceeds counterclockwise around the element is valid, for example (i, j, k), (j, k, i), or (k, i, j).

This numbering convention is important and necessary in order to compute a positive area for A_e.

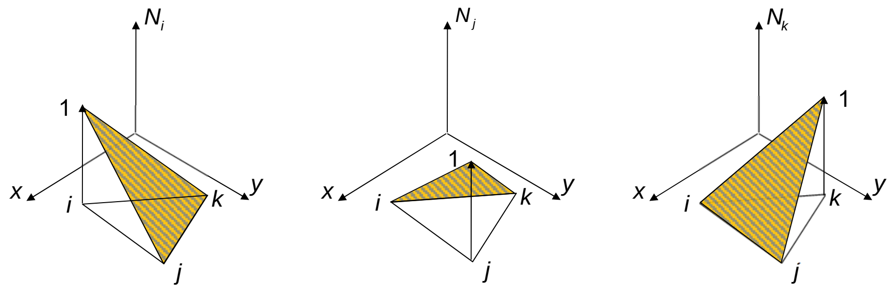
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - In matrix notation, the distribution of the function over the element is:

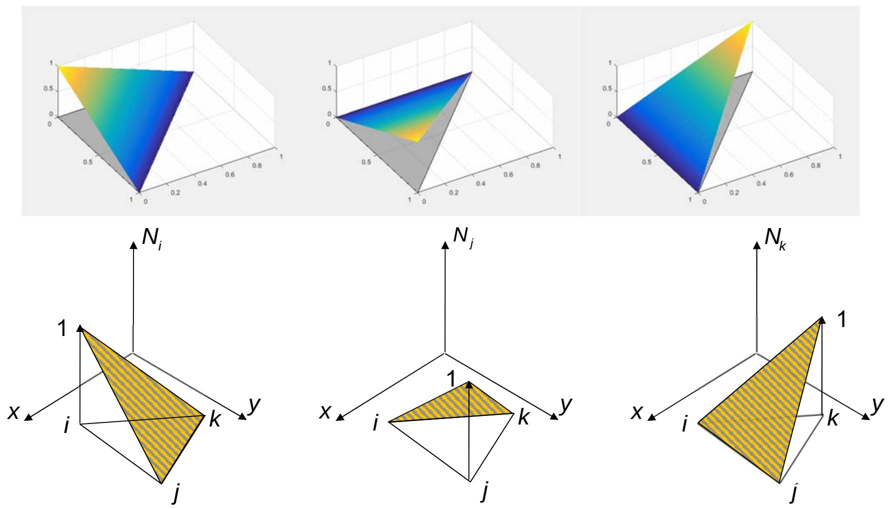
$$u_e(x,y) = \mathbf{u_e}^{\mathbf{T}}\mathbf{N} = \mathbf{N}^{\mathbf{T}}\mathbf{u_e}$$

The linear triangular shape functions are illustrated below:



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - The derivatives of u over the element with respect to both coordinates are:

$$\frac{\partial u_e(x,y)}{\partial x} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial x} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial x}\mathbf{u_e} \qquad \frac{\partial u_e(x,y)}{\partial y} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial y} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial y}\mathbf{u_e}$$

Calculating the derivatives of the shape functions gives:

$$\frac{\partial \mathbf{N}}{\partial x} = \frac{\mathbf{b_e}}{2A_e} \qquad \frac{\partial \mathbf{N}}{\partial y} = \frac{\mathbf{c_e}}{2A_e}$$

$$\mathbf{b_e}^{\mathbf{T}} = \langle b_i \quad b_j \quad b_k \rangle \qquad \mathbf{c_e}^{\mathbf{T}} = \langle c_i \quad c_j \quad c_k \rangle$$

$$b_i = y_j - y_k \qquad c_i = x_k - x_j$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Interpolation - Observing the form of the derivative it is apparent that the partial derivatives of the function *u* will be constant over a linear triangular element.

There are many problems associated with accuracy and convergence for this type of element.

In elasticity analysis, stress and strain are related by a partial differential equation, using a linear triangular element to described stress will result in a constant approximation for strain over the element.

Therefore, elements of this type are called **constant strain elements**.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - The functional for the Poisson equation is:

$$Z(u) \approx \frac{1}{2}\sum_{e} \iint_{A_e} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dA$$

$$+\frac{1}{2}\sum_{e}{}' \int_{\gamma_{2e}} \alpha u^2\, ds - \sum_{e} \iint_{A_e} uf\, dA - \sum_{e}{}' \int_{\gamma_{2e}} uh\, ds = 0$$

We can write the functional in the following form:

$$Z(u) \approx \sum_{e} \frac{Z_{e1}}{2} + \sum_{e}{}' \frac{Z_{e2}}{2} - \sum_{e} Z_{e3} - \sum_{e}{}' Z_{e4}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> – Where the components are defined as:

$$Z_{e1} = \iint_{A_e} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dA \qquad Z_{e2} = \int_{\gamma_{2e}} \alpha u^2 \, ds$$

$$Z_{e3} = \iint_{A_e} uf \, dA \qquad Z_{e4} = \int_{\gamma_{2e}} uh \, ds$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - Evaluation of Z_{e1}:

$$Z_{e1} = \iint_{A} \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} \right] dA$$

Recall the first derivatives of *u* with respect to *x* and *y* are:

$$\frac{\partial u_e(x,y)}{\partial x} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial x} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial x} \mathbf{u_e}$$

$$\frac{\partial u_e(x,y)}{\partial y} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial y} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial y} \mathbf{u_e}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - **Evaluation of Z_{e1}**: Replacing the derivatives with the above approximations gives:

$$Z_{e1} = \iint_A \left[\mathbf{u_e^T} \frac{\partial \mathbf{N}}{\partial x} \frac{\partial \mathbf{N^T}}{\partial x} \mathbf{u_e} + \mathbf{u_e^T} \frac{\partial \mathbf{N}}{\partial y} \frac{\partial \mathbf{N^T}}{\partial y} \mathbf{u_e} \right] dA$$

$$= \mathbf{u_e^T} \iint_A \left[\frac{\partial \mathbf{N}}{\partial x} \frac{\partial \mathbf{N^T}}{\partial x} + \frac{\partial \mathbf{N}}{\partial y} \frac{\partial \mathbf{N^T}}{\partial y} \right] dA \; \mathbf{u_e} = \mathbf{u_e^T} \mathbf{k_e} \mathbf{u_e}$$

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x} \frac{\partial \mathbf{N^T}}{\partial x} + \frac{\partial \mathbf{N}}{\partial y} \frac{\partial \mathbf{N^T}}{\partial y} \right] dA$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - **Evaluation of Z_{e1}**:The integrals defined in $\mathbf{k_e}$ are the elemental “stiffness” matrix.

For the linear triangular element we have discussed the stiffness matrix reduces to:

$$\mathbf{k_e} = \iint_{A_e} \left[\frac{\mathbf{b_e b_e^T} + \mathbf{c_e c_e^T}}{4A_e^2} \right] dA$$

Since the integrand of $\mathbf{k_e}$ is a constant, the elemental stiffness matrix becomes:

$$\mathbf{k_e} = \frac{\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}}{4A_e}$$

The resulting is a 3x3 **elemental stiffness matrix**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - Evaluation of Z_{e2}:

$$Z_{e2} = \int_{\gamma_{2e}} \alpha u^2 \, ds$$

In this case, the interpolation of *u* with respect to *x* and *y* is used to describe the behavior along the boundary:

$$Z_{e2} = \int_{\gamma_{2e}} \mathbf{u_e^T N} \alpha \mathbf{N^T u_e} \, ds = \mathbf{u_e^T} \left(\int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N^T} \, ds \right) \mathbf{u_e} = \mathbf{u_e^T a_e u_e}$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N} \alpha \mathbf{N^T} \, ds$$

The resulting is a 2x2 **elemental stiffness matrix**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - Evaluation of Z_{e3}:

$$Z_{e3} = \iint_{A_e} uf \, dA$$

Substituting the approximation for *u* into the integral results in:

$$Z_{e3} = \mathbf{u_e^T} \left(\iint_{A_e} \mathbf{N} f \, dA \right) = \mathbf{u_e^T f_e} \qquad \mathbf{f_e} = \iint_{A_e} \mathbf{N} f \, dA$$

The resulting is a 3x1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - Evaluation of Z_{e4}:

$$Z_{e4} = \int_{\gamma_{2e}} uh\, ds$$

Substituting the approximation for u into the integral results in:

$$Z_{e4} = \mathbf{u_e^T}\left(\int_{\gamma_{2e}} \mathbf{N}h\, ds\right) = \mathbf{u_e^T h_e} \qquad \mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N}h\, ds$$

The resulting is a 2x1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - In terms of the matrix definitions, the functional may be written in the following form:

$$Z(u_1, u_2, u_3, \ldots, u_N) \approx \sum_e \left(\frac{\mathbf{u_e^T k_e u_e}}{2} - \mathbf{u_e^T f_e}\right) + \sum_e{}' \left(\frac{\mathbf{u_e^T a_e u_e}}{2} - \mathbf{u_e^T h_e}\right)$$

where the first sum is over all the elements that form the domain of the problem and the second sum is over elements that have a straight-line segment on the boundary of the domain.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - In terms of the matrix definitions, the functional may be written in the following form:

$$Z(u_1, u_2, u_3, \ldots, u_N) \approx \sum_e \left(\frac{\mathbf{u_e^T k_e u_e}}{2} - \mathbf{u_e^T f_e} \right) + \sum_e{}' \left(\frac{\mathbf{u_e^T a_e u_e}}{2} - \mathbf{u_e^T h_e} \right)$$

In this formulation, there are two types of "stiffness" components.

The first, the $\mathbf{k_e}$ terms, are associated with the Laplacian differential operator and the second, the $\mathbf{a_e}$ terms, correspond to the prescribed boundary conditions.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

<u>Elemental Formulation</u> - In terms of the matrix definitions, the functional may be written in the following form:

$$Z(u_1, u_2, u_3, \ldots, u_N) \approx \sum_e \left(\frac{\mathbf{u_e^T k_e u_e}}{2} - \mathbf{u_e^T f_e} \right) + \sum_e{}' \left(\frac{\mathbf{u_e^T a_e u_e}}{2} - \mathbf{u_e^T h_e} \right)$$

The right-hand side of the system equations is also formed from two components.

The $\mathbf{f_e}$ terms correspond to the Poisson term of the differential equation.

The $\mathbf{h_e}$ terms handle any nonhomogeneous boundary conditions.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Assembly - The assembly is denoted by the summation in the matrix equation. The global matrix form of the formulation is:

$$Z \approx \frac{\mathbf{u_G}^{\mathbf{T}}\mathbf{K_G}\mathbf{u_G}}{2} - \mathbf{u_G}^{\mathbf{T}}\mathbf{F_G} = Z\left(\mathbf{u_G}\right)$$

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e{}' \; \mathbf{a_G} \qquad \mathbf{F_G} = \sum_e \mathbf{f_G} + \sum_e{}' \; \mathbf{h_G}$$

$$\frac{\partial Z}{\partial u_i} = 0 \qquad \frac{\partial Z}{\partial \mathbf{u_G}} = \frac{\left(\mathbf{K_G} + \mathbf{K_G}^{\mathbf{T}}\right)\mathbf{u}_G}{2} - \mathbf{F_G} \;\rightarrow\; \boxed{\mathbf{K_G}\mathbf{u_G} = \mathbf{F_G}}$$

$$\text{since } \mathbf{K_G} = \mathbf{K_G}^{\mathbf{T}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Constraints - The constraints on the system equations are the forced boundary conditions $u = g(s)$ on the surface Γ_1.

These conditions are applied to the system equations in a manner similar to that discussed for one-dimensional problems.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Solution - Since there are two types of boundary conditions, there are three possible situations to consider when determining a solution to the system equations.

One case is when the entire boundary consists of Dirichlet boundary conditions, $u = g(s)$.

In this situation, a unique solution may be found.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Solution - Since there are two types of boundary conditions, there are three possible situations to consider when determining a solution to the system equations.

A second case where a unique solution is possible is when the boundary is composed of both Dirichlet and Neumann boundary conditions.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Solution - Since there are two types of boundary conditions, there are three possible situations to consider when determining a solution to the system equations.

The one situation where a singular solution is obtained is when Neumann-type conditions are prescribed along the entire boundary.

In this case, only a derivative-type condition exists.

There are infinite solutions in this type of problem.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Computation of Derived Variables - The solution for the nodal values of *u* are often called the **primary variables**, whereas the derivatives and any other values based on the primary variables are called **secondary** or **derived variables**.

In this case the values of the function *u* are the primary variables and $\partial u/\partial n = n_x\, \partial u/\partial x + n_y\, \partial u/\partial y$ is consider a secondary variable.

$$\frac{\partial u_e(x,y)}{\partial x} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial x} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial x}\mathbf{u_e} = \frac{\mathbf{b_e}^{\mathbf{T}}\mathbf{u_e}}{2A_e}$$

$$\frac{\partial u_e(x,y)}{\partial y} = \mathbf{u_e}^{\mathbf{T}} \frac{\partial \mathbf{N}}{\partial y} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial y}\mathbf{u_e} = \frac{\mathbf{c_e}^{\mathbf{T}}\mathbf{u_e}}{2A_e}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Recall the elemental matrices have the following form:

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N^T}}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N^T}}{\partial y} \right] dA \qquad \mathbf{k_e} = \frac{\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}}{4A_e}$$

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N}\alpha\mathbf{N^T}\, ds$$

$$\mathbf{f_e} = \iint_{A_e} \mathbf{N} f\, dA$$

$$\mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h\, ds$$

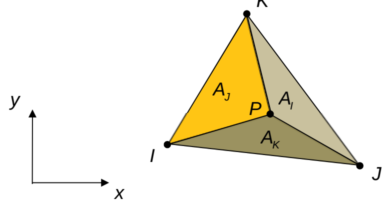
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

The evaluation of these integrals over a general element A_e can be very tedious. In order to simplify the computation we will discuss and introduce a local set of coordinates called ***area coordinates***.

Consider a general linear triangular element:

$$A_I + A_J + A_K = A_e \rightarrow \frac{A_I}{A_e} + \frac{A_J}{A_e} + \frac{A_K}{A_e} = 1$$



$$L_I + L_J + L_K = 1$$

$$x = x_I L_I + x_J L_J + x_K L_K$$

$$y = y_I L_I + y_J L_J + y_K L_K$$

Where L_I, L_J, and L_K are the area coordinates.

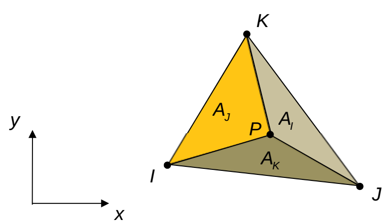
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

As point *P* approaches any point on line *IK*, the area $A_J \to 0$ and therefore, $L_J \to 0$

As point *P* approaches point *K*, then $A_K \to A_e$ and $L_K \to 1$

Solving the three equations, L_I, L_J, and L_K we find these area coordinates are equal to the linear triangular shape functions N_i, N_j, and N_k.



$$L_I + L_J + L_K = 1$$

$$x = x_I L_I + x_J L_J + x_K L_K$$

$$y = y_I L_I + y_J L_J + y_K L_K$$

Where L_I, L_J, and L_K are the area coordinates.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

From the area coordinate relationship:

$$L_I + L_J + L_K = 1$$

We observe that if L_I and L_J are known then $L_K = 1 - L_I - L_J$

Therefore we can write the variation of *x* and *y* with the area coordinates as:

$$x = x_k + (x_i - x_k)L_I + (x_j - x_k)L_J = x(L_I, L_J)$$

$$y = y_k + (y_i - y_k)L_I + (y_j - y_k)L_J = y(L_I, L_J)$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - Using the local or area coordinates in the integrals transform the elemental matrices as follows:

$$\mathbf{k_e} = \iint_A \left[\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N}^T}{\partial x} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N}^T}{\partial y} \right] dA \quad = \quad \iint G(x,y)\, dx\, dy$$

The differential area ***dA*** is a vector with magnitude *dA* and direction normal to the element area, which in this case is ***k***.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - The vector ***dA*** is given by the determinant rule:

$$\mathbf{dA} = \mathbf{dx} \times \mathbf{dy} = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial L_I} & \frac{\partial y}{\partial L_I} & 0 \\ \frac{\partial x}{\partial L_J} & \frac{\partial y}{\partial L_J} & 0 \end{vmatrix} dL_I dL_J = \left(\frac{\partial x}{\partial L_I}\frac{\partial y}{\partial L_J} - \frac{\partial x}{\partial L_J}\frac{\partial y}{\partial L_I} \right) dL_I dL_J \mathbf{k}$$

where:

$$\mathbf{dx} = \frac{\partial x}{\partial L_I} dL_I + \frac{\partial x}{\partial L_J} dL_J$$

$$\mathbf{dy} = \frac{\partial y}{\partial L_I} dL_I + \frac{\partial y}{\partial L_J} dL_J$$

$$dA = |\,\mathbf{J}\,|\, dL_I dL_J$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - where |**J**| is defined as the determinant of:

$$|\,\mathbf{J}\,| = \begin{vmatrix} \frac{\partial x}{\partial L_I} & \frac{\partial y}{\partial L_I} \\ \frac{\partial x}{\partial L_J} & \frac{\partial y}{\partial L_J} \end{vmatrix} = \begin{vmatrix} x_i - x_k & y_i - y_k \\ x_j - x_k & y_j - y_k \end{vmatrix} = 2A_e$$

Therefore, $\boldsymbol{k_e}$ transformed into area coordinates is:

$$\mathbf{k_e} = \int\int \mathbf{G}\left[x\left(L_I, L_J\right) y\left(L_I, L_J\right)\right] |\,\mathbf{J}\,|\, dL_I dL_J$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - To transform the partial derivatives $\partial/\partial x$ and $\partial/\partial y$ to functions of L_I and L_J:

$$\frac{\partial}{\partial L_I} = \frac{\partial}{\partial x}\frac{\partial x}{\partial L_I} + \frac{\partial}{\partial y}\frac{\partial y}{\partial L_I} \qquad \frac{\partial}{\partial \mathbf{L}} = \mathbf{J}\frac{\partial}{\partial \mathbf{x}}$$

$$\frac{\partial}{\partial L_J} = \frac{\partial}{\partial x}\frac{\partial x}{\partial L_J} + \frac{\partial}{\partial y}\frac{\partial y}{\partial L_J}$$

where **J**, $\partial/\partial\mathbf{L}$, and $\partial/\partial\mathbf{x}$ are:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial L_I} & \frac{\partial y}{\partial L_I} \\ \frac{\partial x}{\partial L_J} & \frac{\partial y}{\partial L_J} \end{bmatrix} \qquad \frac{\partial}{\partial \mathbf{L}} = \begin{Bmatrix} \frac{\partial}{\partial L_I} \\ \frac{\partial}{\partial L_J} \end{Bmatrix} \qquad \frac{\partial}{\partial \mathbf{x}} = \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - Where **J** is called the **Jacobian matrix** of the transformation. The matrix form of the transformation may be inverted.

$$\frac{\partial}{\partial \mathbf{L}} = \mathbf{J}\frac{\partial}{\partial \mathbf{x}} \qquad \rightarrow \qquad \frac{\partial}{\partial \mathbf{x}} = \mathbf{J}^{-1}\frac{\partial}{\partial \mathbf{L}}$$

The matrix may be partitioned as:

$$\frac{\partial}{\partial x} = \mathbf{J_1}\frac{\partial}{\partial \mathbf{L}} \qquad \frac{\partial}{\partial y} = \mathbf{J_2}\frac{\partial}{\partial \mathbf{L}}$$

where $\mathbf{J_1}$ and $\mathbf{J_2}$ are the first and second rows of $\mathbf{J^{-1}}$.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Carl Gustav Jacob Jacobi (10 December 1804 – 18 February 1851) was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations, and number theory. His name is occasionally written as Carolus Gustavus Iacobus Iacobi in his Latin books, and his first name is sometimes given as Karl.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - The partial derivatives $\partial/\partial x$ and $\partial/\partial y$ may now be written entirely in terms of **L**. Therefore, the partial derivatives of the shape functions may be written as:

$$\frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial \mathbf{x}} = \mathbf{J_1}\frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial \mathbf{L}} = \mathbf{J_1}\Delta^{\mathbf{T}} \qquad \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial \mathbf{y}} = \mathbf{J_2}\frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial \mathbf{L}} = \mathbf{J_2}\Delta^{\mathbf{T}}$$

$$\frac{\partial \mathbf{N}}{\partial \mathbf{x}} = \Delta \mathbf{J_1}^{\mathbf{T}} \qquad \frac{\partial \mathbf{N}}{\partial \mathbf{y}} = \Delta \mathbf{J_2}^{\mathbf{T}}$$

$$\Delta^{T} = \frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial \mathbf{L}} = \begin{bmatrix} \frac{\partial L_I}{\partial L_I} & \frac{\partial L_J}{\partial L_I} & \frac{\partial L_K}{\partial L_I} \\ \frac{\partial L_I}{\partial L_J} & \frac{\partial L_J}{\partial L_J} & \frac{\partial L_K}{\partial L_J} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{k_e}$ - Substituting all the pieces of the transformation in the $\mathbf{k_e}$ terms gives:

$$\mathbf{k_e} = \int\int \left[\Delta \mathbf{J_1}^{\mathbf{T}}\mathbf{J_1}\Delta^{\mathbf{T}} + \Delta \mathbf{J_2}^{\mathbf{T}}\mathbf{J_2}\Delta^{\mathbf{T}} \right] |\,\mathbf{J}\,|\, dL_I dL_J$$

$$= \int\int \Delta \left[\mathbf{J_1}^{\mathbf{T}}\mathbf{J_1} + \mathbf{J_2}^{\mathbf{T}}\mathbf{J_2} \right] \Delta^{\mathbf{T}} \,|\,\mathbf{J}\,|\, dL_I dL_J = \int\int \Delta \mathbf{JJ}\Delta^{\mathbf{T}} dL_I dL_J$$

where $\mathbf{JJ} = (\mathbf{J_1}^{\mathbf{T}}\mathbf{J_1} + \mathbf{J_2}^{\mathbf{T}}\mathbf{J_2})|\mathbf{J}|$.

The integrand of the above integral is a constant, therefore, $\mathbf{k_e}$ reduces to:

$$\mathbf{k_e} = \frac{\left(\mathbf{b_e}\mathbf{b_e}^{\mathbf{T}} + \mathbf{c_e}\mathbf{c_e}^{\mathbf{T}} \right)}{4A_e}$$

The resulting elemental stiffness matrix $\mathbf{k_e}$ is a 3 x 3

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{f_e}$ - In general, the integral $\mathbf{f_e}$ is:

$$\mathbf{f_e} = \iint_{A_e} \mathbf{N} f\left(x,y\right) dA$$

For a general function $f(x,y)$, the above integral may be quite tedious to evaluate, therefore we will assume that f varies linearly over the element, $f(x,y) = \mathbf{N^T f}$, where the vector $\mathbf{f}$ contains values of the function f at the node points.

With this assumption the integral becomes:

$$\mathbf{f_e} \approx \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\mathbf{f}\, dA = \left(\iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\, dA \right)\mathbf{f}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{f_e}$ – Expanding the integral gives:

$$\mathbf{f_e} \approx \iint_{A_e} \mathbf{N}\mathbf{N}^{\mathbf{T}}\mathbf{f}\, dA \approx \iint \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \rangle \begin{Bmatrix} f_i \\ f_j \\ f_k \end{Bmatrix} dA$$

$$\approx \iint \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2^2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3^2 \end{bmatrix} \begin{Bmatrix} f_i \\ f_j \\ f_k \end{Bmatrix} dA$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{f_e}$ - For integrations of the type:

$$\iint_{A_e} N_I^a N_J^b N_K^c \, dA = a!b!c! \frac{2A_e}{(a+b+c+2)!}$$

Therefore:

$$\mathbf{f_e} \approx \iint \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2^2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3^2 \end{bmatrix} \begin{Bmatrix} f_i \\ f_j \\ f_k \end{Bmatrix} dA = \frac{A_e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} f_i \\ f_j \\ f_k \end{Bmatrix}$$

$$\mathbf{f_e} \approx \frac{A_e}{12} \begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix}$$

The resulting is a 3 x 1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

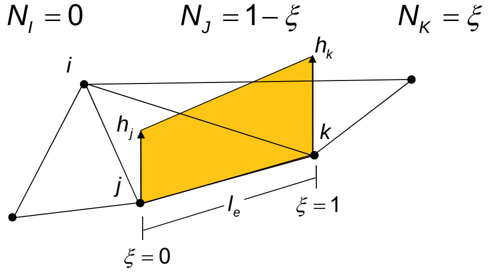
Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{h_e}$ - Consider the integral: $\mathbf{h_e} = \int_{\gamma_{2e}} \mathbf{N} h \, ds$

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the triangular element, the original shape functions reduce to:

$N_I = 0 \qquad N_J = 1 - \xi \qquad N_K = \xi$



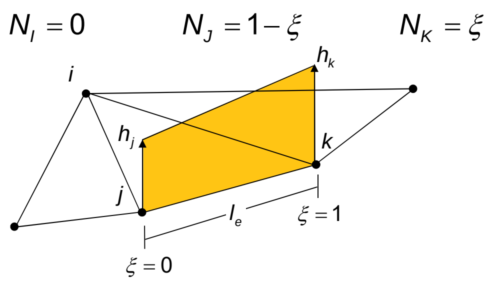
$$\mathbf{h_e} = \int_{\gamma_{2e}} \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} h(\xi) \, l_e \, d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{h_e}$ - For a general function $h(\xi)$ the integral may be tedious to evaluate, therefore we will assume that *h* varies linearly over the boundary.

Consider $h(\xi) = \mathbf{N^T h}$, where the vector **h** contains the values of the function *h* at the boundary node points.



$$\mathbf{h_e} = \int_{\gamma_{2e}} \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} h(\xi)\, l_e\, d\xi$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{h_e}$ - With this assumption the integral becomes:

$$\mathbf{h_e} \approx \int_{\gamma_{2e}} \mathbf{NN^T h}\, l_e d\xi \approx \left(\int_{\gamma_{2e}} \mathbf{NN^T}\, l_e d\xi \right) \mathbf{h}$$

$$\int_{\gamma_{2e}} N_I^a N_J^b\, ds = a!\,b! \frac{l_e}{(a+b+1)!}$$

Therefore:

$$\mathbf{h_e} \approx \frac{l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{h} \approx \frac{l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} h_j \\ h_k \end{Bmatrix} \qquad \mathbf{h_e} \approx \frac{l_e}{6} \begin{Bmatrix} 2h_j + h_k \\ h_j + 2h_k \end{Bmatrix}$$

The resulting is a 2 x 1 **elemental load vector**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{a_e}$ - The evaluation of $\mathbf{a_e}$ is very similar that of $\mathbf{h_e}$ except that there is an extra $\mathbf{N^T}$ in the integrand.

The variation of the function α(s) will be approximated as $\alpha \approx \alpha_j N_j + \alpha_k N_k$. Consider the integral $\mathbf{a_e}$:

$$\mathbf{a_e} = \int_{\gamma_{2e}} \mathbf{N}\alpha\mathbf{N^T}\, ds$$

$$\approx \int_{\gamma_{2e}} \begin{bmatrix} N_j\left(\alpha_j N_j + \alpha_k N_k\right)N_j & N_j\left(\alpha_j N_j + \alpha_k N_k\right)N_k \\ N_k\left(\alpha_j N_j + \alpha_k N_k\right)N_j & N_k\left(\alpha_j N_j + \alpha_k N_k\right)N_k \end{bmatrix} l_e d\xi$$

The resulting is a 2 x 2 **elemental stiffness matrix**

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Evaluation of $\mathbf{a_e}$ - The integration formula for the type of integrals is:

$$\int_{\gamma_{2e}} N_I^a N_J^b\, ds = a!\,b!\frac{l_e}{(a+b+1)!}$$

$$\mathbf{a_e} \approx \frac{l_e}{12}\begin{bmatrix} 3\alpha_j + \alpha_k & \alpha_j + \alpha_k \\ \alpha_j + \alpha_k & \alpha_j + 3\alpha_k \end{bmatrix}$$

The resulting 2 x 2 **stiffness matrix** contributes to the global system equations when the element has a side as part of the boundary.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Evaluation of Matrices - Linear Triangular Elements

Recall, the global system equations are composed from the following summations:

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \sum_e{}' \mathbf{a_G} \qquad \mathbf{F_G} = \sum_e \mathbf{f_G} + \sum_e{}' \mathbf{h_G}$$

The resulting system equations are, in matrix form, given as:

$$\boxed{\mathbf{K_G u_G} = \mathbf{F_G}}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

PROBLEM #17 - For a linear interpolation, verify that the two expressions for the elemental stiffness $\mathbf{k_e}$, given as:

$$\mathbf{k_e} = \int\int \left[\Delta \mathbf{J_1}^{\mathbf{T}} \mathbf{J_1} \Delta^{\mathbf{T}} + \Delta \mathbf{J_2}^{\mathbf{T}} \mathbf{J_2} \Delta^{\mathbf{T}} \right] |\,\mathbf{J}\,|\, dL_I dL_J$$

and

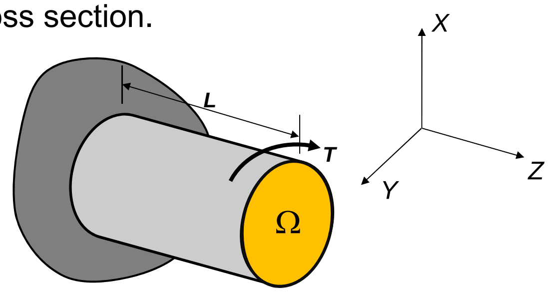
$$\mathbf{k_e} = \frac{\left(\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}\right)}{4A_e}$$

are exactly the same.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solid Mechanics Application – Torsion of a prismatic bar

An important application form solid mechanics is the problem of torsion of a homogeneous isotropic prismatic bar of arbitrary cross section.

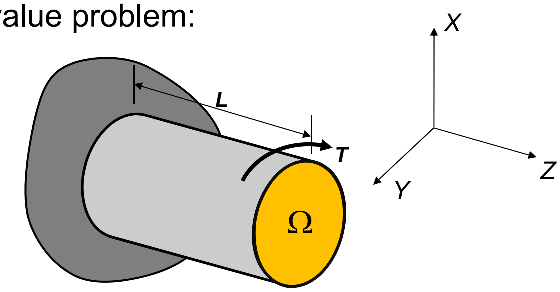


The three basic ideas of solid mechanics – kinematics, kinetics, and constitution – are used to develop the strain-displacement relations, equilibrium equations, and the stress-strain relationships.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solid Mechanics Application – Torsion of a prismatic bar

Determining the deformations and stresses in a bar of arbitrary cross section can be reduced to the flowing two-dimensional boundary value problem:



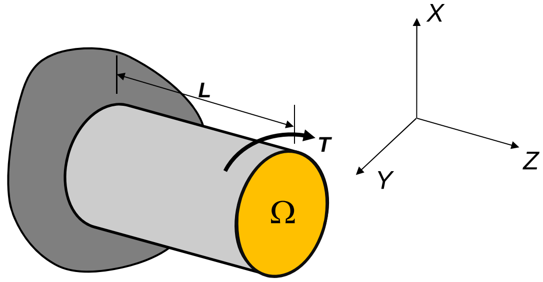
$$\nabla^2\phi(X,Y) + 2G\theta = 0 \qquad \text{in } \Omega$$

$$\phi = 0 \qquad \text{on } \Gamma$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solid Mechanics Application – Torsion of a prismatic bar

Where ϕ is the Prandtl stress function, G is the shear modulus, and θ is the constant rate of twist along the axis of the bar.



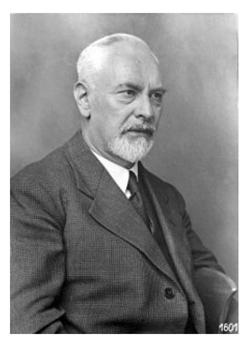
$$\nabla^2\phi\left(X,Y\right)+2G\theta=0 \qquad \text{in } \Omega$$

$$\phi=0 \qquad \text{on } \Gamma$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solid Mechanics Application – Torsion of a prismatic bar

Ludwig Prandtl (4 February 1875 – 15 August 1953 was a German engineer. He was a pioneer in the development of rigorous systematic mathematical analyses which he used for underlying the science of aerodynamics, which have come to form the basis of the applied science of aeronautical engineering. His studies identified the boundary layer, thin-airfoils, and lifting-line theories. The Prandtl number was named after him.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solid Mechanics Application – Torsion of a prismatic bar

The nonzero stress components are given in terms of the stress function:

$$\tau_{xz} = \frac{\partial \phi}{\partial y} \qquad\qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

The total torque transmitted along the bar is determined from:

$$T = 2\iint_{\Omega} \phi \, dA$$

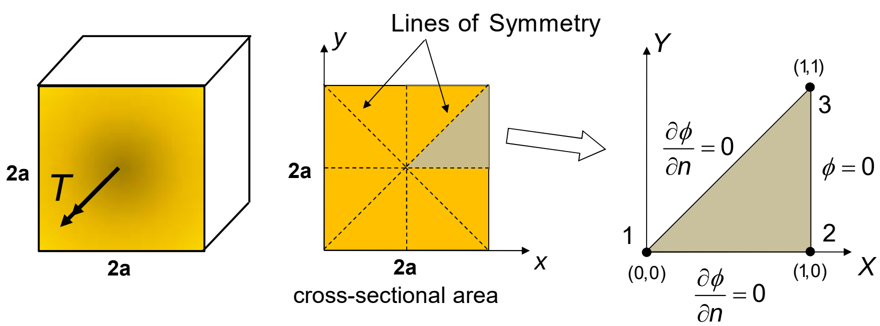
Generally, the determine of the stress function ϕ and then the stresses and the applied torque T as follows:

1. determine ϕ by solving the differential equation
2. determine the stresses and the torque T as function of ϕ

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - The simplest model for torsion of a square bar, utilizing symmetry is a single triangular element.

The general problem domain and the FEM mesh are shown below.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Before beginning the FEM model, it is desirable to non-dimensionalize the problem.

$$X = \frac{x}{a} \qquad Y = \frac{y}{a} \qquad \Psi = \frac{\phi}{2G\theta a^2}$$

Therefore, the governing differential equation becomes:

$$\nabla^2\Psi\left(X, Y\right) + 1 = 0 \quad \text{in } \Omega \quad -1 \le X \le 1$$

$$\Psi = 0 \quad \text{on } \Gamma \quad -1 \le Y \le 1$$

The stresses and torque for the Prandtl stress function are:

$$\tau_{xz} = 2G\theta a \frac{\partial \Psi}{\partial Y} \qquad \tau_{yz} = -2G\theta a \frac{\partial \Psi}{\partial X}$$

$$T = 4G\theta a^4 \iint \Psi \; dXdY$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

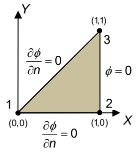
Example - **Elemental Formulation** - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e} = \frac{\left(\mathbf{b_e}\mathbf{b_e}^{\mathbf{T}} + \mathbf{c_e}\mathbf{c_e}^{\mathbf{T}}\right)}{4A_e} \qquad b_i = y_j - y_k \qquad c_i = x_k - x_j$$

where

$$\mathbf{b_e} = \begin{Bmatrix} Y_2 - Y_3 \\ Y_3 - Y_1 \\ Y_1 - Y_2 \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} X_3 - X_2 \\ X_1 - X_3 \\ X_2 - X_1 \end{Bmatrix}$$

In this example, node 1 is located at (X, Y) = (0, 0), node 2 at (1, 0), and node 3 at (1, 1), therefore:



$$\mathbf{b_e} = \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix} \qquad A_e = \frac{1}{2} \qquad \mathbf{k_e} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Elemental Formulation - A loading function of $f = 1$ gives a load vector of:

$$\mathbf{f_e} \approx \frac{A_e}{12}\begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Assembly - Since there is only one element in the model the assembly is simple:

$$\mathbf{K_G} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{F_G} = \frac{1}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Constraints - For this model, $\Psi = 0$ on the boundary, therefore, Ψ_2 and $\Psi_3 = 0$.

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

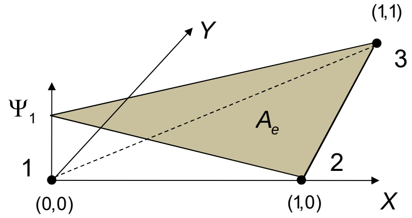
Solution - In this case, the solution is quite simple:

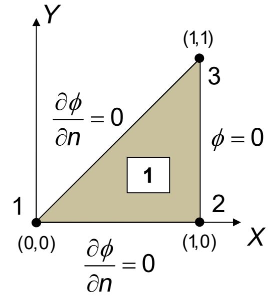
$$\Psi_1 = \frac{1}{3} \qquad \rightarrow \qquad \phi_1 = \frac{2G\theta a^2}{3}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solution – The variation of f over the element is by the linear interpolation functions.

$$\Psi_1 = \frac{1}{3} \qquad \rightarrow \qquad \phi_1 = \frac{2G\theta a^2}{3}$$





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Computation of Derived Variables** - The partial derivatives with respect to *x* and *y* that define the stress components are:

$$\tau_{xz} = 2G\theta a \frac{\partial \Psi}{\partial Y} = 2G\theta a \frac{\mathbf{c_e}^{\mathbf{T}} \Psi_{\mathbf{e}}}{2A_e}$$

$$= \frac{2G\theta a}{2A_e} \left\langle X_3 - X_2 \quad X_1 - X_3 \quad X_2 - X_1 \right\rangle \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{Bmatrix}$$

$$= \frac{2G\theta a}{2A_e} \left\langle 0 \quad -1 \quad 1 \right\rangle \begin{Bmatrix} 1/3 \\ 0 \\ 0 \end{Bmatrix} = 0$$

$$\tau_{yz} = -2G\theta a \frac{\partial \Psi}{\partial X} = -2G\theta a \frac{\mathbf{b_e}^{\mathbf{T}} \Psi_{\mathbf{e}}}{2A_e} = \frac{2G\theta a}{3}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Computation of Derived Variables** - The total torque on the cross-section is:

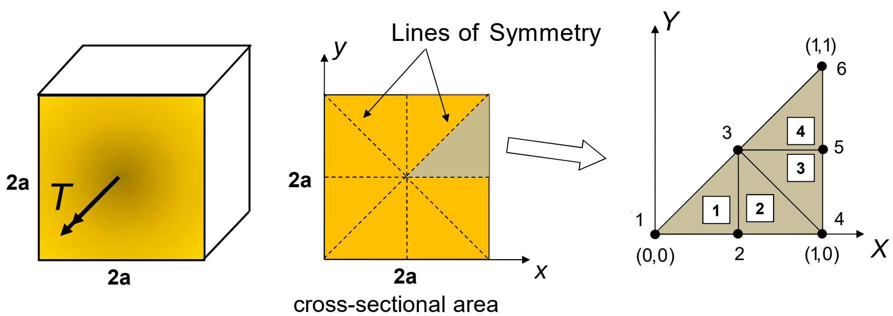
$$T = 4G\theta a^4 \iint \Psi \; dXdY = 8\left(4G\theta a^4 \iint_{A_e} \Psi_e^T N \, dX \, dY \right)$$

$$= 8\left(\frac{4G\theta a^4 A_e}{3} \left(\Psi_1 + \Psi_2 + \Psi_3 \right) \right)$$

$$T = \frac{16G\theta a^4}{9} = 1.7778G\theta a^4 \quad \rightarrow \quad T_{exact} = 2.2496G\theta a^4$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Consider the same problem of torsion of a homogeneous isotropic prismatic bar as above except using a more refined mesh.



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Recall, the non-dimensional Poisson equation governing this problem.

$$\nabla^2\Psi(X,Y)+1=0 \qquad \text{in } \Omega \qquad -1\le X\le 1$$
$$\Psi=0 \qquad \text{on } \Gamma \qquad -1\le Y\le 1$$

with

$$X=\frac{x}{a} \qquad Y=\frac{y}{a} \qquad \Psi=\frac{\phi}{2G\theta a^2}$$

The stresses and torque for the Prandtl stress function are:

$$\tau_{xz}=2G\theta a\frac{\partial\Psi}{\partial Y} \qquad \tau_{yz}=-2G\theta a\frac{\partial\Psi}{\partial X}$$

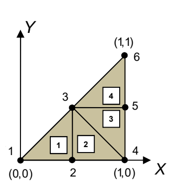
$$T=4G\theta a^4\iint\Psi\, dXdY$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Elemental Formulation** - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e}=\frac{\left(\mathbf{b_e b_e}^T+\mathbf{c_e c_e}^T\right)}{4A_e} \qquad \mathbf{b_e}=\begin{Bmatrix}Y_2-Y_3\\Y_3-Y_1\\Y_1-Y_2\end{Bmatrix} \qquad \mathbf{c_e}=\begin{Bmatrix}X_3-X_2\\X_1-X_3\\X_2-X_1\end{Bmatrix}$$

For element 1: node 1 is located at (X, Y) = (0, 0); node 2 at (0.5, 0); and node 3 at (0.5, 0.5).



$$\mathbf{b_1}=\frac{1}{2}\begin{Bmatrix}-1\\1\\0\end{Bmatrix} \qquad A_1=\frac{1}{8} \qquad \mathbf{k}_1=\frac{1}{2}\begin{array}{c}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\end{array}\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}$$

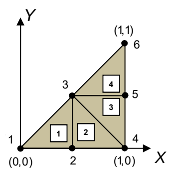
$$\mathbf{c_1}=\frac{1}{2}\begin{Bmatrix}0\\-1\\1\end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e} = \frac{\left(\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}\right)}{4A_e} \qquad \mathbf{b_e} = \begin{Bmatrix} Y_4 - Y_3 \\ Y_3 - Y_2 \\ Y_2 - Y_4 \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} X_3 - X_4 \\ X_2 - X_3 \\ X_4 - X_2 \end{Bmatrix}$$

For element 2: node 2 is located at (X, Y) = (0.5, 0); node 4 at (1, 0); and node 3 at (0.5, 0.5).



$$\mathbf{b}_2 = \frac{1}{2}\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \qquad A_2 = \frac{1}{8} \qquad \mathbf{k}_2 = \frac{1}{2}\begin{array}{c} \left[\begin{array}{ccc} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \end{array}\begin{array}{c} \boxed{2} \\ \boxed{4} \\ \boxed{3} \end{array}$$

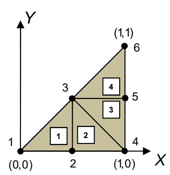
$$\mathbf{c}_2 = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e} = \frac{\left(\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}\right)}{4A_e} \qquad \mathbf{b_e} = \begin{Bmatrix} Y_4 - Y_5 \\ Y_5 - Y_3 \\ Y_3 - Y_4 \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} X_5 - X_4 \\ X_3 - X_5 \\ X_4 - X_3 \end{Bmatrix}$$

For element 3: node 3 is located at (X, Y) = (0.5, 0.5); node 4 at (1, 0); and node 5 at (1, 0.5).



$$\mathbf{b}_3 = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad A_3 = \frac{1}{8} \qquad \mathbf{k}_3 = \frac{1}{2}\left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{array}\right]\begin{array}{c} \boxed{3} \\ \boxed{4} \\ \boxed{5} \end{array}$$

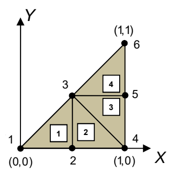
$$\mathbf{c}_3 = \frac{1}{2}\begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:

$$\mathbf{k_e} = \frac{\left(\mathbf{b_e b_e}^{\mathbf{T}} + \mathbf{c_e c_e}^{\mathbf{T}}\right)}{4A_e} \qquad \mathbf{b_e} = \begin{Bmatrix} Y_5 - Y_6 \\ Y_6 - Y_3 \\ Y_3 - Y_5 \end{Bmatrix} \qquad \mathbf{c_e} = \begin{Bmatrix} X_6 - X_5 \\ X_3 - X_6 \\ X_5 - X_3 \end{Bmatrix}$$

For element 4: node 3 is located at (X, Y) = (0.5, 0.5); node 5 at (1, 0.5); and node 6 at (1, 1).



$$\mathbf{b}_4 = \frac{1}{2}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad \mathbf{c}_4 = \frac{1}{2}\begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix} \qquad A_4 = \frac{1}{8}$$

$$\mathbf{k}_4 = \frac{1}{2}\begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \boxed{3} \\ \boxed{5} \\ \boxed{6} \end{array}$$

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

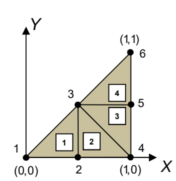
<u>Example</u> - Elemental Formulation - The loading function of f = 1 gives a series of elemental load vectors of:

$$\mathbf{f_e} \approx \frac{A_e}{12}\begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{f_1} = \mathbf{f_2} = \mathbf{f_3} = \mathbf{f_4}$$

<u>Assembly</u> - Since there are four elements in the model the assembly is not difficult:

Element 1

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 2 \\ 1 \end{Bmatrix}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

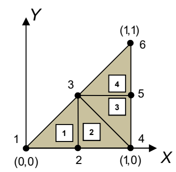
Example - **Elemental Formulation** - The loading function of $f = 1$ gives a series of elemental load vectors of:

$$\mathbf{f_e} \approx \frac{A_e}{12}\begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{f_1} = \mathbf{f_2} = \mathbf{f_3} = \mathbf{f_4}$$

Assembly - Since there are four elements in the model the assembly is not difficult:

Element 2

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 2 \\ 1 \end{Bmatrix}$$



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

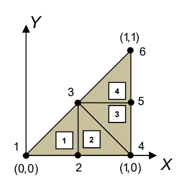
Example - **Elemental Formulation** - The loading function of $f = 1$ gives a series of elemental load vectors of:

$$\mathbf{f_e} \approx \frac{A_e}{12}\begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{f_1} = \mathbf{f_2} = \mathbf{f_3} = \mathbf{f_4}$$

Assembly - Since there are four elements in the model the assembly is not difficult:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 2 \\ 1 \end{Bmatrix}$$

Element 3



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

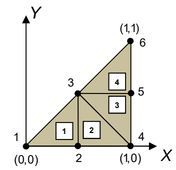
<u>Example</u> - Elemental Formulation - The loading function of *f* = 1 gives a series of elemental load vectors of:

$$\mathbf{f_e} \approx \frac{A_e}{12}\begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{f_1} = \mathbf{f_2} = \mathbf{f_3} = \mathbf{f_4}$$

<u>Assembly</u> - Since there are four elements in the model the assembly is not difficult:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 2 \\ 1 \end{Bmatrix}$$

Element 4



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Elemental Formulation - The loading function of *f* = 1 gives a series of elemental load vectors of:

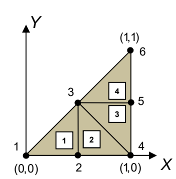
$$\mathbf{f_e} \approx \frac{A_e}{12}\begin{Bmatrix} 2f_i + f_j + f_k \\ f_i + 2f_j + f_k \\ f_i + f_j + 2f_k \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{f_1} = \mathbf{f_2} = \mathbf{f_3} = \mathbf{f_4}$$

<u>Assembly</u> - Since there are four elements in the model the assembly is not difficult:

Element 1 Element 2

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 2 \\ 1 \end{Bmatrix}$$

Element 3 Element 4



TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - **Constraints** - For this model, Ψ = 0 on the boundary, therefore, Ψ_4, Ψ_5, and Ψ_6 = 0.

$$\frac{1}{2}\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \\ 4 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solution - Solving the above equations gives:

$$\Psi_1 = \frac{14}{48} \qquad \Psi_2 = \frac{10}{48} \qquad \Psi_3 = \frac{9}{48} \qquad \Psi = \frac{\phi}{2G\theta a^2}$$

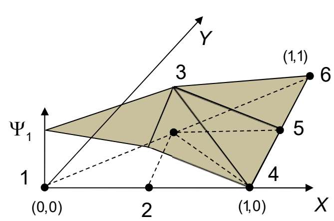
$$\phi_1 = \frac{28G\theta a^2}{48} \qquad \phi_2 = \frac{20G\theta a^2}{48} \qquad \phi_3 = \frac{18G\theta a^2}{48}$$

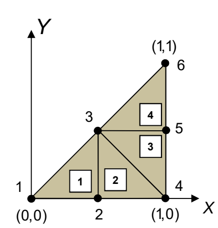
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Solution - Solving the above equations gives:

$$\Psi_1 = \frac{14}{48} \qquad \Psi_2 = \frac{10}{48} \qquad \Psi_3 = \frac{9}{48} \qquad \Psi = \frac{\phi}{2G\theta a^2}$$

$$\phi_1 = \frac{28G\theta a^2}{48} \qquad \phi_2 = \frac{20G\theta a^2}{48} \qquad \phi_3 = \frac{18G\theta a^2}{48}$$





TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

<u>Example</u> - Computation of Derived Variables - The total torque may be calculated as:

$$T = 4G\theta a^4 \iint \Psi \, dXdY = 8\left(4G\theta a^4 \iint_{A_e} \Psi_e^T N \, dX \, dY \right)$$

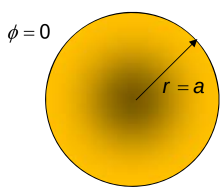
$$T = 8\sum_{e=1}^{4} \left(\frac{4G\theta a^4 A_e}{3} \left(\Psi_{e1} + \Psi_{e2} + \Psi_{e3} \right) \right) = \frac{2,240G\theta a^4}{1,152}$$

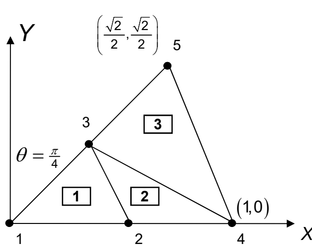
$$T = 1.9444G\theta a^4 \qquad \rightarrow \qquad T_{exact} = 2.2496G\theta a^4$$

$T = 1.7778G\theta a^4$ One three-node triangle model

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

PROBLEMS #18 - For the mesh shown below set up and solve the torsion problem for the circle. Compare your results for the maximum shear stress and the total torque *T* with the exact solution.





$$X = \frac{r}{a}\cos\theta$$

$$Y = \frac{r}{a}\sin\theta$$

Should the answers depend upon the angle θ?

What boundary conditions should be used on the radial lines of the model?

Check to see how well these boundary conditions are satisfied.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

PROBLEM #19 - Repeat Problem #18 using 4, 8, and 16 triangular elements.

Utilize the program **POIS36** given out in the class to perform your analysis.

Compare your results for the maximum shear stress and the total torque *T* with the exact solution.

End of Chapter 3a