One of the most useful differential equations for engineers is Laplace’s equation.

Pierre-Simon, marquis de Laplace (March 23, 1749 – March 5, 1827) was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics. He formulated Laplace’s equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in applied mathematics, is also named after him.

The nonhomogeneous form of Laplace’s equation is called the Poisson equation:

$$\nabla^2 u + f = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f = 0$$

Siméon-Denis Poisson (June 21, 1781 – April 25, 1840), was a French mathematician, geometer, and physicist. Poisson’s well-known correction of Laplace’s second order partial differential equation for potential was first published in the Bulletin de la société philomatique (1813).

Where $\Omega$ is the interior domain, and $\Gamma_1$ and $\Gamma_2$ form the boundary of the domain.

A boundary condition that specifies the value of the function $u$ on the surface $\Gamma_1$ is called a Dirichlet (dee ree KLAY) boundary condition or type one condition.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Johann Peter Gustave Lejeune Dirichlet (February 13, 1805 – May 5, 1859) was a German mathematician credited with the modern formal definition of a function.

\[ u(x, y) + f(x, y) = 0 \]

\[ u = g(s) \]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

A boundary condition prescribed in the form of a derivative of the function \( u \) on the surface \( \Gamma_2 \) is called a type two or a Neumann boundary condition.

\[ \frac{\partial u}{\partial n} = a(s)u - h(s) \]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Carl Gottfried Neumann (May 7, 1832 - March 27, 1925) was a German mathematician. Neumann worked on the Dirichlet principle, and can be considered one of the initiators of the theory of integral equations.

\[ \frac{\partial u}{\partial n} = a(s)u - h(s) \]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

If the value of \( \alpha \) is not zero then the condition is called a Robins boundary condition.

\[ \frac{\partial u}{\partial n} = a(s)u - h(s) \]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Victor Gustave Robin (1855-1897) was a French mathematical analyst and applied mathematician who lectured in mathematical physics at the Sorbonne in Paris and also worked in the area of thermodynamics.

\[ \frac{\partial u}{\partial n} = a(s)u - h(s) \]

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

If the entire boundary is a type one boundary condition, then the boundary value problem is called a Dirichlet Problem.

\[ \frac{\partial u}{\partial n} = a(s)u - h(s) \]
If the entire boundary is a type two with $\alpha = 0$ then the boundary value problem is called a Neumann Problem.

**Ritz Finite Element Model**

The Ritz finite element formulation is based on the functional for the Poisson equation.

If you are interested in how the energy functional is developed refer to your textbook.

The Poisson functional is:

$$Z(u) = \frac{1}{2} \iint_{\Omega} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \, d\Omega - \iiint_{\Omega} uf \, d\Omega$$

$$+ \frac{1}{2} \int_{\gamma} au^2 \, ds + \int_{\gamma} uh \, ds = 0$$

**Discretization**

- The first step in developing a finite element model, just as in one-dimensional analysis, is discretization.

The first type of two-dimensional discretization we will discuss utilizes straight-sided or linear triangular elements.

**Discretization**

- Another difficulty is that the boundary conditions which are generally described as continuous function over the boundary are distributed over a piecewise linear representation of the surface.

**Discretization**

- Both of these difficulties may be modeled more accurately by using most sophisticated elements, for example a triangular element with curved sides.
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

**Discretization** - In terms of the discretization, the functional $Z$ may be represented by a sum of the integrals over each element area $A_e$ and each elemental surface $\gamma_e$ as:

$$Z(u) \approx \frac{1}{2} \sum \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dA_e - \sum \int uf dA_e$$

$$+ \frac{1}{2} \sum \int au^2 ds - \sum \int uh ds = 0$$

Where the sum is over all the elements, and $\Sigma$ is over each elemental segment $\gamma_{ke}$ of the $\Gamma_2$ portion of the surface.

---

**Interpolation** - The simplest interpolation over a straight-sided three node triangular element is to assume the function $u(x, y)$ is represented by a linear plane.

$$u(x, y) = \alpha + \beta x + \gamma y$$

where $\alpha$, $\beta$, and $\gamma$ are constants determined by matching the function $u_e$ with the nodal values of the element:

$$u_e(x_i, y_i) = \alpha + \beta x_i + \gamma y_i = u_i$$

$$u_e(x_j, y_j) = \alpha + \beta x_j + \gamma y_j = u_j$$

$$u_e(x_k, y_k) = \alpha + \beta x_k + \gamma y_k = u_k$$

---

**Interpolation** - In a manner similar to that used to develop the linear, quadratic, and cubic shape functions for one-dimensional problems, we may describe the variation of $u$ over the element as:

$$u(x, y) = N_{ij}u_i + N_{jk}u_j + N_{ki}u_k$$

where $i$, $j$, and $k$ are permuted cyclically.

The determinant of the coefficients is:

$$2A_e = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

where $A_e$ is the area of the element.

Any numbering scheme that proceeds counterclockwise around the element is valid, for example $(i, j, k)$, $(j, k, i)$, or $(k, i, j)$.

This numbering convention is important and necessary in order to compute a positive area for $A_e$.

---

**Interpolation** - Solving the three equations for $\alpha$, $\beta$, and $\gamma$ and substituting back into the expression representing the variation of $u$ over the element results in:

$$u_e(x, y) = N_{ij}u_i + N_{jk}u_j + N_{ki}u_k$$

where:

$$N_i = \frac{a_i + b_i x + c_i y}{2A_e}$$

with:

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

where $i$, $j$, and $k$ are permuted cyclically.

**Interpolation** - In matrix notation, the distribution of the function over the element is:

$$u_e(x, y) = N^T u_s$$

The linear triangular shape functions are illustrated below:
Two-Dimensional Boundary Value Problems

Ritz Finite Element Model

Interpolation - The derivatives of $u$ over the element with respect to both coordinates are:

$$\frac{\partial u_0(x,y)}{\partial x} = u_0 \frac{\partial N}{\partial x}, \quad \frac{\partial u_0(x,y)}{\partial y} = u_0 \frac{\partial N}{\partial y}$$

Calculating the derivatives of the shape functions gives:

$$\frac{\partial N}{\partial x} = \frac{b_s}{2A_e}, \quad \frac{\partial N}{\partial y} = \frac{c_s}{2A_e}$$

$$b_s = \{b_i, b_j, b_k\}, \quad c_s = \{c_i, c_j, c_k\}$$

$$b_i = y_j - y_k \quad c_i = x_k - x_i$$

Elemental Formulation - The functional for the Poisson equation is:

$$Z(u) = \int_{A_e} \left[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} \right] dA$$

We can write the functional in the following form:

$$Z(u) = \sum_{s=1}^{4} \frac{Z_s}{2} = \sum_{s=1}^{4} \left( \frac{Z_{s1}}{2} + \frac{Z_{s2}}{2} - \frac{Z_{s3}}{2} - \frac{Z_{s4}}{2} \right)$$

Elemental Formulation - Evaluation of $Z_{s1}$:

$$Z_{s1} = \int_{A_e} \left[ \frac{\partial \tilde{u}}{\partial x} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \frac{\partial \tilde{u}}{\partial y} \right] dA$$

Recall the first derivatives of $u$ with respect to $x$ and $y$ are:

$$\frac{\partial u_0(x,y)}{\partial x} = u_0 \frac{\partial N}{\partial x}, \quad \frac{\partial u_0(x,y)}{\partial y} = u_0 \frac{\partial N}{\partial y}$$

$$\frac{\partial \tilde{u}_0(x,y)}{\partial x} = u_0 \left( \frac{\partial N}{\partial x} \right)^T, \quad \frac{\partial \tilde{u}_0(x,y)}{\partial y} = u_0 \left( \frac{\partial N}{\partial y} \right)^T$$

Elemental Formulation - Evaluation of $Z_{s1}$: Replacing the derivatives with the above approximations gives:

$$Z_{s1} = \int_{A_e} \left[ \frac{\partial \tilde{N}}{\partial x} \frac{\partial \tilde{N}}{\partial x} + \frac{\partial \tilde{N}}{\partial y} \frac{\partial \tilde{N}}{\partial y} \right] dA$$

$$= u_0^T \left[ \frac{\partial \tilde{N}}{\partial x} \frac{\partial \tilde{N}}{\partial x} + \frac{\partial \tilde{N}}{\partial y} \frac{\partial \tilde{N}}{\partial y} \right] dA = u_0^T \left[ \frac{\partial \tilde{N}}{\partial x} \frac{\partial \tilde{N}}{\partial x} + \frac{\partial \tilde{N}}{\partial y} \frac{\partial \tilde{N}}{\partial y} \right] dA$$

$$k_s = \int_{A_e} \left[ \frac{\partial \tilde{N}}{\partial x} \frac{\partial \tilde{N}}{\partial x} + \frac{\partial \tilde{N}}{\partial y} \frac{\partial \tilde{N}}{\partial y} \right] dA$$
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - Evaluation of $Z_{e_1}$:
The integrals defined in $k_e$ are the elemental “stiffness” matrix.
For the linear triangular element we have discussed the stiffness matrix reduces to:

$$k_e = \int_A \left[ \frac{b_i b_i^T + c_i c_i^T}{4A_e} \right] dA$$

Since the integrand of $k_e$ is a constant, the elemental stiffness matrix becomes:

$$k_e = \frac{b_i b_i^T + c_i c_i^T}{4A_e}$$

The resulting is a 3x3 elemental stiffness matrix

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - Evaluation of $Z_{e_2}$:

$$Z_{e_2} = \int_{\Gamma_e} \alpha\partial u\partial n d\Gamma$$
In this case, the interpolation of $u$ with respect to $x$ and $y$ is used to describe the behavior along the boundary:

$$Z_{e_2} = \int_{\Gamma_e} u_i N_{\alpha} N_i^T d\Gamma = u_i \left( \int_{\Gamma_e} N_{\alpha} N_i^T d\Gamma \right) u_{i\alpha} = u_i a_{\alpha}$$

The resulting is a 2x2 elemental stiffness matrix

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - Evaluation of $Z_{e_3}$:

$$Z_{e_3} = \int_A u_h dA$$
Substituting the approximation for $u$ into the integral results in:

$$Z_{e_3} = u_i \left( \int_A N f dA \right) = u_i f_e$$

The resulting is a 3x1 elemental load vector

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - Evaluation of $Z_{e_4}$:

$$Z_{e_4} = \int_{\Gamma_e} u_h d\Gamma$$
Substituting the approximation for $u$ into the integral results in:

$$Z_{e_4} = u_i \left( \int_{\Gamma_e} N h d\Gamma \right) = u_i h_e$$

The resulting is a 2x1 elemental load vector

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - In terms of the matrix definitions, the functional may be written in the following form:

$$Z(u_i, u_j, \ldots, u_k) = \sum_i \frac{u_i k_i u_i}{2} - u_i^T f_e + \sum_i \frac{u_i a_i u_i}{2} - u_i^T h_e$$

where the first sum is over all the elements that form the domain of the problem and the second sum is over elements that have a straight-line segment on the boundary of the domain.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Ritz Finite Element Model

Elemental Formulation - In terms of the matrix definitions, the functional may be written in the following form:

$$Z(u_i, u_j, \ldots, u_k) = \sum_i \frac{u_i k_i u_i}{2} - u_i^T f_e + \sum_i \frac{u_i a_i u_i}{2} - u_i^T h_e$$

In this formulation, there are two types of “stiffness” components.
The first, the $k_e$ terms, are associated with the Laplacian differential operator and the second, the $a_e$ terms, correspond to the prescribed boundary conditions.
Ritz Finite Element Model

**Elemental Formulation** - In terms of the matrix definitions, the functional may be written in the following form:

\[
Z(u_i, u_j, u_{i'}, u_{j'}) = \sum_t \left( u_i^k u_j - u_{i'}^k u_{j'} \right) + \sum_t \left( u_i^l u_j - u_{i'}^l u_{j'} \right) h_t
\]

The right-hand side of the system equations is also formed from two components.

The \( f_t \) terms correspond to the Poisson term of the differential equation.

The \( h_t \) terms handle any nonhomogeneous boundary conditions.

**Constraints** - The constraints on the system equations are the forced boundary conditions \( u = g(s) \) on the surface \( \Gamma \).

These conditions are applied to the system equations in a manner similar to that discussed for one-dimensional problems.

**Solution** - Since there are two types of boundary conditions, there are three possible situations to consider when determining a solution to the system equations.

A second case where a unique solution is possible is when the boundary is composed of both Dirichlet and Neumann boundary conditions.

The one situation where a singular solution is obtained is when Neumann-type conditions are prescribed along the entire boundary.

In this case, only a derivative-type condition exists.

There are infinite solutions in this type of problem.
Ritz Finite Element Model

Computation of Derived Variables - The solution for the nodal values of $u$ are often called the primary variables, whereas the derivatives and any other values based on the primary variables are called secondary or derived variables.

In this case the values of the function $u$ are the primary variables and $\frac{\partial u}{\partial n} = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y}$ is considered a secondary variable.

Evaluation of Matrices - Linear Triangular Elements

Recall the elemental matrices have the following form:

$$ A = \int \left[ \frac{\partial N^j}{\partial x} \frac{\partial N^l}{\partial x} + \frac{\partial N^j}{\partial y} \frac{\partial N^l}{\partial y} \right] dA $$

$$ k_e = \int \left[ b_1, b_2, c_1, c_2 \right] \cdot \left[ b_1, b_2, c_1, c_2 \right]^T \frac{1}{A_e} dA $$

Evaluation of $a_e$ - Using the local or area coordinates in the integrals transform the elemental matrices as follows:

$$ a_e = \int N^j N^l dA $$

Where $L_1, L_2, L_3$ are the area coordinates.

Evaluation of $f_e$ - Using the local or area coordinates in the integrals transform the elemental matrices as follows:

$$ f_e = \int N^j N^l \frac{dA}{A_e} $$

Using the local or area coordinates in the integrals transform the elemental matrices as follows:

$$ h_e = \int N^j h^l dA $$

The differential area $dA$ is a vector with magnitude $dA$ and direction normal to the element area, which in this case is $k$.
Evaluation of Matrices - Linear Triangular Elements

Evaluation of \( k_e \) - The vector \( \mathbf{dA} \) is given by the determinant rule:

\[
\mathbf{dA} = \mathbf{dx} \times \mathbf{dy} - \left| \begin{array}{ccc}
\frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} & 0 \\
\frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} & 0 \\
0 & 0 & 0
\end{array} \right| \mathbf{dL} \mathbf{dL}
\]

where:

\[
\mathbf{dx} = \frac{\partial x}{\partial \mathbf{L}} \mathbf{dL} + \frac{\partial x}{\partial \mathbf{L}} \mathbf{dL} \\
\mathbf{dy} = \frac{\partial y}{\partial \mathbf{L}} \mathbf{dL} + \frac{\partial y}{\partial \mathbf{L}} \mathbf{dL}
\]

\[ dA = |J| dL dL \]

Therefore, \( k_e \) transformed into area coordinates is:

\[ k_e = \int \left| \mathbf{J} \right| \left[ \mathbf{G} \left( \mathbf{x} \left( \mathbf{L}, L_j \right) \right) \right] \left| \mathbf{J} \right| dL dL \]

Evaluation of Matrices - Linear Triangular Elements

Evaluation of \( \mathbf{dA} \) - To transform the partial derivatives \( \frac{\partial x}{\partial \mathbf{L}} \) and \( \frac{\partial y}{\partial \mathbf{L}} \) to functions of \( L_i \) and \( L_j \):

\[
\frac{\partial \mathbf{L}}{\partial \mathbf{L}} = \begin{bmatrix} \frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} \\ \frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} \\ \frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} \end{bmatrix} = J \mathbf{J}^{-1}
\]

where

\[
J = \begin{bmatrix} \frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} \\ \frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} \\ \frac{\partial x}{\partial \mathbf{L}} & \frac{\partial y}{\partial \mathbf{L}} \end{bmatrix}, \quad J_1 = \begin{bmatrix} \frac{\partial x}{\partial \mathbf{L}} \\ \frac{\partial x}{\partial \mathbf{L}} \\ \frac{\partial x}{\partial \mathbf{L}} \end{bmatrix}, \quad J_2 = \begin{bmatrix} \frac{\partial y}{\partial \mathbf{L}} \\ \frac{\partial y}{\partial \mathbf{L}} \\ \frac{\partial y}{\partial \mathbf{L}} \end{bmatrix}
\]

The resulting elemental stiffness matrix \( k_e \) is a 3 x 3 matrix.
**TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS**

**Evaluation of Matrices - Linear Triangular Elements**

**Evaluation of \( f_s \) - In general, the integral \( f_s \) is:**

\[
f_s = \iint_N f(x,y) \, dA
\]

For a general function \( f(x,y) \), this integral may be quite tedious to evaluate, therefore we will assume that \( f \) varies linearly over the element, \( f(x,y) = N^T f \), where the vector \( f \) contains values of the function \( f \) at the node points.

With this assumption the integral becomes:

\[
f_s = \iint_N N^T f \, dA = \left( \int_N N^T \, dA \right) f
\]

**Evaluation of \( h_s \) - Consider the integral:**

\[
h_s = \int_{\gamma_s} h \, ds
\]

where the integration is along a boundary segment of the element.

Since, the integration is computed along a single side of the triangular element, the original shape functions reduce to:

\[
N_j = \begin{cases} 0 & j = 1 - \xi \\ 1 - \xi & j = \xi \\ 1 & j = 1 \end{cases}
\]

Therefore:

\[
h_s \approx \int_{\gamma_s} N^T h \, ds
\]

**The resulting is a 2 x 1 element load vector**

**Evaluation of \( a_s \) - The evaluation of \( a_s \) is very similar that of \( h_s \) except that there is an extra \( N^T \) in the integrand.**

The variation of the function \( a(\xi) \) will be approximated as

\[
a(\xi) \approx a(\xi) N_j + a(\xi) N_k
\]

Consider \( h(\xi) = N^T h \), where the vector \( h \) contains the values of the function \( h \) at the boundary node points.

\[
h_s = \int_{\gamma_s} \left(\begin{array}{c} 1 - \xi \\ \xi \end{array}\right) h(\xi) \, ds
\]

**The resulting is a 2 x 2 element stiffness matrix**

**Evaluation of Matrices - Linear Triangular Elements**

**Evaluation of \( f_s \) - The formula for integrations of the type is given without proof as:**

\[
\iint_N N_j^T N_k^T f \, dA = \frac{2A_s}{(a+b+c+2)!} f_{jk}
\]

Therefore:

\[
f_s \approx \frac{A_s}{12} \left(\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right) f
\]

The resulting is a 3 x 1 element load vector.
Evaluation of Matrices - Linear Triangular Elements

Evaluation of $a_s$ - The integration formula for the type of integrals is:

$$\int N_i^j N_k^l \, ds = a! b! \frac{1}{(a+b+1)!}$$

$$a_s = \frac{1}{12} \left[ 3 \alpha_j + \alpha_k \quad \alpha_j + \alpha_k \quad \alpha_j + \alpha_k \right]$$

The resulting 2 x 2 stiffness matrix contributes to the global system equations when the element has a side as part of the boundary.

Example - Consider the problem of torsion of a homogeneous isotropic prismatic bar. The general two-dimensional boundary-value problem is:

$$\nabla^2 \phi(x, y) + 2G \frac{\partial \theta}{\partial z} = 0 \quad \text{in } \Omega$$

$$\phi = 0 \quad \text{on } \Gamma$$

where the dependent variable $\phi$ is the Prandlt stress function, $G$ is the shear modulus, and $\theta$ is the constant rate of twist along the axis of the bar. The stress components are given in terms of the derivatives of the Prandlt stress function:

$$\tau_{xz} = \frac{\partial \phi}{\partial y} \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

The total torque transmitted along the bar is determined from:

$$T = 2 \int_\Omega \phi \, dA$$
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Before beginning the FEM model, it is desirable to non-dimensionalize the problem.

\[ X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \psi = \frac{\phi}{2Gh} \]

Therefore, the governing differential equation becomes:

\[ \nabla^2 \psi (X, Y) + 1 = 0 \quad \text{in} \quad \Omega \quad -1 \leq X \leq 1 \]

\[ \psi = 0 \quad \text{on} \quad \Gamma \quad -1 \leq Y \leq 1 \]

The stresses and torque for the Prandtl stress function are:

[Equations for \( \tau_{xx}, \tau_{yy}, T \) are presented.]

Example - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:

[Stiffness matrix equations with specific values and node coordinates are shown.]

Example - Computation of Derived Variables - The partial derivatives with respect to \( x \) and \( y \) that define the stress components are:

[Equations for \( \tau_{xx}, \tau_{yy}, T \) are presented.]

Example - Constraints - For this model, \( \psi = 0 \) on the boundary, therefore, \( \psi_1 \) and \( \psi_3 \) are 0.

Solution - In this case, the solution is quite simple:

[Solution equations are shown.]
Example - Consider the same problem of torsion of a homogeneous isotropic prismatic bar as above except using a more refined mesh.

Example - Recall, the non-dimensional Poisson equation governing this problem:
\[
V^2 \Psi (X, Y) + 1 = 0 \quad \text{in } \Omega \quad \text{with } -1 \leq X \leq 1
\]
\[
\Psi = 0 \quad \text{on } \Gamma \quad -1 \leq Y \leq 1
\]
\[
X = \frac{x}{a} \quad Y = \frac{y}{a} \quad \Psi = \frac{\phi}{2Gt}\]

The stresses and torque for the Prandtl stress function are:
\[
\tau_{xx} = 2Gt\frac{\partial^2 \Psi}{\partial Y^2} \quad \tau_{yy} = -2Gt\frac{\partial^2 \Psi}{\partial X^2}
\]
\[
T = 4Gt\int \int \Psi \, dXdY
\]

Example - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:
\[
k_e = \frac{(b \cdot b_e^T + c \cdot c_e^T)}{4A_e}
\]
\[
b_e = \begin{bmatrix}
Y_2 - Y_1 \\
Y_3 - Y_1 \\
Y_4 - Y_1
\end{bmatrix} \quad c_e = \begin{bmatrix}
X_2 - X_1 \\
X_3 - X_1 \\
X_4 - X_1
\end{bmatrix}
\]

For element 1: node 1 is located at (X, Y) = (0, 0); node 2 at (0.5, 0); and node 3 at (0.5, 0.5).

Example - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:
\[
k_e = \frac{(b \cdot b_e^T + c \cdot c_e^T)}{4A_e}
\]
\[
b_e = \begin{bmatrix}
Y_2 - Y_1 \\
Y_3 - Y_1 \\
Y_4 - Y_1
\end{bmatrix} \quad c_e = \begin{bmatrix}
X_2 - X_1 \\
X_3 - X_1 \\
X_4 - X_1
\end{bmatrix}
\]

For element 2: node 2 is located at (X, Y) = (0.5, 0); node 4 at (1, 0); and node 3 at (0.5, 0.5).

Example - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:
\[
k_e = \frac{(b \cdot b_e^T + c \cdot c_e^T)}{4A_e}
\]
\[
b_e = \begin{bmatrix}
Y_2 - Y_1 \\
Y_3 - Y_1 \\
Y_4 - Y_1
\end{bmatrix} \quad c_e = \begin{bmatrix}
X_2 - X_1 \\
X_3 - X_1 \\
X_4 - X_1
\end{bmatrix}
\]

For element 3: node 3 is located at (X, Y) = (0.5, 0.5); node 4 at (1, 0); and node 5 at (1, 0.5).

Example - Elemental Formulation - Using a linear triangular element the elemental stiffness matrix components are:
\[
k_e = \frac{(b \cdot b_e^T + c \cdot c_e^T)}{4A_e}
\]
\[
b_e = \begin{bmatrix}
Y_2 - Y_1 \\
Y_3 - Y_1 \\
Y_4 - Y_1
\end{bmatrix} \quad c_e = \begin{bmatrix}
X_2 - X_1 \\
X_3 - X_1 \\
X_4 - X_1
\end{bmatrix}
\]

For element 4: node 3 is located at (X, Y) = (0.5, 0.5); node 5 at (1, 0.5); and node 6 at (1, 1).
TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Elemental Formulation - The loading function of $T=1$ gives a series of elemental load vectors of:

$$f_1 = \frac{2f_1 + f_2 + f_3}{12}, \quad f_2 = \frac{2f_2 + f_3 + f_4}{12}, \quad f_3 = f_1 = f_4.$$  

Assembly - Since there are four elements in the model the assembly is not difficult:

<table>
<thead>
<tr>
<th>1</th>
<th>-1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Example - Constraints - For this model, $\Psi = 0$ on the boundary, therefore, $\Psi_x, \Psi_y$, and $\Psi_\theta = 0$.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 4 & -2 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ \end{bmatrix}.$$

Solution - Solving the above equations gives:

$$\Psi_1 = \frac{14}{48}, \quad \Psi_2 = \frac{10}{48}, \quad \Psi_3 = \frac{9}{48}, \quad \Psi = \frac{\phi}{2G/\alpha^2}.$$  

$$\phi = \frac{28G/\alpha^2}{48}, \quad \phi_2 = \frac{20G/\alpha^2}{48}, \quad \phi_3 = \frac{18G/\alpha^2}{48}.$$  

PROBLEMS #17 - For the mesh shown below set up and solve the torsion problem for the circle. Compare your results for the maximum shear stress and the total torque $T$ with the exact solution.

Should the answers depend upon the angle $\theta$? What boundary conditions should be used on the radial lines of the model? Check to see how well these boundary conditions are satisfied.

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Example - Computation of Derived Variables - The total torque may be calculated as:

$$T = 4G/\alpha^3 \int \left( \Psi_1 N dX dY + \Psi_2 N dX dY + \Psi_3 N dX dY \right) = 8 \left( 4G/\alpha^3 \int \Psi_1 dX dY + \Psi_2 dX dY + \Psi_3 dX dY \right) = 2.240 G/\alpha^3 + \frac{1}{152}.$$  

$$T = 1.944G/\alpha^3 \rightarrow \tau_{yxt} = \frac{16G/\alpha^3}{6} = 2.6667G/\alpha^3$$

PROBLEM #18 - Repeat Problem #17 using 4, 8, and 16 triangular elements.

Utilize the program POIS36 given out in the class to perform your analysis. Compare your results for the maximum shear stress and the total torque $T$ with the exact solution.