EIGENVALUE PROBLEMS

Sturm-Liouville problems arise that are eigenvalue problems rather than inhomogeneous boundary value problems.

The development and application of finite element models to these eigenvalue problems will be discussed in this section.

The standard form of the eigenvalue problem associated with the Sturm-Liouville problem can be expressed as:

$$(pu')' + (\lambda r - q)u = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

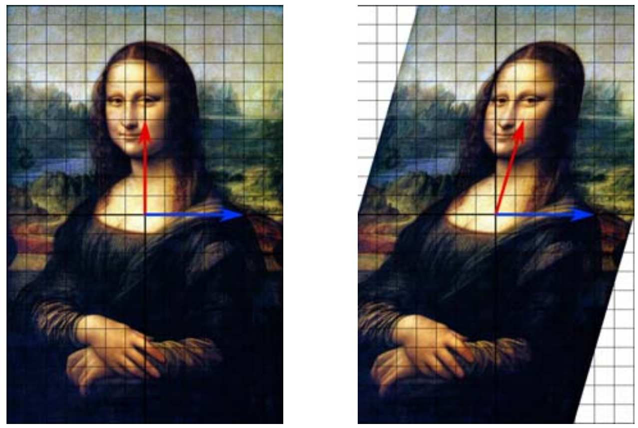
EIGENVALUE PROBLEMS

Eigenvalues are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, characteristic values, proper values, or latent roots.

The terms characteristic vector, characteristic value, and characteristic space are also used for these concepts.

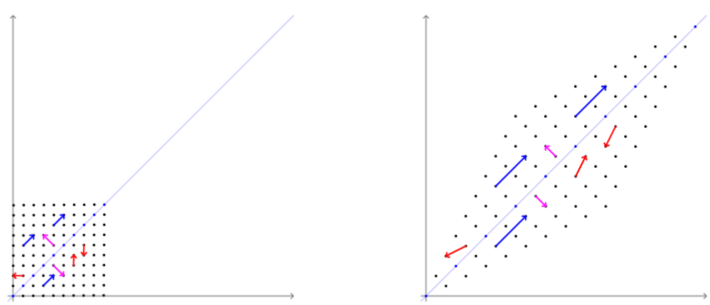
The prefix *eigen*- is adopted from the German word eigen for "self" or "proper".

EIGENVALUE PROBLEMS



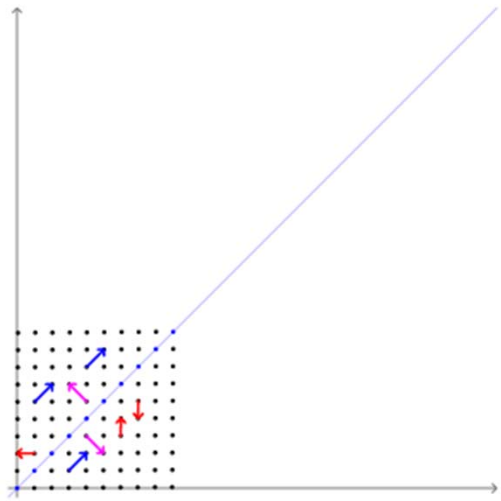
In this shear mapping the red arrow changes direction but the blue arrow does not. The blue arrow is an eigenvector of this shear mapping, and since its length is unchanged its eigenvalue is 1.

EIGENVALUE PROBLEMS



The transformation matrix preserves the direction of vectors parallel to eigenvector (in blue) and (in violet). The points that lie on the line through the origin, parallel to an eigenvector, remain on the line after the transformation. The vectors in red are not eigenvectors, therefore their direction is altered by the transformation.

EIGENVALUE PROBLEMS



EIGENVALUE PROBLEMS

The task confronting us is to determine the special values of the parameter λ for which there are corresponding nontrivial solutions *u*.

The λ 's and corresponding *u*'s are termed ***eigenvalues*** and ***eigenfunctions***, respectively.

$$(pu')' + (\lambda r - q)u = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

EIGENVALUE PROBLEMS

To this end we assume an approximate solution of the form:

$$v(x) = \sum_{i=1}^{N+1} v_i n_i(x)$$

To this end we assume an approximate solution of the form:

$$(pv')' + (\lambda r - q)v = \left(p\sum_{i=1}^{N+1} v_i n_i'(x)\right)' + (\lambda r - q)\sum_{i=1}^{N+1} v_i n_i(x)$$

$$= E(x, v_1, v_2, \cdots, v_{N+1})$$

where $n_i(x)$ are the linear nodal interpolation functions introduced earlier and ***E*** is the error arising from the fact that the approximate solution v does not (in general) satisfy the differential equation.

EIGENVALUE PROBLEMS

It can be shown that carrying through the integration by parts and the subsequent development leads to:

$$\mathbf{A_G u_G} - \lambda \mathbf{B_G u_G} = \mathbf{0}$$

$$\mathbf{A_G} = \sum_e (\mathbf{p_G} + \mathbf{q_G}) + \mathbf{BT_G} \qquad \mathbf{B_G} = \sum_e \mathbf{r_G}$$

where

$$\mathbf{p_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}' p(x) \mathbf{N}'^{\mathbf{T}}\, dx \qquad \mathbf{q_e} = \int_{x_i}^{x_{i+1}} \mathbf{N} q(x) \mathbf{N}^{\mathbf{T}}\, dx$$

$$\mathbf{r_e} = \int_{x_i}^{x_{i+1}} \mathbf{N} r(x) \mathbf{N}^{\mathbf{T}}\, dx \qquad \mathbf{BT_G} = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta \end{bmatrix}$$

EIGENVALUE PROBLEMS

Constraints arising from essential boundary conditions are enforced by deleting from both **A** and **B** the row and column corresponding to the constrained variable.

We write the constrained set of equations as:

$$(\mathbf{A} - \lambda\mathbf{B})u = 0$$

where **A** and **B** are now reduced *M* x *M* matrices with $M = N + 1 - m$, *m* being the number of essential boundary conditions that have been imposed.

EIGENVALUE PROBLEMS

Constraints arising from essential boundary conditions are enforced by deleting from both **A** and **B** the row and column corresponding to the constrained variable.

We write the constrained set of equations as:

$$(\mathbf{A} - \lambda\mathbf{B})u = 0$$

The equation above is an example of the ***generalized linear algebraic eigenvalue problem***.

It is very similar in character to the ***algebraic eigenvalue problem***:

$$(\mathbf{A} - \lambda_i\mathbf{I})u = 0$$

EIGENVALUE PROBLEMS

The scalars λ_i are the ***eigenvalues*** and the corresponding nontrivial vectors u_i satisfying: $\left(\mathbf{A} - \lambda_i \mathbf{B}\right) u_i = 0$ are the ***eigenvectors***.

For small hand-calculated finite element models, the λ_i and u_i are frequently obtained in the classical manner by expanding the determinant: $\left|\mathbf{A} - \lambda \mathbf{B}\right| = 0$

to obtain an M^{th} order polynomial whose roots are the approximate eigenvalues.

These M roots are then substituted one at a time into the equations: $\left(\mathbf{A} - \lambda_i \mathbf{B}\right) u_i = 0$

to determine the corresponding approximate ***eigenvectors***.

EIGENVALUE PROBLEMS

With the matrices $\mathbf{p_G}$, $\mathbf{q_G}$, $\mathbf{r_G}$, and $\mathbf{BT_G}$ symmetric, **A** and **B** are also symmetric.

In such a case, the theory can be used to show that all the eigenvalues λ_i are real and that eigenvectors u_i and u_j corresponding to distinct eigenvalues λ_i and λ_j satisfy a bi-orthogonality relationship given by:

$$\mathbf{u_i B u_j^T} = 0 \qquad i \neq j$$

These general results can be used as checks on the calculations when determining the eigenvalues and eigenvectors.

EIGENVALUE PROBLEMS

For eigenvalue problems of dimension larger than three or four, it is essential to have available a reliable computer code for extracting the eigenvalues and eigenvectors.

Appendix C (in your textbook) contains a discussion of and listings for several routines appropriate for this task.

In addition, MathCAD has functions for determining eigenvalues and eigenvectors:

`eigenvals(A)` and **`eigenvecs(A)`**

In Matlab the functions are:

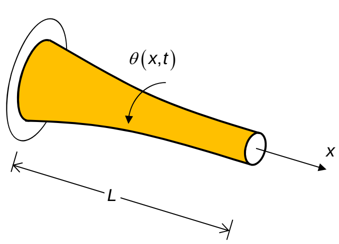
`[V,D]=eig(A)`

EIGENVALUE PROBLEMS

Torsional Vibrations

Consider the problem of the torsional vibrations of a uniform circular-cross-section bar.

The relationship governing twist θ is given as:



$$\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right) = \rho J \frac{\partial^2 \theta}{\partial t^2}$$

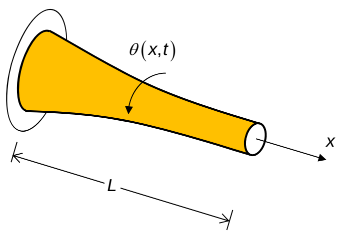
where G is the shear modulus, J is the polar moment of inertia, and ρ is the mass density.

EIGENVALUE PROBLEMS

Torsional Vibrations

Consider the problem of the torsional vibrations of a uniform circular-cross-section bar.

Torsional vibration is angular vibration of an object, commonly a shaft along its axis of rotation.



$$\theta(x,t) = \psi(x)e^{i\omega t}$$

$$\left(JG\psi'\right)' - \omega^2 \rho J \psi = 0$$

$$\psi\left(0\right) = 0$$

$$\psi'\left(L\right) = 0$$

With $\lambda = \omega^2$, comparison with the standard form shows that $p = JG$, $r = \rho J$, $q = 0$, and $\alpha = \beta = 0$.

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Two-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:

$$\mathbf{p_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}' JG \, \mathbf{N}'^{\mathbf{T}} \, dx \qquad \mathbf{r_e} = \int_{x_i}^{x_{i+1}} \mathbf{N} \, \rho J \, \mathbf{N}^{\mathbf{T}} \, dx$$

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N}' JG \, \mathbf{N}'^{\mathbf{T}} \, d\xi \qquad \mathbf{r_e} = \int_0^1 \mathbf{N} \, \rho J \, \mathbf{N}^{\mathbf{T}} \, l_e d\xi$$

For the present physical problem with $q = 0$, $\mathbf{k_e} = \mathbf{p_e}$ are the **elemental mechanical stiffness matrices**.

The $\mathbf{r_e}$ are the corresponding **elemental mass matrices** and will be denoted by $\mathbf{m_e}$.

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

In an analogous fashion we will use **K** rather than **A**, and **M** rather than **B** at the global level.

If we consider linear interpolation functions:

$$\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

with $l_e = L/2$ for each element:

$$\mathbf{k}_{e1} = \frac{2JG}{L}\int_0^1 \mathbf{N}'\mathbf{N}'^{\mathbf{T}}\, d\xi = \frac{2JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k}_{e2} = \mathbf{k}_{e1}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

In an analogous fashion we will use **K** rather than **A**, and **M** rather than **B** at the global level.

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with $l_e = L/2$ for each element:

$$\mathbf{m}_{e1} = \int_0^1 \mathbf{N}\rho J\mathbf{N}^{\mathbf{T}} l_e\, d\xi = \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \rho J \left\langle 1-\xi \quad \xi \right\rangle l_e d\xi = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{m}_{e2} = \mathbf{m}_{e1}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Expanded to the global level:

Element 1

$$\mathbf{k_{G1}} = \frac{2JG}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{k_{G2}} = \frac{2JG}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Element 2

Element 1

$$\mathbf{m_{G1}} = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{m_{G2}} = \frac{\rho JL}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Element 2

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

With $\mathbf{BT_G} = 0$, the assembled matrices are:

Element 1 Element 1

$$\mathbf{K_G} = \frac{2JG}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Element 2 Element 2

The constraint $\psi_1 = 0$ arises from the essential boundary condition $\psi(0) = 0$.

Denoting **K** and **M** as $\mathbf{K_G}$ and $\mathbf{M_G}$ with the first row and column deleted, there results:

$$(\mathbf{K} - \phi\mathbf{M})\psi = \begin{bmatrix} 2-4\phi & -1-\phi \\ -1-\phi & 1-2\phi \end{bmatrix}\begin{Bmatrix} \psi_2 \\ \psi_3 \end{Bmatrix} = 0 \qquad \phi = \frac{\omega^2 L^2 \rho}{24G}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Requiring the determinant of **K**-ϕ**M** to vanish yields:

$$2(1-2\phi)^2 = (1+\phi)^2 \qquad \phi = \frac{\omega^2 L^2 \rho}{24G}$$

The roots are: ϕ_1 = 0.1082 and ϕ_2 = 1.3204.

The corresponding frequencies are given by:

$$\omega_1^2 = \frac{24G\phi_1}{\rho L^2} = \frac{2.5968G}{\rho L^2} \qquad \left(\omega_1^2\right)_{exact} = \left(\frac{\pi}{2L}\right)^2 \frac{G}{\rho} = \frac{2.4674G}{\rho L^2}$$

$$\omega_2^2 = \frac{24G\phi_2}{\rho L^2} = \frac{31.690G}{\rho L^2} \qquad \left(\omega_2^2\right)_{exact} = \left(\frac{3\pi}{2L}\right)^2 \frac{G}{\rho} = \frac{22.207G}{\rho L^2}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

The estimate of the lowest eigenvalue is quite acceptable (5.1% error).

However, the estimate of the second eigenvalue is much less satisfactory (42.7% error).

$$\omega_1^2 = \frac{24G\phi_1}{\rho L^2} = \frac{2.5968G}{\rho L^2} \qquad \left(\omega_1^2\right)_{exact} = \left(\frac{\pi}{2L}\right)^2 \frac{G}{\rho} = \frac{2.4674G}{\rho L^2}$$

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EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

The eigenvectors are obtained by substituting the ϕ_i's, one at a time, back into the constrained equations.

For the first eigenvalue-eigenvector pair, the first equation becomes:

$$(2-4\phi_1)\psi_{12} - (1+\phi_1)\psi_{13} = 0$$

where $\psi^T = [\psi_{12} \quad \psi_{13}]$ is the constrained first eigenvector.

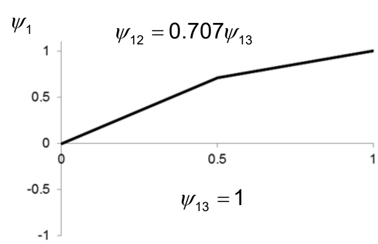
Solving for ψ_{12} yields: $\psi_{12} = 0.707\psi_{13}$

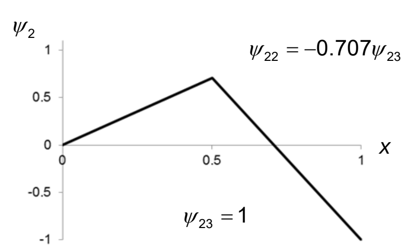
Repeating for the ϕ_2 yields: $\psi_{22} = -0.707\psi_{23}$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

The corresponding approximate eigenfunctions or mode shapes are shown below:





These two eigenfunctions or mode shapes are approximations to the exact eigenfunctions:

$$\psi_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right) \qquad \psi_1 = \sin\left(\frac{\pi x}{2L}\right) \qquad \psi_2 = \sin\left(\frac{3\pi x}{2L}\right)$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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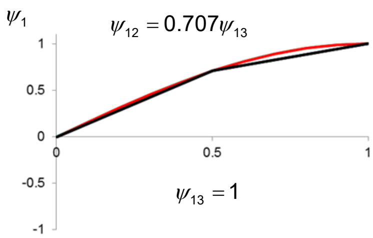

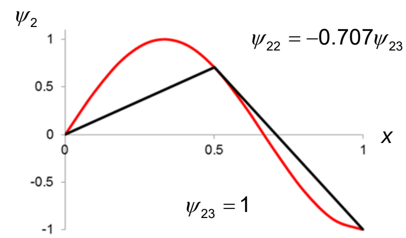

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EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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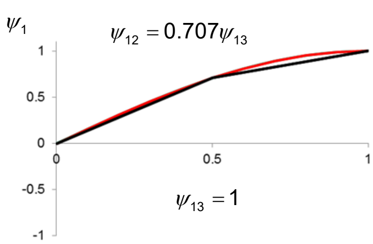

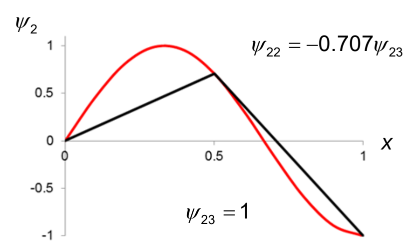

Note that the approximate eigenvectors can be made, by the appropriate choice of the arbitrary constant arising in the solution, to coincide at the nodes with the eigenfunctions they are trying to represent, and that they have the correct number of interior zeros (n -1) as required by the theory.

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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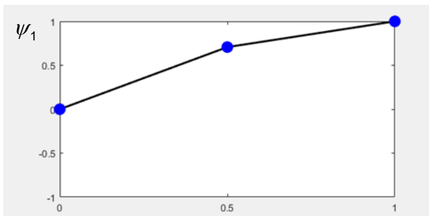

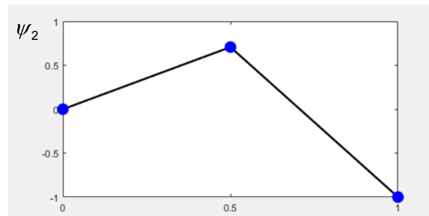

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EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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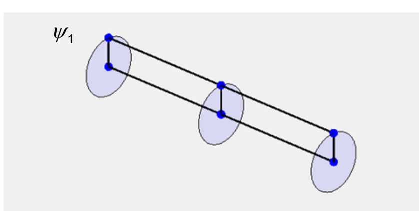

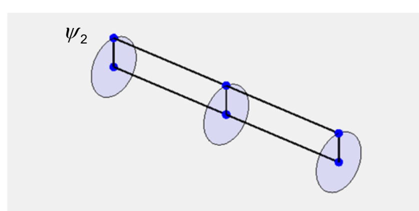

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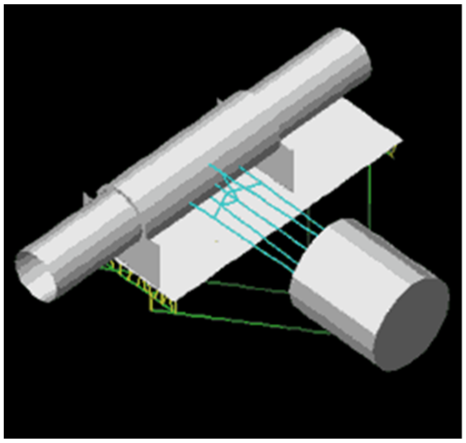
EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Example of torsional vibration mode shape



EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Four-element solution - The elemental $\mathbf{k_e}$ and $\mathbf{m_e}$ matrices have exactly the same form as in the two-element model with l_e now taken as L/4.

Omitting some of the details, the constrained 4 x 4 eigenvalue problem is: $\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$

where:

$$\mathbf{K_G} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{96G}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Four-element solution - Requiring the determinant of **K**-ϕ**M** to vanish yields four roots; using Matlab function **eig($M_G^{-1}K_G$)** the eigenvalues are displayed below along with the corresponding exact eigenvalues, and percent errors.

i	ϕ_i	$\omega_i^2 L^2 \rho / G$	$\left(\omega_i^2 L^2 \rho / G\right)_{exact}$	% error
1	0.026034	2.4993	2.4674	1.30
2	0.259085	24.872	22.207	12.05
3	0.854924	82.073	61.685	33.05
4	1.787792	171.62	120.90	41.96

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Four-element solution – The Matlab function **eig($M_G^{-1}K_G$)** gives:

$$\psi_1 = \begin{Bmatrix} 0.242030 \\ 0.447214 \\ 0.584313 \\ 0.632456 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.584313 \\ -0.447214 \\ 0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.584313 \\ -0.447214 \\ -0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.242030 \\ 0.447214 \\ -0.584313 \\ 0.632456 \end{Bmatrix}$$

Scaling the Matlab results to make the largest value equal to 1 gives:

$$\psi_1 = \begin{Bmatrix} 0.382656 \\ 0.707057 \\ 0.923814 \\ 1.000000 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} 0.923814 \\ 0.707057 \\ -0.382656 \\ -1.000000 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.923814 \\ -0.707057 \\ -0.382656 \\ 1.000000 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.382656 \\ 0.707057 \\ -0.923814 \\ 1.000000 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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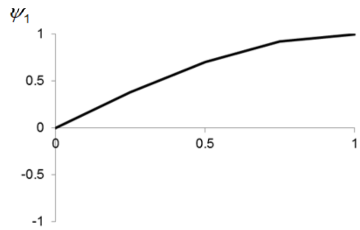

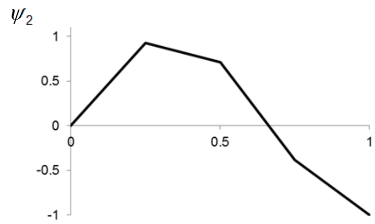

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EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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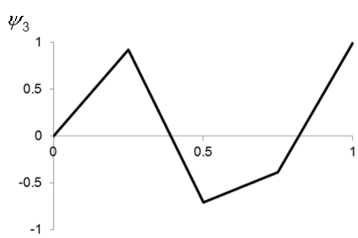

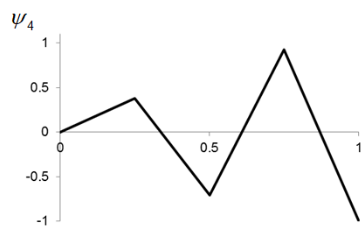

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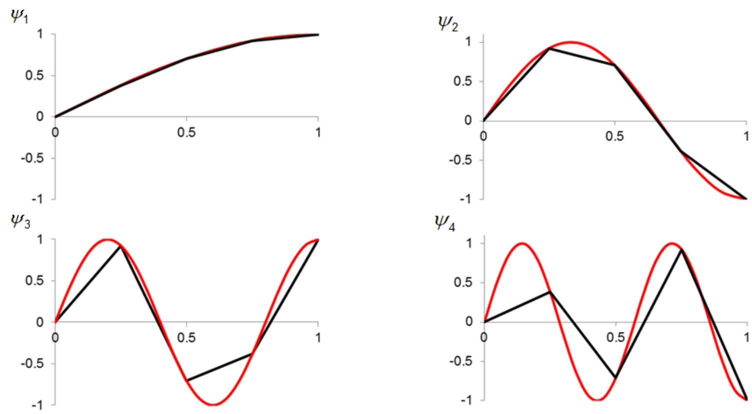
EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Four-element solution - For the two-element model, the number of constrained degrees of freedom is two.

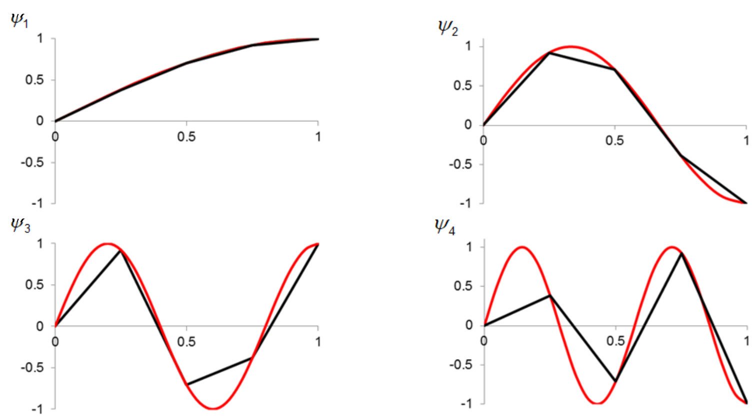
The lowest eigenvalue predicted by that model is 5.2% in error, a good estimate.



EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Four-element solution - With four degrees of freedom, the two lowest eigenvalue estimates are 1.3% and 12.0% in error respectively, again quite reasonable.

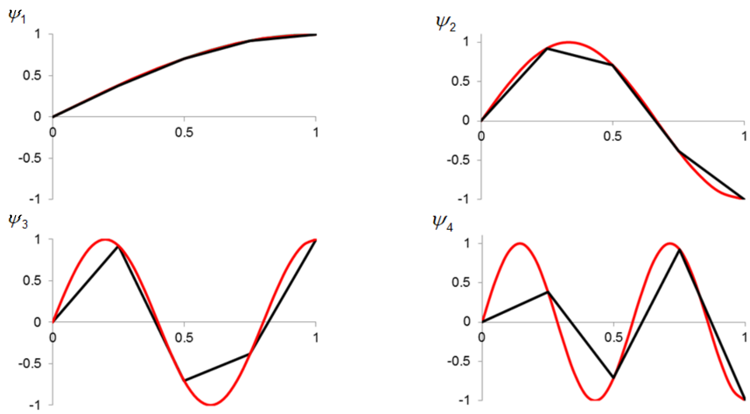


EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

This observation can be stated as a rule of thumb:

***For an algebraic eigenvalue problem of the type considered in this section, a model with 2N constrained degrees of freedom is necessary to obtain good estimates for the first N eigenvalues*.**



EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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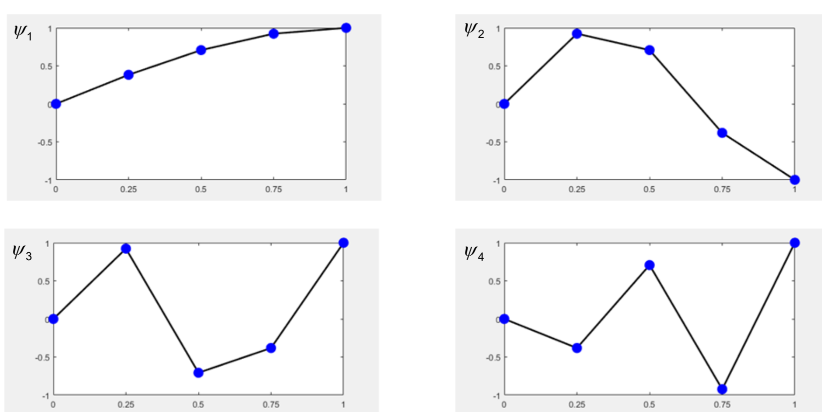

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

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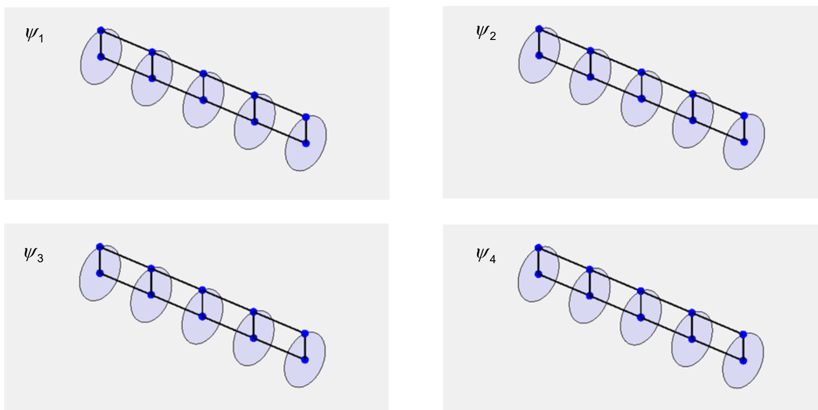


EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Eight-element solution - The elemental $\mathbf{k_e}$ and $\mathbf{m_e}$ matrices have exactly the same form as in the two- and four-element models with l_e now taken as L/8.

The constrained 8 x 8 eigenvalue problem is: $\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$

$$\mathbf{K_G} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \phi = \frac{\omega^2 L^2 \rho}{384G}$$

$$\boldsymbol{\psi}^T = \langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6 \quad \psi_7 \quad \psi_8 \quad \psi_9 \rangle$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Eight-element solution - The elemental $\mathbf{k_e}$ and $\mathbf{m_e}$ matrices have exactly the same form as in the two- and four-element models with l_e now taken as L/8.

The constrained 8 x 8 eigenvalue problem is: $\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$

$$\mathbf{M_G} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \phi = \frac{\omega^2 L^2 \rho}{384G}$$

$$\boldsymbol{\psi}^T = \left\langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6 \quad \psi_7 \quad \psi_8 \quad \psi_9 \right\rangle$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Eight-element solution - Requiring the determinant of $\mathbf{K}$-$\phi\mathbf{M}$ to vanish yields eight roots; using Matlab function **eig($\mathbf{M_G}^{-1}\mathbf{K_G}$).**

The first four eigenvalues are displayed below along with the corresponding exact eigenvalues, and percent errors.

i	ϕ_i	$\omega_i^2 L^2 \rho / G$	$\left(\omega_i^2 L^2 \rho / G\right)_{exact}$	% error
1	0.0064462	2.4753	2.4674	0.32%
2	0.0595205	22.8559	22.2066	2.92%
3	0.1739060	66.7799	61.6850	8.26%
4	0.3666860	140.8074	120.9026	16.46%

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

Compared to the four-element solution, the error in the values of the first four eigenvalues is significantly lower than those computed using the four-element approximation.

Using Matlab, the corresponding eigenvectors using **eig($M_G^{-1}K_G$)** are:

$$\psi_1 = \begin{Bmatrix} 0.091966 \\ 0.180399 \\ 0.261898 \\ 0.333333 \\ 0.391959 \\ 0.435521 \\ 0.462347 \\ 0.471405 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.261898 \\ -0.435521 \\ -0.462347 \\ -0.333333 \\ -0.091966 \\ 0.180399 \\ 0.391959 \\ 0.471405 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.391959 \\ 0.435521 \\ 0.091966 \\ -0.333333 \\ -0.462347 \\ -0.180399 \\ 0.261898 \\ 0.471405 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.462347 \\ -0.180399 \\ 0.391959 \\ 0.333333 \\ -0.261898 \\ -0.435521 \\ 0.091966 \\ 0.471405 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

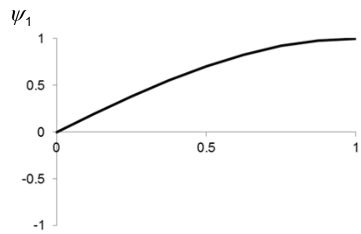
Compared to the four-element solution, the error in the values of the first four eigenvalues is significantly lower than those computed using the four-element approximation.

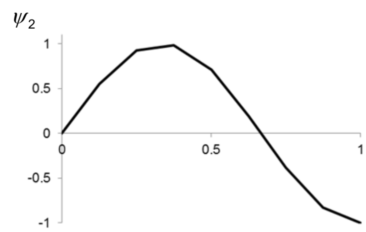
Scaling the results of Matlab as that the max is 1 gives:

$$\psi_1 = \begin{Bmatrix} 0.195090 \\ 0.382683 \\ 0.555570 \\ 0.707106 \\ 0.831469 \\ 0.923879 \\ 0.980785 \\ 1.000000 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.555571 \\ -0.923880 \\ -0.980786 \\ -0.707107 \\ -0.195091 \\ 0.382683 \\ 0.831469 \\ 1.000000 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.831469 \\ 0.923879 \\ 0.195093 \\ -0.707106 \\ -0.980785 \\ -0.382683 \\ 0.555570 \\ 1.000000 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} 0.980785 \\ 0.382683 \\ -0.831470 \\ -0.707107 \\ 0.555570 \\ 0.923879 \\ -0.195091 \\ -1.000000 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

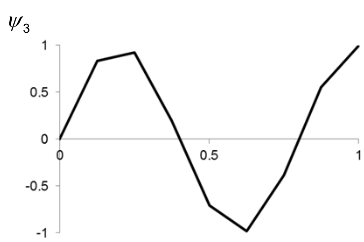


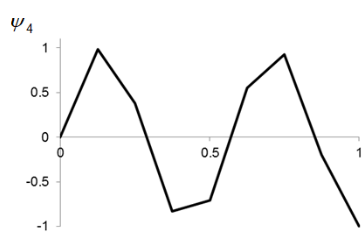


$$\psi_1 = \left\{\begin{matrix} 0.195090 \\ 0.382683 \\ 0.555570 \\ 0.707106 \\ 0.831469 \\ 0.923879 \\ 0.980785 \\ 1.000000 \end{matrix}\right\} \quad \psi_2 = \left\{\begin{matrix} -0.555571 \\ -0.923880 \\ -0.980786 \\ -0.707107 \\ -0.195091 \\ 0.382683 \\ 0.831469 \\ 1.000000 \end{matrix}\right\} \quad \psi_3 = \left\{\begin{matrix} 0.831469 \\ 0.923879 \\ 0.195093 \\ -0.707106 \\ -0.980785 \\ -0.382683 \\ 0.555570 \\ 1.000000 \end{matrix}\right\} \quad \psi_4 = \left\{\begin{matrix} 0.980785 \\ 0.382683 \\ -0.831470 \\ -0.707107 \\ 0.555570 \\ 0.923879 \\ -0.195091 \\ -1.000000 \end{matrix}\right\}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1





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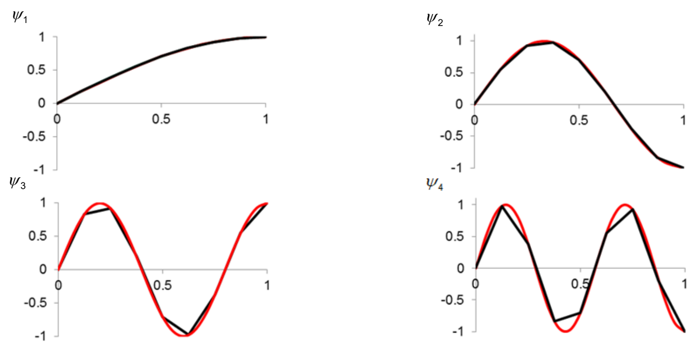
EIGENVALUE PROBLEMS

Torsional Vibrations – Example 1

These eigenvectors are similar in shape to those developed in the four-element solution; however, with an eight-element solution there is more detail in the curves.

The sinusoidal nature of the first four modes of vibration are more apparent in these plots than those using the four-element solution.



EIGENVALUE PROBLEMS

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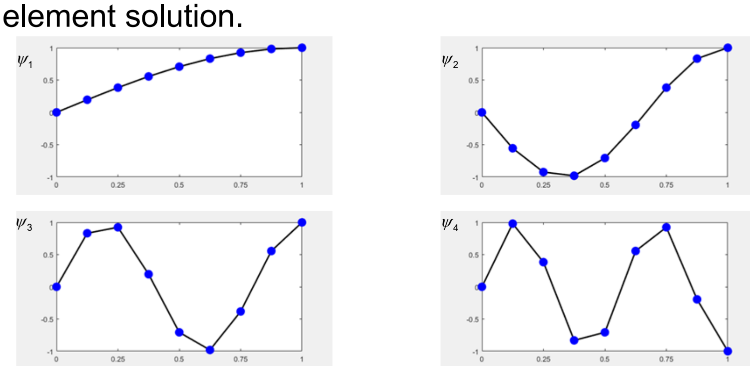

EIGENVALUE PROBLEMS

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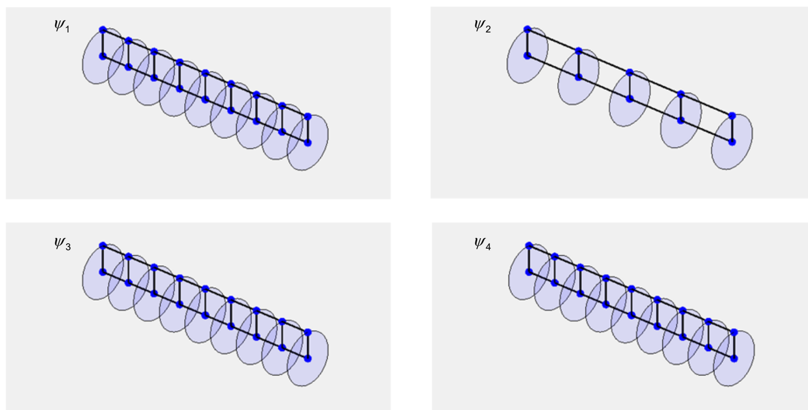
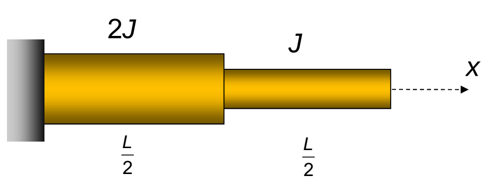


EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

Consider the torsional vibration of a non-uniform bar shown below.



G and ρ are constant

Four-element solution - The constrained 4 x 4 eigenvalue problem is:

$$\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The elemental $\mathbf{k_e}$ matrices, with l_e = L/4 are:

$$\mathbf{k_1} = \frac{8JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k_2} = \mathbf{k_1}$$

$$\mathbf{k_3} = \frac{4JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k_4} = \mathbf{k_3}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The elemental $\mathbf{m_e}$ matrices, with l_e = L/4 are:

$$\mathbf{m_1} = \frac{\rho JL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m_2} = \mathbf{m_1}$$

$$\mathbf{m_3} = \frac{\rho JL}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m_4} = \mathbf{m_3}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The unconstrained 5 x 5 eigenvalue problem is:

$$\left(\mathbf{K_G} - \phi\mathbf{M_G}\right)\boldsymbol{\psi} = 0$$

Element 1 Element 2 Element 3 Element 4

$$\mathbf{K_G} = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 \\ 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & 0 \\ 0 & 2 & 6 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \langle \psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{96G}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The constrained 4 x 4 eigenvalue problem with $\Psi_1 = 0$ is:

$$\left(\mathbf{K_G} - \phi\mathbf{M_G}\right)\boldsymbol{\psi} = 0$$

$$\mathbf{K_G} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \begin{bmatrix} 8 & 2 & 0 & 0 \\ 2 & 6 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{96G}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

Requiring the determinant of **K**-ϕ**M** to vanish yields four roots; using Matlab function **eig(M$_G^{-1}$K$_G$)** the eigenvalues are displayed below.

i	ϕ_i	$\omega_i^2 L^2 \rho / G$
1	0.0388	3.7248
2	0.2197	21.0912
3	0.9477	90.9792
4	1.6980	163.0080

Note that the eigenvalues for the four-element non-uniform bar are slightly larger than the four-element solution for the uniform bar.

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The eigenvalues can be found using a variety of available solution techniques.

In Matlab, **eig(M$_G^{-1}$K$_G$)** gives:

$$\psi_1 = \begin{Bmatrix} 0.217788 \\ 0.386824 \\ 0.595009 \\ 0.670000 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.450968 \\ -0.414621 \\ 0.330131 \\ 0.718144 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.450968 \\ -0.414621 \\ -0.330131 \\ 0.718144 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.217788 \\ 0.386824 \\ -0.595009 \\ 0.670000 \end{Bmatrix}$$

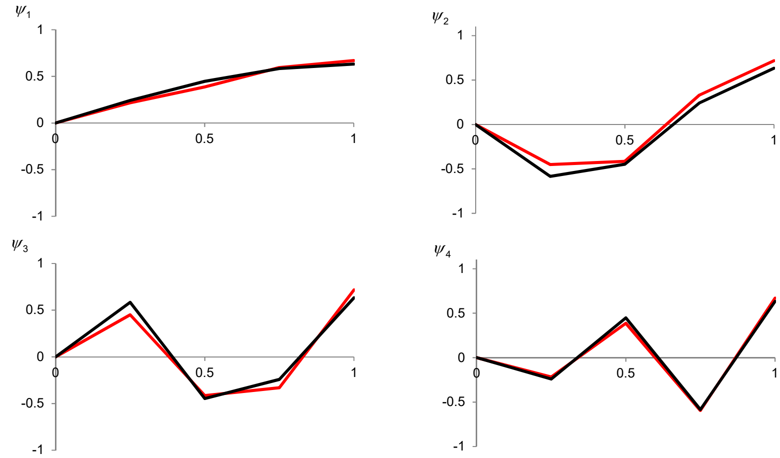
The eigenvectors for the four-element solution with a constant value of J are:

$$\psi_1 = \begin{Bmatrix} 0.242030 \\ 0.447214 \\ 0.584313 \\ 0.632456 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.584313 \\ -0.447214 \\ 0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.584313 \\ -0.447214 \\ -0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.242030 \\ 0.447214 \\ -0.584313 \\ 0.632456 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

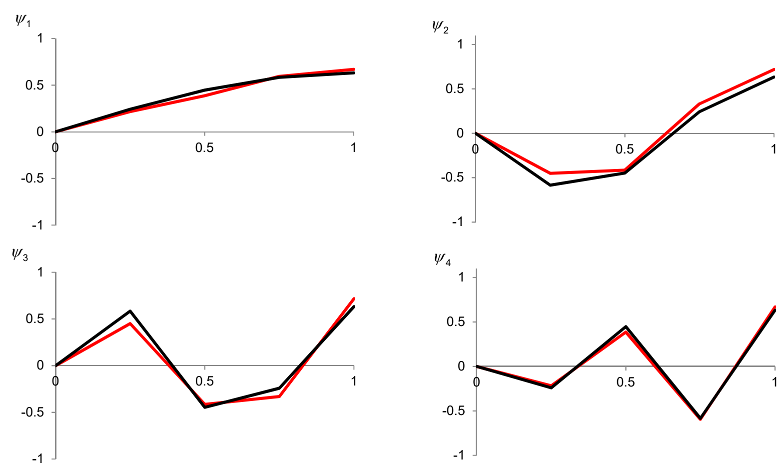
The corresponding scaled eigenfunctions are shown in the figures below (the eigenvalues for $\boldsymbol{J_{1,2} = 2J}$ are shown by the solid **red** lines).



EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

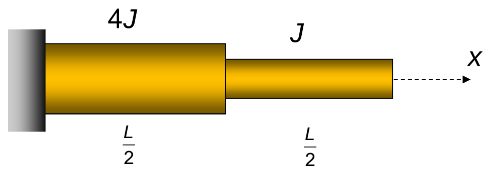
The values of the scaled eigenvectors are slightly smaller over the first half of the bar reflecting the increased value of $\boldsymbol{J}$ in that portion of the bar.



EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

To examine this effect further, consider the same non-uniform bar; however, the polar moment of inertia is $\boldsymbol{J_{1,2} = 4J}$ over the first half of the bar.



Four-element solution - The constrained 4 x 4 eigenvalue problem is:

$$\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The polar moment of inertia is $\boldsymbol{J_{1,2} = 4J}$ over the first half of the bar.

The elemental $\mathbf{k_e}$ matrices, with l_e = L/4 are:

$$\mathbf{k_1} = \frac{16JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k_2} = \mathbf{k_1}$$

$$\mathbf{k_3} = \frac{4JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k_4} = \mathbf{k_3}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The polar moment of inertia is $\boldsymbol{J_{1,2} = 4J}$ over the first half of the bar.

The elemental $\mathbf{m_e}$ matrices, with l_e = L/4 are:

$$\mathbf{m_1} = \frac{\rho JL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m_2} = \mathbf{m_1}$$

$$\mathbf{m_3} = \frac{\rho JL}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m_4} = \mathbf{m_3}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The constrained 4 x 4 eigenvalue problem is:

$$\left(\mathbf{K_G} - \phi\mathbf{M_G}\right)\boldsymbol{\psi} = 0$$

$$\mathbf{K_G} = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 5 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \begin{bmatrix} 16 & 4 & 0 & 0 \\ 4 & 10 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{96G}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

Requiring the determinant of **K**-ϕ**M** to vanish yields four roots; using Matlab function **eig($M_G^{-1}K_G$)** the eigenvalues are displayed below.

i	ϕ_i	$\omega_i^2 L^2 \rho / G$
1	0.05239	5.02944
2	0.18777	18.0263
3	1.03491	99.35136
4	1.61017	154.5763

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

The eigenvalues can be found using a variety of available solution techniques.

In Matlab, use **eig($M_G^{-1}K_G$)** gives:

$$\psi_1 = \begin{Bmatrix} 0.186214 \\ 0.316806 \\ 0.602601 \\ 0.708400 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.330386 \\ -0.347388 \\ 0.408379 \\ 0.776784 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.330386 \\ -0.347388 \\ -0.408379 \\ 0.776784 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.186214 \\ 0.316806 \\ -0.602601 \\ 0.708400 \end{Bmatrix}$$

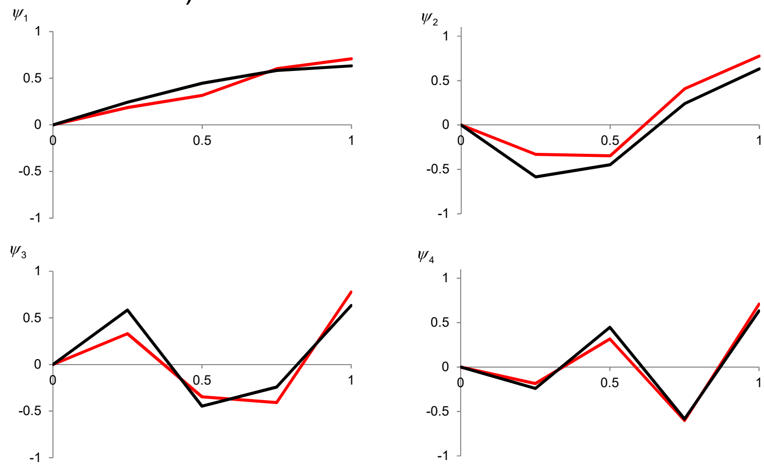
The eigenvectors for the four-element solution with a constant value of ***J*** are:

$$\psi_1 = \begin{Bmatrix} 0.242030 \\ 0.447214 \\ 0.584313 \\ 0.632456 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.584313 \\ -0.447214 \\ 0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.584313 \\ -0.447214 \\ -0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.242030 \\ 0.447214 \\ -0.584313 \\ 0.632456 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Torsional Vibrations – Example 2

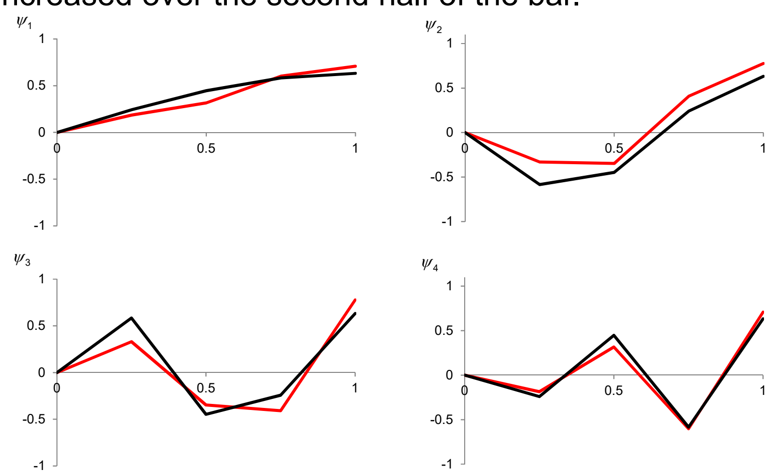
The corresponding scaled eigenfunctions are shown in the figures below (the eigenvalues for $\boldsymbol{J_{1,2} = 4J}$ are shown by the solid lines).



EIGENVALUE PROBLEMS

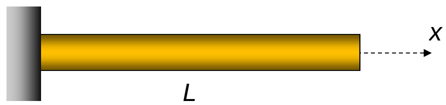
Torsional Vibrations – Example 2

The values of the scaled eigenvectors are smaller over the first half of the bar (increased value of ***J***) and slightly increased over the second half of the bar.



EIGENVALUE PROBLEMS

PROBLEM #16 - Consider the torsional vibration of a uniform bar shown below and develop a solution to one-dimensional eigenvalue problems using a quadratic interpolation function.



E, J, and ρ are constant

Use the resulting formulation to solve for the problem using:

a) one quadratic element with equally spaced nodes,
b) two equal length quadratic elements with equally spaced nodes, and
c) compare the quadratic solution with those using linear interpolation functions.

EIGENVALUE PROBLEMS

Axial Vibrations

For axial vibrations the elemental stiffness and mass matrices are given by:

$$\mathbf{k_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}' AE\, \mathbf{N}'^{\mathbf{T}}\, dx \qquad \mathbf{m_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}\, \rho A\, \mathbf{N}^{\mathbf{T}}\, dx$$

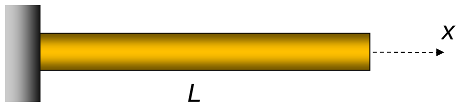
Concentrated masses are handled by simply adding the mass **M** to the corresponding main diagonal element of the global **M** matrix.

The rest of the basic steps in the finite element method are carried out in exactly the same manner as described in the previous section for the corresponding torsion problem.

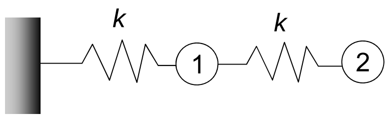
EIGENVALUE PROBLEMS

Axial Vibrations

Two-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:



A, E, and ρ are constant

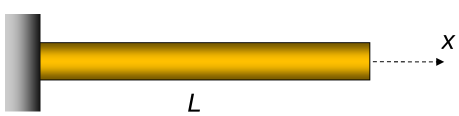


$$k = \frac{2AE}{L}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Two-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:



A, E, and ρ are constant

Then with:

$$\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$$

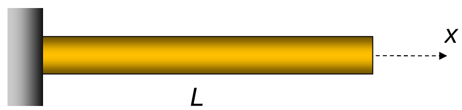
and $l_e = x_{i+1} - x_i = L/2$ for each element:

$$\mathbf{k_e} = \int_{x_i}^{x_{i+1}} \mathbf{N}' AE\, \mathbf{N}'^{\mathbf{T}}\, dx \quad \Rightarrow \quad \frac{1}{l_e}\int_0^1 \mathbf{N}' AE\, \mathbf{N}'^{\mathbf{T}}\, d\xi$$

EIGENVALUE PROBLEMS

Axial Vibrations

Two-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:



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Then with: $\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$

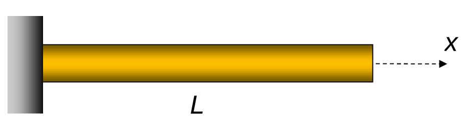
and $l_e = x_{i+1} - x_i = L/2$ for each element:

$$\mathbf{k}_{e1} = \frac{1}{l_e}\int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} AE\, d\xi = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k}_{e2} = \mathbf{k}_{e1}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Two-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:



A, E, and ρ are constant

Then with: $\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$

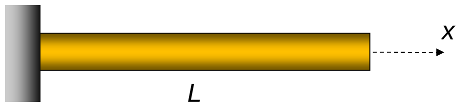
and $l_e = x_{i+1} - x_i = L/2$ for each element:

$$\mathbf{m}_{e1} = \int_0^1 \mathbf{N}\rho A \mathbf{N}^{\mathbf{T}}\, l_e d\xi = \int_0^1 \begin{Bmatrix} N_i \\ N_{i+1} \end{Bmatrix} \rho A \left\langle N_i \quad N_{i+1} \right\rangle l_e d\xi$$

EIGENVALUE PROBLEMS

Axial Vibrations

Two-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:



A, E, and ρ are constant

Then with:

$$\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$$

and $l_e = x_{i+1} - x_i = L/2$ for each element: $\quad \mathbf{m_{e2}} = \mathbf{m_{e1}}$

$$\mathbf{m_{e1}} = \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} A\rho \left\langle 1-\xi \quad \xi \right\rangle l_e d\xi = \frac{A\rho L}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

With $\mathbf{BT_G} = 0$, the assembled matrices are:

$$\mathbf{K_G} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The constraint $\psi_1 = 0$ arises from the essential boundary condition $\psi(0) = 0$.

$$\mathbf{K_G} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Denoting **K** and **M** as $\mathbf{K_G}$ and $\mathbf{M_G}$ with the first row and column deleted, there results:

$$(\mathbf{K} - \phi\mathbf{M})\psi = \begin{bmatrix} 2-4\phi & -1-\phi \\ -1-\phi & 1-2\phi \end{bmatrix} \begin{Bmatrix} \psi_2 \\ \psi_3 \end{Bmatrix} = 0 \qquad \phi = \frac{\omega^2 L^2 \rho}{24E}$$

Requiring the determinant of **K**-ϕ**M** to vanish yields:

$$2(1-2\phi)^2 = (1+\phi)^2$$

with roots ϕ_1 = 0.1082 and ϕ_2 = 1.3204

EIGENVALUE PROBLEMS

Axial Vibrations

The corresponding frequencies are given by:

$$\omega_1^2 = \frac{24E\phi_1}{\rho L^2} = \frac{2.5968\,E}{\rho L^2}$$

$$\left(\omega_1^2\right)_{exact} = \left(\frac{\pi}{2L}\right)^2 \frac{E}{\rho} = \frac{2.4674\,E}{\rho L^2}$$

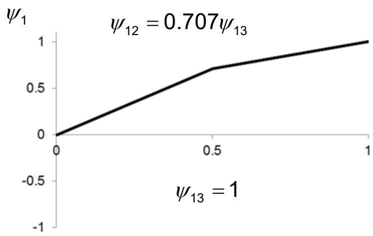
$$\omega_2^2 = \frac{24E\phi_2}{\rho L^2} = \frac{31.690\,E}{\rho L^2}$$

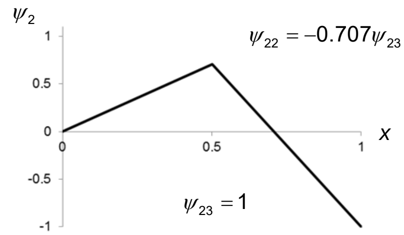
$$\left(\omega_2^2\right)_{exact} = \left(\frac{3\pi}{2L}\right)^2 \frac{E}{\rho} = \frac{22.207\,E}{\rho L^2}$$

EIGENVALUE PROBLEMS

Axial Vibrations

The corresponding approximate eigenfunctions or mode shapes are shown below.





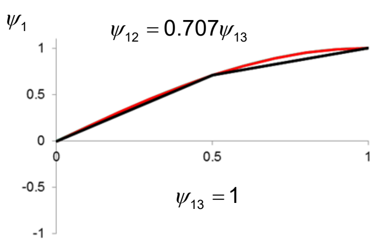
These two eigenfunctions or mode shapes are approximations to the exact eigenfunctions:

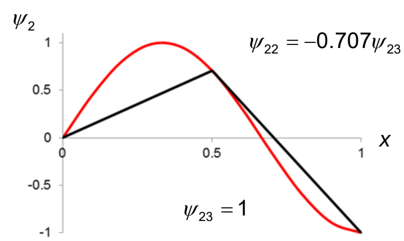
$$\psi_n = \frac{\sin(2n-1)\pi x}{2L}$$

EIGENVALUE PROBLEMS

Axial Vibrations

The corresponding approximate eigenfunctions or mode shapes are shown below.





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EIGENVALUE PROBLEMS

Axial Vibrations

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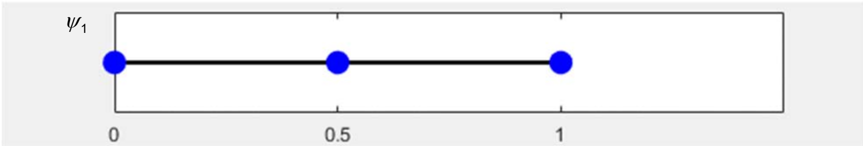
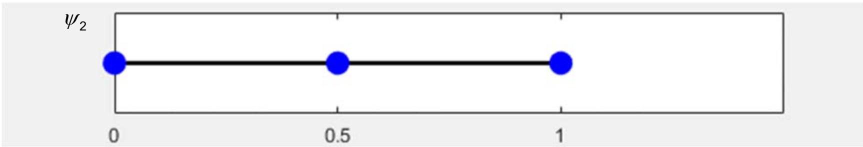
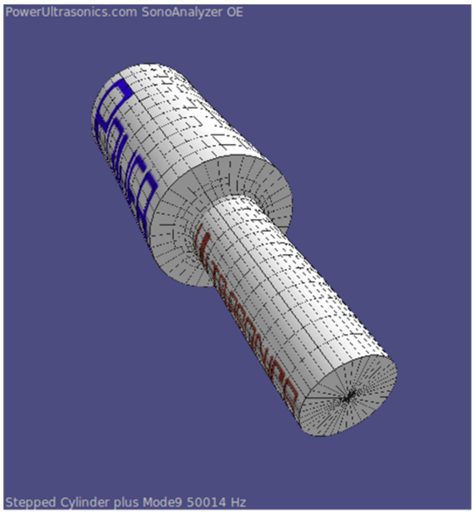




EIGENVALUE PROBLEMS

Axial Vibrations

Example of axial vibration mode shape

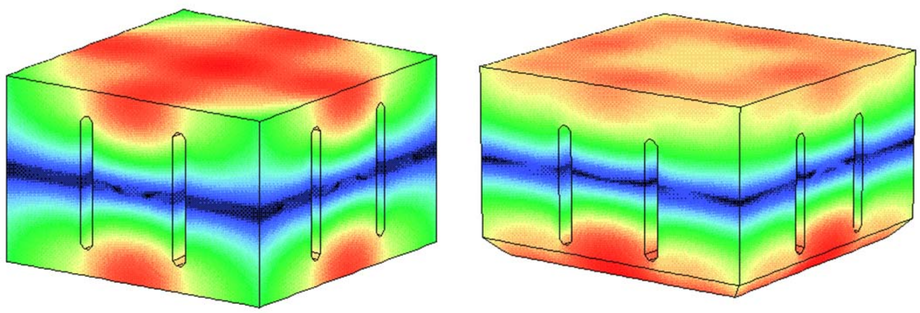


EIGENVALUE PROBLEMS

Axial Vibrations

The following example shows a 20 kHz 8" square block horn.

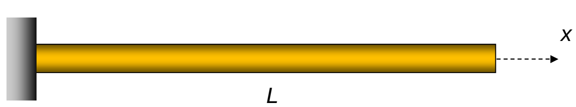
The horn is one half-wavelength long at the axial resonance (the desired resonance), as indicated by the single node that is generally transverse to the principal direction of vibration.



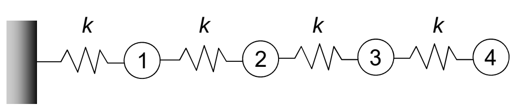
EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - Consider the following four-element solution of the vibration of an axial rod using the lumped-masses (similar to that used in our discussion for springs) shown below.



A, E, and ρ are constant



$$k = \frac{4AE}{L}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:

Then with: $\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$

and $l_e = L/4$ for each element:

$$\mathbf{k_e} = \frac{1}{l_e}\int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} AE\, d\xi = \frac{4AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:

Then with: $\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$

and $l_e = L/4$ for each element:

$$\mathbf{m_e} = \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} A\rho \langle 1-\xi \quad \xi \rangle l_e d\xi = \frac{A\rho L}{24}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - The elemental $\mathbf{k_e}$ and $\mathbf{m_e}$ matrices have exactly the same form as in the two-element model with l_e now taken as L/4.

Omitting some of the details, the constrained 4 x 4 eigenvalue problem is: $(\mathbf{K} - \phi\mathbf{M})\boldsymbol{\psi} = 0$

where:

$$\mathbf{K_G} = \frac{4AE}{L}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{M_G} = \frac{A\rho L}{24}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{96E}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - Requiring the determinant of $\mathbf{K}$-$\phi\mathbf{M}$ to vanish yields four roots; using the Matlab function **eig($\mathbf{M_G}^{-1}\mathbf{K_G}$)** the eigenvalues are displayed below along with the corresponding exact eigenvalues, and percent errors.

i	ϕ_i	$\omega_i^2 L^2 \rho / E$	$(\omega_i^2 L^2 \rho / E)_{exact}$	% error
1	0.026034	2.4993	2.4674	1.29
2	0.259084	24.8721	22.2066	12.00
3	0.854924	82.0727	61.6851	33.05
4	1.787791	171.6279	120.9026	41.93

EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution –The Matlab function $\mathbf{eig(M_G^{-1}K_G)}$ gives:

$$\psi_1 = \begin{Bmatrix} 0.242030 \\ 0.447214 \\ 0.584313 \\ 0.632456 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.584313 \\ -0.447214 \\ 0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.584313 \\ -0.447214 \\ -0.242030 \\ 0.632456 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} -0.242030 \\ 0.447214 \\ -0.584313 \\ 0.632456 \end{Bmatrix}$$

Scaling the eigenvectors so that the largest value is 1 gives:

$$\psi_1 = \begin{Bmatrix} 0.382683 \\ 0.707106 \\ 0.923879 \\ 1.000000 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} 0.923879 \\ 0.707106 \\ -0.382683 \\ -1.000000 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.923879 \\ -0.707106 \\ -0.382683 \\ 1.000000 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} 0.382683 \\ -0.707106 \\ 0.923879 \\ -1.000000 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

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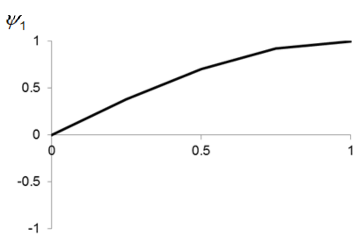

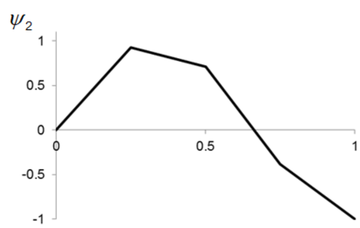

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EIGENVALUE PROBLEMS

Axial Vibrations

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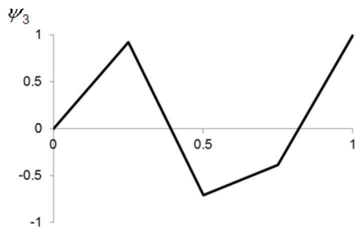

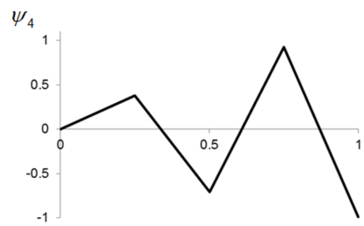

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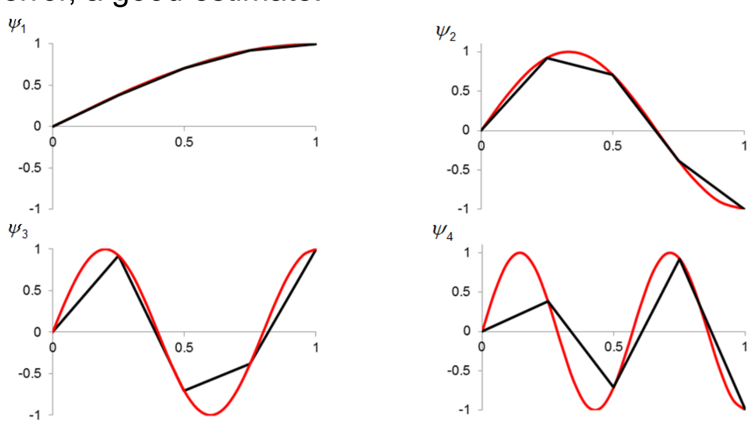
EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - For the two-element model, the number of constrained degrees of freedom is two.

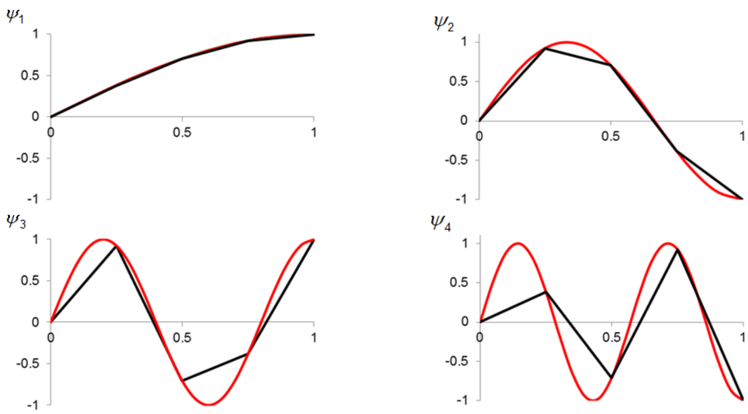
The lowest eigenvalue predicted by that model is 5.2% in error, a good estimate.



EIGENVALUE PROBLEMS

Axial Vibrations

Four-element solution - For the four-element model, with four degrees of freedom, the two lowest eigenvalue estimates are 1.3% and 12% in error respectively, again quite reasonable.



EIGENVALUE PROBLEMS

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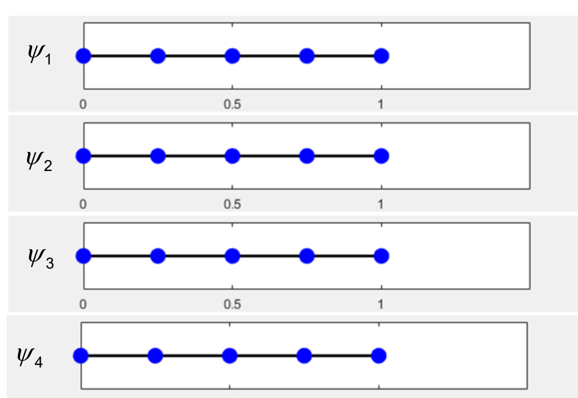


EIGENVALUE PROBLEMS

Axial Vibrations

Eight-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:

Then with: $\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$

and l_e = *L/8* for each element:

$$\mathbf{k_e} = \frac{1}{l_e}\int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} AE\, d\xi = \frac{8AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Eight-element solution - Consider a two-element model with equal-length elements. The elemental matrices are:

Then with: $\mathbf{N} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad dx = l_e d\xi$

and l_e = *L/8* for each element:

$$\mathbf{m_e} = \int_0^1 \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} A\rho \left\langle 1-\xi \quad \xi \right\rangle l_e d\xi = \frac{A\rho L}{48}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Eight-element solution - The elemental $\mathbf{k_e}$ and $\mathbf{m_e}$ matrices have exactly the same form as in the two- and four-element models with l_e now taken as L/8.

The constrained 8 x 8 eigenvalue problem is: $\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$

$$\mathbf{K_G} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \left\langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6 \quad \psi_7 \quad \psi_8 \quad \psi_9 \right\rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{384E}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Eight-element solution - The elemental $\mathbf{k_e}$ and $\mathbf{m_e}$ matrices have exactly the same form as in the two- and four-element models with l_e now taken as L/8.

The constrained 8 x 8 eigenvalue problem is: $\left(\mathbf{K} - \phi\mathbf{M}\right)\boldsymbol{\psi} = 0$

$$\mathbf{M_G} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boldsymbol{\psi}^T = \left\langle \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6 \quad \psi_7 \quad \psi_8 \quad \psi_9 \right\rangle \qquad \phi = \frac{\omega^2 L^2 \rho}{384E}$$

EIGENVALUE PROBLEMS

Axial Vibrations

Eight-element solution - Requiring the determinant of **K**-ϕ**M** to vanish yields four roots; using Matlab function **eig($M_G^{-1}K_G$)** the eigenvalues are displayed below along with the corresponding exact eigenvalues, and percent errors.

i	ϕ_i	$\omega_i^2 L^2 \rho / E$	$\left(\omega_i^2 L^2 \rho / E\right)_{exact}$	% error
1	0.006446	2.4753	2.4674	0.32%
2	0.059520	22.8557	22.2066	2.92%
3	0.173906	66.7799	61.6850	8.26%
4	0.366686	140.8074	120.9027	16.46%

EIGENVALUE PROBLEMS

Axial Vibrations

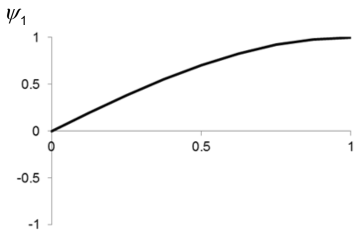
Compared to the four-element solution, the error in the values of the first four eigenvalues is significantly lower than those computed using the four-element approximation.

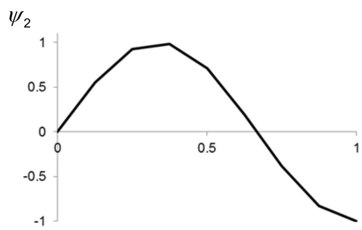
Using Matlab, the corresponding eigenvectors using **eig($M_G^{-1}K_G$)** are:

$$\psi_1 = \begin{Bmatrix} 0.091966 \\ 0.180398 \\ 0.261898 \\ 0.333333 \\ 0.391958 \\ 0.435520 \\ 0.462346 \\ 0.471404 \end{Bmatrix} \quad \psi_2 = \begin{Bmatrix} -0.261898 \\ -0.435520 \\ -0.462346 \\ -0.333333 \\ -0.091966 \\ 0.180398 \\ 0.391958 \\ 0.471404 \end{Bmatrix} \quad \psi_3 = \begin{Bmatrix} 0.391958 \\ 0.435520 \\ 0.091966 \\ -0.333333 \\ -0.462346 \\ -0.180398 \\ 0.261898 \\ 0.471404 \end{Bmatrix} \quad \psi_4 = \begin{Bmatrix} 0.462346 \\ 0.180398 \\ -0.391958 \\ -0.333333 \\ 0.261898 \\ 0.435520 \\ -0.091966 \\ -0.471404 \end{Bmatrix}$$

EIGENVALUE PROBLEMS

Axial Vibrations



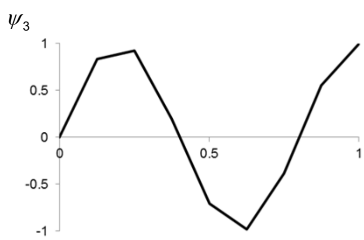


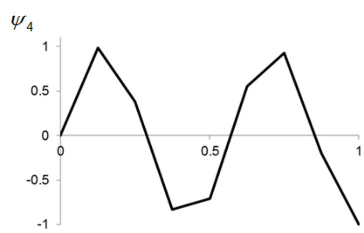
The scaled eigenvectors are:

$$\psi_1 = \left\{\begin{matrix} 0.195090 \\ 0.382683 \\ 0.555570 \\ 0.707106 \\ 0.831469 \\ 0.923879 \\ 0.980785 \\ 1.000000 \end{matrix}\right\} \quad \psi_2 = \left\{\begin{matrix} -0.555570 \\ -0.923879 \\ -0.980785 \\ -0.707106 \\ -0.195090 \\ 0.382683 \\ 0.831469 \\ 1.000000 \end{matrix}\right\} \quad \psi_3 = \left\{\begin{matrix} 0.831469 \\ 0.923879 \\ 0.195090 \\ -0.707106 \\ -0.980785 \\ -0.382683 \\ 0.555570 \\ 1.000000 \end{matrix}\right\} \quad \psi_4 = \left\{\begin{matrix} 0.980785 \\ 0.382683 \\ -0.831469 \\ -0.707106 \\ 0.555570 \\ 0.923879 \\ -0.195090 \\ -1.000000 \end{matrix}\right\}$$

EIGENVALUE PROBLEMS

Axial Vibrations





The scaled eigenvectors are:

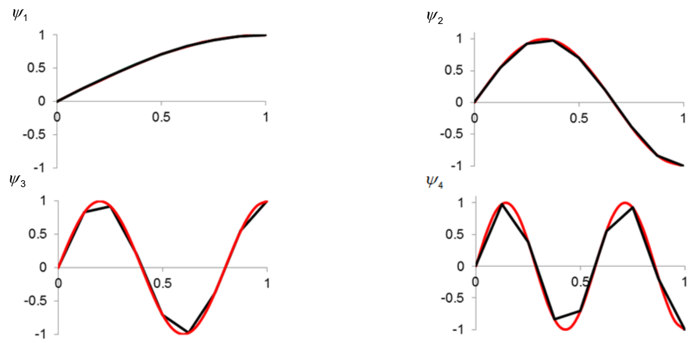
$$\psi_1 = \left\{\begin{matrix} 0.195090 \\ 0.382683 \\ 0.555570 \\ 0.707106 \\ 0.831469 \\ 0.923879 \\ 0.980785 \\ 1.000000 \end{matrix}\right\} \quad \psi_2 = \left\{\begin{matrix} -0.555570 \\ -0.923879 \\ -0.980785 \\ -0.707106 \\ -0.195090 \\ 0.382683 \\ 0.831469 \\ 1.000000 \end{matrix}\right\} \quad \psi_3 = \left\{\begin{matrix} 0.831469 \\ 0.923879 \\ 0.195090 \\ -0.707106 \\ -0.980785 \\ -0.382683 \\ 0.555570 \\ 1.000000 \end{matrix}\right\} \quad \psi_4 = \left\{\begin{matrix} 0.980785 \\ 0.382683 \\ -0.831469 \\ -0.707106 \\ 0.555570 \\ 0.923879 \\ -0.195090 \\ -1.000000 \end{matrix}\right\}$$

EIGENVALUE PROBLEMS

Axial Vibrations

These eigenvectors are similar in shape to those developed in the four-element solution; however, with an eight-element solution there is more detail in the curves.

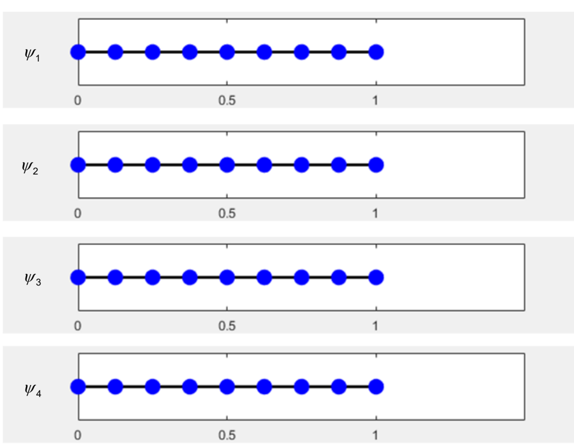
The sinusoidal nature of the first four modes of vibration are more apparent obvious in these plots than those using the four-element solution.



EIGENVALUE PROBLEMS

Axial Vibrations

These eigenvectors are similar in shape to those developed in the four-element solution; however, with an eight-element solution there is more detail in the curves.



End of
1-D Eigenvalues