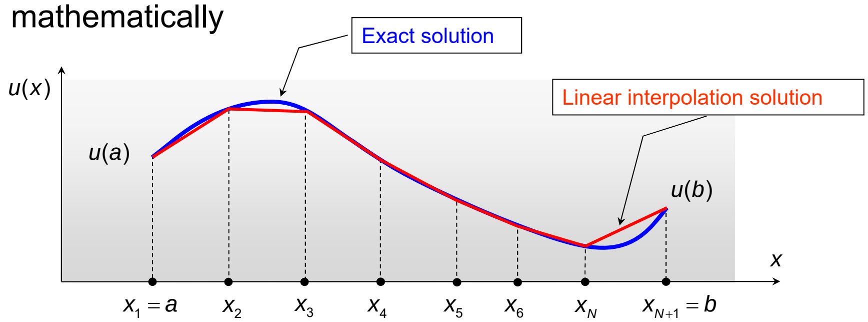
1-D FEM - Higher Order Interpolation Functions

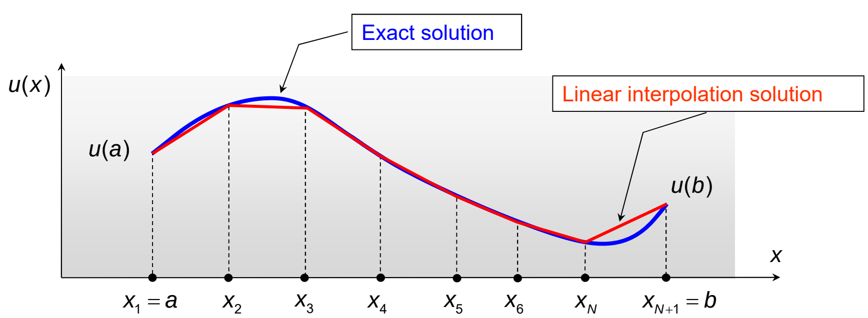
In our discussions about finite element methods up to this point we have used a linear elemental interpolation to describe the variation of the function *u* over an element.

The main advantage of linear interpolation functions is that terms involving the derivative are relatively simple mathematically



1-D FEM - Higher Order Interpolation Functions

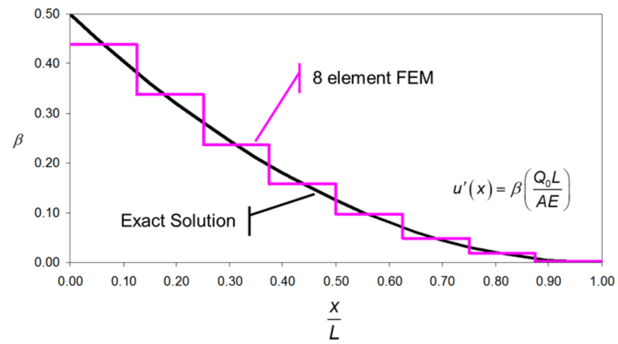
In addition, the transformation from elemental coordinates, ξ, to global coordinates, *x*, is very straightforward and does not increase the complexity of the problem.



1-D FEM - Higher Order Interpolation Functions

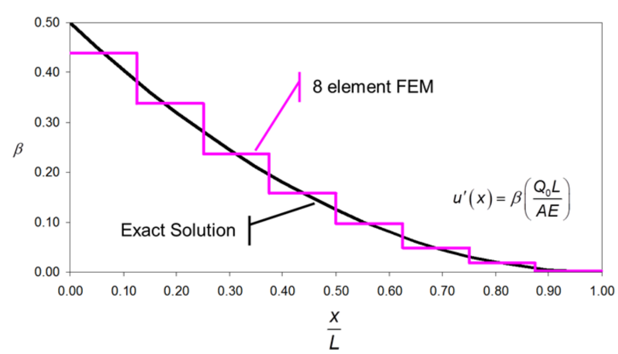
However, the advantages may be overshadowed by the fact that the derivative of the solution, *u*, over an element is constant.

This fact introduces a relatively large discontinuity in any derived variables at interelement boundaries.



1-D FEM - Higher Order Interpolation Functions

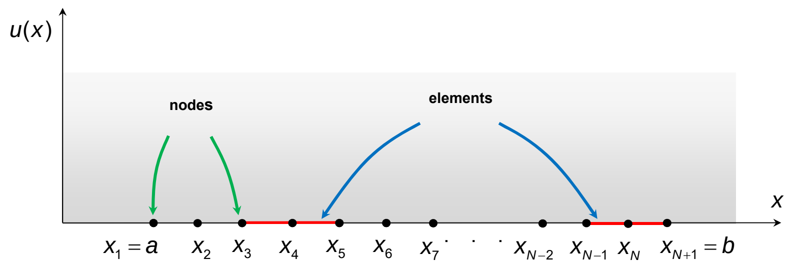
In this section, to improve the accuracy of our solution and avoid the disadvantages associated with a linear element, we will introduce and discuss ***quadratic elemental interpolation***.



1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - A quadratic curve is uniquely defined by three points.

Therefore, a set of elemental shape functions constituting a quadratic interpolation will be defined over a three-node element.

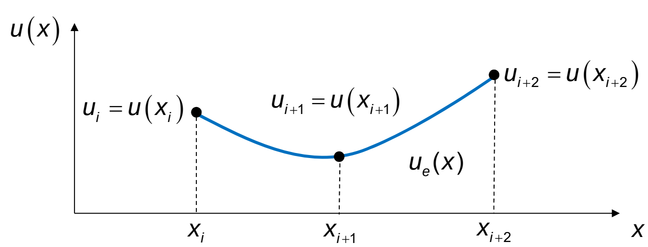


1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - The quadratic variation of the unknown function over an element may be written in global coordinates as:

$$u_e = c_1 + c_2 x + c_3 x^2$$

Matching the values of the function at the endnodes of each element require u_e to be:



1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - The quadratic variation of the unknown function over an element may be written in global coordinates as:

$$u_e = c_1 + c_2 x + c_3 x^2$$

Matching the values of the function at the endnodes of each element require u_e to be:

$$u_e = c_1 + c_2 x_i + c_3 x_i^2 = u_i$$

$$u_e = c_1 + c_2 x_{i+1} + c_3 x_{i+1}^2 = u_{i+1}$$

$$u_e = c_1 + c_2 x_{i+2} + c_3 x_{i+2}^2 = u_{i+2}$$

where u_i, u_{i+1} and u_{i+2} are the values of the unknown function at x_i, x_{i+1} and x_{i+2} respectively.

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - Solving for c_1, c_2, and c_3 results in the following equation:

$$u_e = N_i u_i + N_{i+1} u_{i+1} + N_{i+2} u_{i+2}$$

where:

$$N_i = \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})}$$

$$N_{i+1} = \frac{(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}$$

$$N_{i+2} = \frac{(x - x_i)(x - x_{i+1})}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})}$$

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - The derivative of *u* may be computed as:

$$u_e' = N_i' u_i + N_{i+1}' u_{i+1} + N_{i+2}' u_{i+2}$$

where:

$$N_i' = \frac{\left(2x - x_{i+1} - x_{i+2}\right)}{\left(x_i - x_{i+1}\right)\left(x_i - x_{i+2}\right)}$$

$$N_{i+1}' = \frac{\left(2x - x_i - x_{i+2}\right)}{\left(x_{i+1} - x_i\right)\left(x_{i+1} - x_{i+2}\right)}$$

$$N_{i+2}' = \frac{\left(x - x_i - x_{i+1}\right)}{\left(x_{i+2} - x_i\right)\left(x_{i+2} - x_{i+1}\right)}$$

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - In matrix form we may write the variation of u_e and u_e' over an element as:

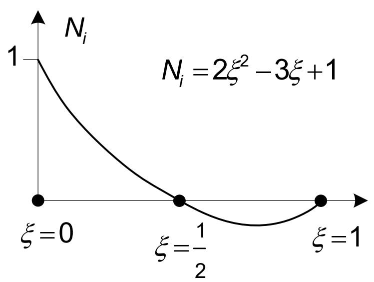
$$u_e = \mathbf{N}^\mathbf{T} \mathbf{u_e} \qquad u_e' = \mathbf{N}'^\mathbf{T} \mathbf{u_e}$$

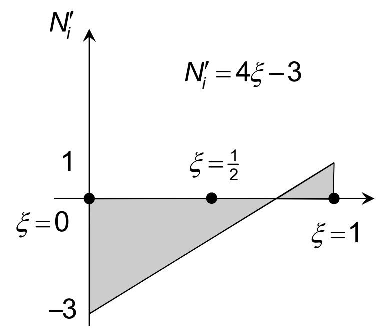
where the vectors **N**, **N'**, and $\mathbf{u_e}$ are:

$$\mathbf{N} = \begin{Bmatrix} N_i \\ N_{i+1} \\ N_{i+2} \end{Bmatrix} \qquad \mathbf{N}' = \begin{Bmatrix} N_i' \\ N_{i+1}' \\ N_{i+2}' \end{Bmatrix} \qquad \mathbf{u_e} = \begin{Bmatrix} u_i \\ u_{i+1} \\ u_{i+2} \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - It is often more convenient to express interpolation functions in terms of an elemental coordinate system ξ. Therefore, the quadratic interpolation functions may be written as:





1-D FEM - Higher Order Interpolation Functions

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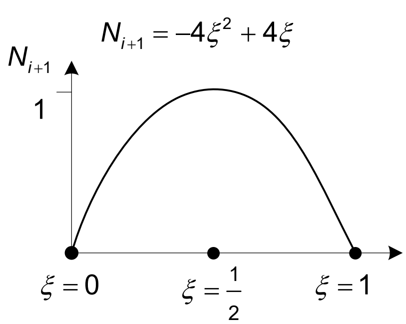

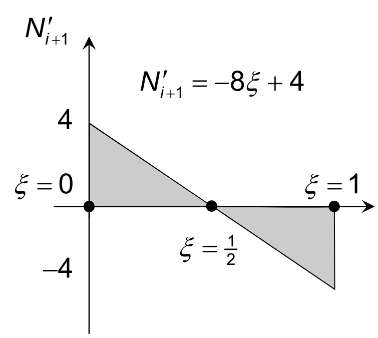

1-D FEM - Higher Order Interpolation Functions

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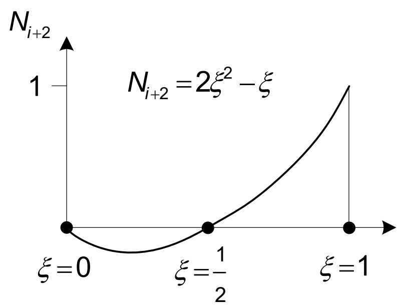
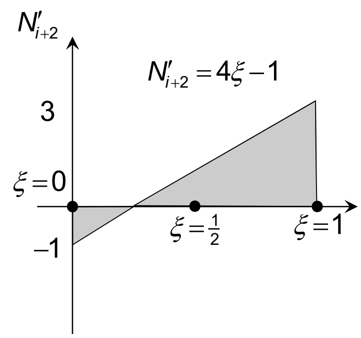
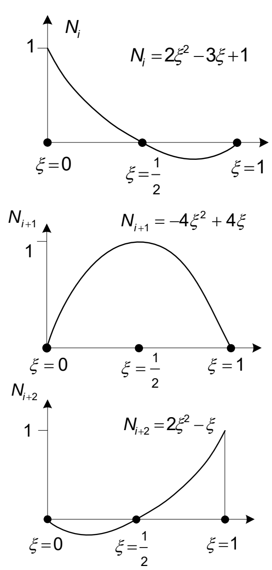




1-D FEM - Higher Order Interpolation Functions

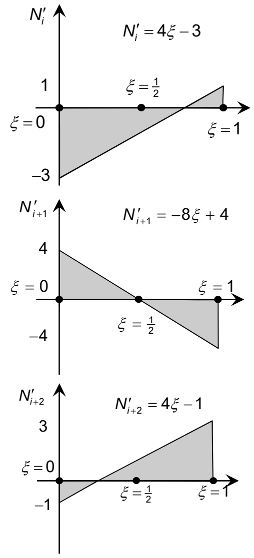
Quadratic Interpolation



$$\sum N_i = 1$$

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation



$$\sum N_i' = 0$$

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - In a manner identical to the linear elemental transformation we discussed in an early section, the quadratic element transformation from global coordinates, x, to element coordinates, ξ, may be written as:

$$x_e = \mathbf{N}^\mathbf{T}\mathbf{x_e} = \langle N_i \quad N_{i+1} \quad N_{i+2} \rangle \begin{Bmatrix} x_i \\ x_{i+1} \\ x_{i+2} \end{Bmatrix}$$

$$= \left(2\xi^2 - 3\xi + 1\right)x_i + \left(-4\xi^2 + 4\xi\right)x_{i+1} + \left(2\xi^2 - \xi\right)x_{i+2}$$

1-D FEM - Higher Order Interpolation Functions

Quadratic Interpolation - Differentiating x with respect to ξ gives:

$$dx = \frac{dx}{d\xi}d\xi = \left[(4\xi - 3)x_i + (-8\xi + 4)x_{i+1} + (4\xi - 1)x_{i+2}\right]d\xi$$

At first observation, the quadratic transformation seems to be more complicated than the linear transformation.

However, if we assume the three nodes defining the quadratic element are equally spaced, it can be shown that the transformation becomes:

$$dx = l_e d\xi \qquad l_e = x_{i+2} - x_i = \text{length of the element}$$

$$x = x_i + \xi l_e$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Recall the Sturm-Liouville boundary value problem we have discussed previously:

$$pu'' - qu + \lambda\rho u + f = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

with the corresponding functional:

$$Z(u) = \int_a^b \left[\frac{p(u')^2 - qu^2}{2} - uf\right]dx + \frac{\alpha u(a)^2}{2} + \frac{\beta u(b)^2}{2} + Au(a) + Bu(b)$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - By discretization of the functional using a quadratic elemental interpolation results in:

$$Z(u) = \sum_e \int_{x_i}^{x_{i+2}} \left[\frac{p(u')^2 - qu^2}{2} - uf \right] dx + \frac{\alpha u(a)^2}{2} + \frac{\beta u(b)^2}{2} + Au(a) + Bu(b)$$

The approximation of the energy functional may be written in the following form:

$$Z(u) = \sum_e \left(\frac{Z_{pe} + Z_{qe}}{2} - Z_{fe} \right) + \frac{\alpha u(a)^2}{2} + \frac{\beta u(b)^2}{2} - Au(a) - Bu(b)$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - The integrals Z_{pe}, Z_{qe}, and Z_{fe} are defined as:

$$Z_{pe} = \int_{x_i}^{x_{i+2}} u'p(x)u' \, dx \qquad Z_{qe} = \int_{x_i}^{x_{i+2}} uq(x)u \, dx$$

$$Z_{fe} = \int_{x_i}^{x_{i+2}} uf(x) \, dx$$

Substituting the coordinate transformation for x in the integrals Z_{pe}, Z_{qe}, and Z_{fe} result in:

$$Z_{pe} = \frac{1}{l_e^2} \int_0^1 u'p(x)u' \, l_e \, d\xi \qquad Z_{qe} = \int_0^1 uq(x)u \, l_e \, d\xi$$

$$Z_{fe} = \int_0^1 uf(x) l_e \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Now we replace the function u and it's derivative u' with the quadratic elemental approximation using the shape functions in the elemental coordinate ξ.

For example, consider the integral Z_{pe}:

$$Z_{pe} \approx \frac{1}{l_e^2}\int_0^1 \mathbf{u_e^T N'} p(x) \mathbf{N'^T u_e} \; l_e \, d\xi$$

where $\mathbf{p_e}$ is defined as:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N'} p(x) \mathbf{N'^T} \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - The integrals Z_{qe} and Z_{fe} may be written in a similar manner:

$$Z_{qe} \approx \mathbf{u_e^T}\left(\int_0^1 \mathbf{N} q(x) \mathbf{N^T} \, l_e \, d\xi\right)\mathbf{u_e} \quad = \quad \mathbf{u_e^T q_e u_e}$$

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N^T} \, l_e d\xi$$

$$Z_{fe} \approx \mathbf{u_e^T}\left(\int_0^1 \mathbf{N} f(x) l_e \, d\xi\right)\mathbf{u_e} \quad = \quad \mathbf{u_e^T f_e}$$

$$\mathbf{f_e} = \int_0^1 \mathbf{N} f(x) l_e d\xi$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Let's examine in detail the integrals $\mathbf{p_e}$, $\mathbf{q_e}$, and $\mathbf{f_e}$. Consider the integral $\mathbf{p_e}$:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N}' p(x) \mathbf{N}'^{\mathbf{T}}\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} N_i' \\ N_{i+1}' \\ N_{i+2}' \end{Bmatrix} p(x) \left\langle N_i' \quad N_{i+1}' \quad N_{i+2}' \right\rangle d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} 4\xi - 3 \\ -8\xi + 4 \\ 4\xi - 1 \end{Bmatrix} p(x) \left\langle 4\xi - 3 \quad -8\xi + 4 \quad 4\xi - 1 \right\rangle d\xi$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Let's examine in detail the integrals $\mathbf{p_e}$, $\mathbf{q_e}$, and $\mathbf{f_e}$. Consider the integral $\mathbf{p_e}$:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N}' p(x) \mathbf{N}'^{\mathbf{T}}\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{bmatrix} (4\xi-3)^2 & (-8\xi+4)(4\xi-3) & (4\xi-1)(4\xi-3) \\ (-8\xi+4)(4\xi-3) & (-8\xi+4)^2 & (-8\xi+4)(4\xi-1) \\ (4\xi-1)(4\xi-3) & (-8\xi+4)(4\xi-1) & (4\xi-1)^2 \end{bmatrix} p(x) d\xi$$

$$x = x_i + \xi\, l_e$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Let's examine in detail the integrals $\mathbf{p_e}$, $\mathbf{q_e}$, and $\mathbf{f_e}$. Consider the integral $\mathbf{q_e}$:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N}^{\mathbf{T}} \, l_e \, d\xi$$

$$= \int_0^1 \begin{Bmatrix} N_i \\ N_{i+1} \\ N_{i+2} \end{Bmatrix} q(x) \left\langle N_i \quad N_{i+1} \quad N_{i+2} \right\rangle l_e \, d\xi$$

$$= \int_0^1 \begin{Bmatrix} 2\xi^2 - 3\xi + 1 \\ -4\xi^2 + 4\xi \\ 2\xi^2 - \xi \end{Bmatrix} q(x) \left\langle 2\xi^2 - 3\xi + 1 \quad -4\xi^2 + 4\xi \quad 2\xi^2 - \xi \right\rangle l_e \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Let's examine in detail the integrals $\mathbf{p_e}$, $\mathbf{q_e}$, and $\mathbf{f_e}$. Consider the integral $\mathbf{q_e}$:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N}^{\mathbf{T}} \, l_e \, d\xi$$

$$= \int_0^1 \begin{bmatrix} \left(2\xi^2 - 3\xi + 1\right)^2 & \left(2\xi^2 - 3\xi + 1\right)\left(-4\xi^2 + 4\xi\right) & \left(2\xi^2 - 3\xi + 1\right)\left(2\xi^2 - \xi\right) \\ \left(-4\xi^2 + 4\xi\right)\left(2\xi^2 - 3\xi + 1\right) & \left(-4\xi^2 + 4\xi\right)^2 & \left(-4\xi^2 + 4\xi\right)\left(2\xi^2 - \xi\right) \\ \left(2\xi^2 - \xi\right)\left(2\xi^2 - 3\xi + 1\right) & \left(2\xi^2 - \xi\right)\left(-4\xi^2 + 4\xi\right) & \left(2\xi^2 - \xi\right)^2 \end{bmatrix} q(x) \, l_e \, d\xi$$

$$x = x_i + \xi \, l_e$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - Let's examine in detail the integrals $\mathbf{p_e}$, $\mathbf{q_e}$, and $\mathbf{f_e}$. Consider the integral $\mathbf{f_e}$**:**

$$\mathbf{f_e} = \int_0^1 \mathbf{N} f(x) l_e \, d\xi$$

$$= \int_0^1 \begin{Bmatrix} N_i \\ N_{i+1} \\ N_{i+2} \end{Bmatrix} f(x) l_e \, d\xi$$

$$x = x_i + \xi l_e$$

$$= \int_0^1 \begin{Bmatrix} 2\xi^2 - 3\xi + 1 \\ -4\xi^2 + 4\xi \\ 2\xi^2 - \xi \end{Bmatrix} f(x) l_e \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements - The integrals and the boundary conditions are handled in exactly the same manner as we have discussed before.

Therefore, the functional Z is now a function of the nodal values u_i:

$$Z\left(u_1, u_2, u_3, \ldots, u_{N+1}\right) = \frac{\mathbf{u_G}^{\mathbf{T}} \mathbf{K_G} \mathbf{u_G}}{2} - \mathbf{u_G}^{\mathbf{T}} \mathbf{F_G}$$

where: $\mathbf{K_G} = \sum_e \mathbf{k_G} + \mathbf{BT_G}$ $\qquad$ $\mathbf{F_G} = \sum_e \mathbf{f_G} + \mathbf{bt_G}$

$$\mathbf{BT_G} = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta \end{bmatrix} \qquad \mathbf{bt_G} = \begin{Bmatrix} A \\ 0 \\ 0 \\ \vdots \\ B \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Variational Formulation using Quadratic Elements -

Recall the energy functional $Z(u)$ has the form:

$$Z\left(u_1, u_2, u_3, \ldots, u_{N+1}\right)$$

and has a stationary value that is obtained by requiring each partial derivative to vanish:

$$\frac{\partial Z}{\partial u_i} = 0 \qquad i = 1,\ 2,\ \ldots,\ N+1$$

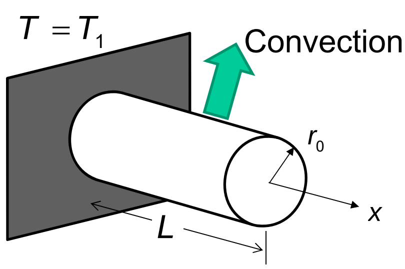
in matrix notation this relationship is:

$$\frac{\partial Z}{\partial \mathbf{u_G}} = 0 \qquad \rightarrow \qquad \boxed{\mathbf{K_G u_G} = \mathbf{F_G}}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Now we will investigate the problem of one-dimensional heat transfer for a circular-cross section bar conducting heat along the axis of the bar as shown below.



We will assume that convection occurs along the length of the bar and at the end

1-D FEM - Higher Order Interpolation Functions

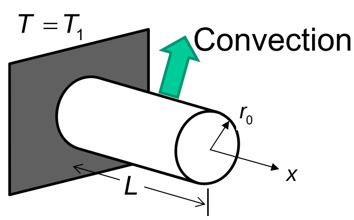
One-Dimensional Heat Conduction with Convection

With T the temperature, the governing equation and boundary conditions can be expressed as:

$$\left(k\pi r_0^2 T'\right)' - h2\pi r_0\left(T - T_0\right) = 0 \qquad 0 < x < L$$

$$T\left(0\right) = T_1$$

$$k\left(L\right)T'\left(L\right) + h_L T\left(L\right) = 0$$



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Nondimensionalize by taking $u = (T - T_0)/ (T_1 - T_0)$ and $x/L = s$, after which the problem can be restated as:

$$\left(u'\right)' - \phi^2 u = 0 \qquad 0 < s < 1$$

$$u\left(0\right) = 1$$

$$u'\left(1\right) + \psi u\left(1\right) = 0$$

where: $\psi = \dfrac{hL}{k} \qquad \phi^2 = \dfrac{2\psi L}{r_0}$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Comparing with the standard form, p = 1, q = ϕ^2, f = 0, A = α = B = 0, and β = ψ.

$$pu'' - qu + \lambda\rho u + f = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

Consider first a one-element quadratically interpolated model.

The numerical results will be based on the specific values: ψ = 1 and ϕ^2 = 10.

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using a single quadratic element, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{p_e} = \int_0^1 \mathbf{N'N'^T}\, d\xi$$

$$\mathbf{q_e} = \int_0^1 \mathbf{N}(10)\mathbf{N^T}\, d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using a single quadratic element, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N'N'^T}\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} N_i' \\ N_{i+1}' \\ N_{i+2}' \end{Bmatrix} \left\langle N_i' \quad N_{i+1}' \quad N_{i+2}' \right\rangle d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} 4\xi - 3 \\ -8\xi + 4 \\ 4\xi - 1 \end{Bmatrix} \left\langle 4\xi - 3 \quad -8\xi + 4 \quad 4\xi - 1 \right\rangle d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

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$$= \frac{1}{l_e}\int_0^1 \begin{bmatrix} (4\xi - 3)^2 & (-8\xi + 4)(4\xi - 3) & (4\xi - 1)(4\xi - 3) \\ (-8\xi + 4)(4\xi - 3) & (-8\xi + 4)^2 & (-8\xi + 4)(4\xi - 1) \\ (4\xi - 1)(4\xi - 3) & (-8\xi + 4)(4\xi - 1) & (4\xi - 1)^2 \end{bmatrix} d\xi$$

$$= \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using a single quadratic element, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N^T}\, l_e d\xi$$

$$= \int_0^1 \begin{Bmatrix} N_i \\ N_{i+1} \\ N_{i+2} \end{Bmatrix} (10) \left\langle N_i \quad N_{i+1} \quad N_{i+2} \right\rangle l_e \, d\xi$$

$$= 10 \int_0^1 \begin{Bmatrix} 2\xi^2 - 3\xi + 1 \\ -4\xi^2 + 4\xi \\ 2\xi^2 - \xi \end{Bmatrix} \left\langle 2\xi^2 - 3\xi + 1 \quad -4\xi^2 + 4\xi \quad 2\xi^2 - \xi \right\rangle l_e \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using a single quadratic element, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N^T}\, l_e d\xi$$

$$= 10 \int_0^1 \begin{bmatrix} \left(2\xi^2 - 3\xi + 1\right)^2 & \left(2\xi^2 - 3\xi + 1\right)\left(-4\xi^2 + 4\xi\right) & \left(2\xi^2 - 3\xi + 1\right)\left(2\xi^2 - \xi\right) \\ \left(-4\xi^2 + 4\xi\right)\left(2\xi^2 - 3\xi + 1\right) & \left(-4\xi^2 + 4\xi\right)^2 & \left(-4\xi^2 + 4\xi\right)\left(2\xi^2 - \xi\right) \\ \left(2\xi^2 - \xi\right)\left(2\xi^2 - 3\xi + 1\right) & \left(2\xi^2 - \xi\right)\left(-4\xi^2 + 4\xi\right) & \left(2\xi^2 - \xi\right)^2 \end{bmatrix} d\xi$$

$$= \frac{1}{3} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

With $A = \alpha = B = 0$, and $\beta = \psi$, $bt_G = 0$ and

$$\mathbf{BT_G} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{bt_G} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

$$\mathbf{K_G} = \sum_e \mathbf{k_G} + \mathbf{BT_G}$$

$$= \underbrace{\frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}}_{\mathbf{p_e}} + \underbrace{\frac{1}{3}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}}_{\mathbf{q_e}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{BT_G}}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The unconstrained global equations can be expressed in augmented form as:

$$\mathbf{K_G u_G} = \mathbf{F_G} \quad \Rightarrow \quad \frac{1}{3}\begin{bmatrix} 11 & -6 & 0 \\ -6 & 32 & -6 \\ 0 & -6 & 14 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The essential boundary condition $u(0) = 1$ is enforced as the constraint $u_1 = 1$, which leads to:

$$\frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & -6 \\ 0 & -6 & 14 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Solution</u> - The equations are ready to be solved.

$$u_1 = 1 \qquad u_2 = 0.203883 \qquad u_3 = 0.087379$$

The variation of *u* over the element is described as:

$$u_e = N_i u_1 + N_{i+1} u_2 + N_{i+2} u_3$$

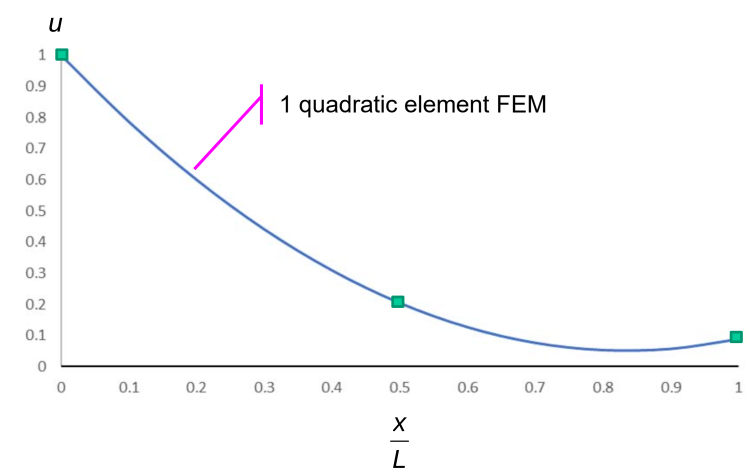
$$= \Big[\left(2\xi^2 - 3\xi + 1\right) + 0.203883\left(-4\xi^2 + 4\xi\right)$$

$$= +0.087379\left(2\xi^2 - \xi\right)\Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using a single quadratic element.



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Computation of Derived Variables</u> - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u'* the quadratic elemental interpolation functions:

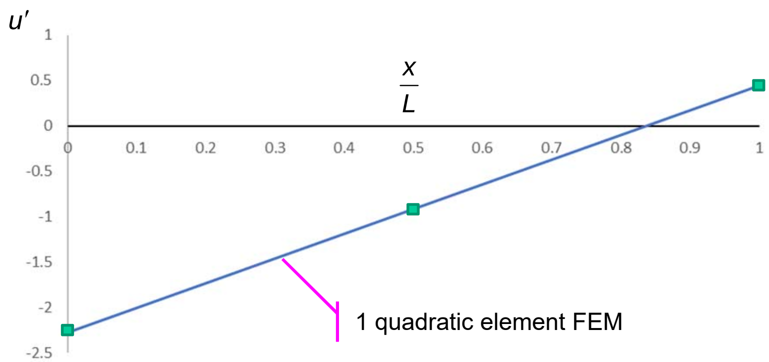
$$u'_e = N'_i u_1 + N'_{i+1} u_2 + N'_{i+2} u_3$$

$$= \left[(4\xi - 3) + 0.203883(-8\xi + 4) + 0.087379(4\xi - 1) \right]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using a single quadratic element.



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The essential boundary condition is satisfied automatically by enforcing the corresponding constraint.

The natural boundary condition is only satisfied in the limit as the number of nodes and elements is increased.

With $\psi = 1$, the nondimensional form of the boundary condition at $s = 1$ is: $u'(1) + \psi u(1) = 0$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

$$u'(1) + \psi u(1) = 0$$

$$\underbrace{\langle 1.0 \quad 0.203883 \quad 0.087378 \rangle}_{\{u\}^T} \left[\underbrace{\begin{Bmatrix} 4(1)-3 \\ -8(1)+4 \\ 4(1)-1 \end{Bmatrix}}_{\{N'(1)\}} + \underbrace{\begin{Bmatrix} 2(1)^2 - 3(1) + 1 \\ -4(1)^2 + 4(1) \\ 2(1)^2 - (1) \end{Bmatrix}}_{\{N(1)\}} \right] = 0.5340 \neq 0$$

This calculation shows the 1-element solution show a quite large error in satisfying the natural boundary condition at $s = 1$.

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **two quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N}'\mathbf{N}'^{\mathbf{T}}\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} N_i' \\ N_{i+1}' \\ N_{i+2}' \end{Bmatrix} \langle N_i' \quad N_{i+1}' \quad N_{i+2}' \rangle\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} 4\xi - 3 \\ -8\xi + 4 \\ 4\xi - 1 \end{Bmatrix} \langle 4\xi - 3 \quad -8\xi + 4 \quad 4\xi - 1 \rangle\, d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **two quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N}'\mathbf{N}'^{\mathbf{T}}\, d\xi \qquad l_e = \frac{1}{2}$$

$$= 2\int_0^1 \begin{bmatrix} (4\xi-3)^2 & (-8\xi+4)(4\xi-3) & (4\xi-1)(4\xi-3) \\ (-8\xi+4)(4\xi-3) & (-8\xi+4)^2 & (-8\xi+4)(4\xi-1) \\ (4\xi-1)(4\xi-3) & (-8\xi+4)(4\xi-1) & (4\xi-1)^2 \end{bmatrix} d\xi$$

$$= \frac{2}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **two quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N^T} \, l_e d\xi$$

$$= \int_0^1 \begin{Bmatrix} N_i \\ N_{i+1} \\ N_{i+2} \end{Bmatrix} (10) \left\langle N_i \quad N_{i+1} \quad N_{i+2} \right\rangle l_e \, d\xi$$

$$= 10 \int_0^1 \begin{Bmatrix} 2\xi^2 - 3\xi + 1 \\ -4\xi^2 + 4\xi \\ 2\xi^2 - \xi \end{Bmatrix} \left\langle 2\xi^2 - 3\xi + 1 \quad -4\xi^2 + 4\xi \quad 2\xi^2 - \xi \right\rangle l_e \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **two quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N^T} \, l_e d\xi \qquad l_e = \frac{1}{2}$$

$$= 5 \int_0^1 \begin{bmatrix} (2\xi^2 - 3\xi + 1)^2 & (2\xi^2 - 3\xi + 1)(-4\xi^2 + 4\xi) & (2\xi^2 - 3\xi + 1)(2\xi^2 - \xi) \\ (-4\xi^2 + 4\xi)(2\xi^2 - 3\xi + 1) & (-4\xi^2 + 4\xi)^2 & (-4\xi^2 + 4\xi)(2\xi^2 - \xi) \\ (2\xi^2 - \xi)(2\xi^2 - 3\xi + 1) & (2\xi^2 - \xi)(-4\xi^2 + 4\xi) & (2\xi^2 - \xi)^2 \end{bmatrix} d\xi$$

$$= \frac{1}{6} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

With A = α = B = 0, and β = ψ, bt_G = 0 and

$$\mathbf{BT_G} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{bt_G} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

$$\mathbf{K_G} = \sum_e \left(\mathbf{p_e} + \mathbf{q_e}\right) + \mathbf{BT_G}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The unconstrained global equations can be expressed in augmented form as: $\mathbf{K_G u_G} = \mathbf{F_G}$

Element 1

$$\frac{1}{6}\begin{bmatrix} 32 & -30 & 3 & 0 & 0 \\ -30 & 80 & -30 & 0 & 0 \\ 3 & -30 & 64 & -30 & 3 \\ 0 & 0 & -30 & 80 & -30 \\ 0 & 0 & 3 & -30 & 38 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Element 2

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The essential boundary condition $u(0) = 1$ is enforced as the constraint $u_1 = 1$, which leads to:

$$\frac{1}{6}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 80 & -30 & 0 & 0 \\ 0 & -30 & 64 & -30 & 3 \\ 0 & 0 & -30 & 80 & -30 \\ 0 & 0 & 3 & -30 & 38 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 5 \\ -0.5 \\ 0 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Solution</u> - The equations are ready to be solved.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1.000000 \\ 0.454417 \\ 0.211779 \\ 0.103911 \\ 0.065315 \end{Bmatrix}$$

The variation of u over the element 1 is described as:

$$u_{e_1} = N_i u_1 + N_{i+1} u_2 + N_{i+2} u_3$$

$$= \Big[\left(2\xi^2 - 3\xi + 1\right) + 0.454417\left(-4\xi^2 + 4\xi\right)$$

$$= +0.211779\left(2\xi^2 - \xi\right)\Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Solution - The equations are ready to be solved.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1.000000 \\ 0.454417 \\ 0.211779 \\ 0.103911 \\ 0.065315 \end{Bmatrix}$$

The variation of *u* over the element 2 is described as:

$$u_{e_2} = N_i u_3 + N_{i+1} u_4 + N_{i+2} u_5$$

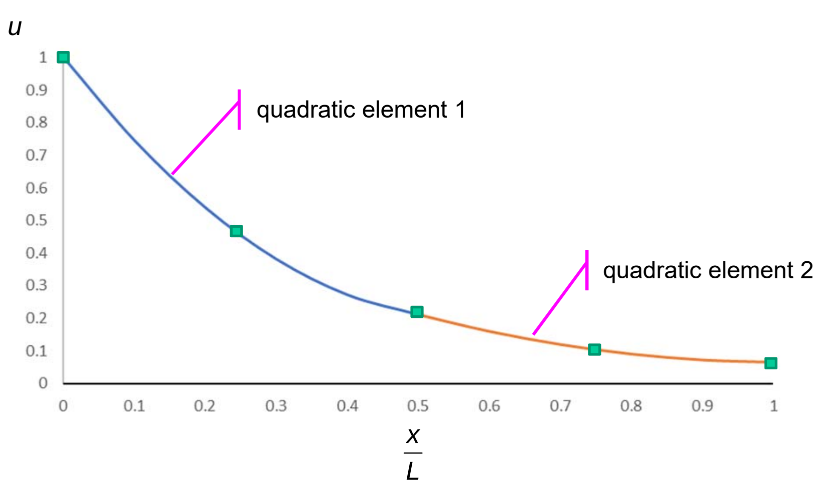
$$= \Big[0.211779\left(2\xi^2 - 3\xi + 1\right) + 0.103911\left(-4\xi^2 + 4\xi\right)$$

$$= +0.065315\left(2\xi^2 - \xi\right) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using two quadratic elements.



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Computation of Derived Variables</u> - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u*' using the quadratic elemental interpolation functions:

$$u'_{e_1} = N'_i u_1 + N'_{i+1} u_2 + N'_{i+2} u_3$$

$$= \Big[(4\xi - 3) + 0.454417(-8\xi + 4) + 0.211779(4\xi - 1) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Computation of Derived Variables</u> - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u*' using the quadratic elemental interpolation functions:

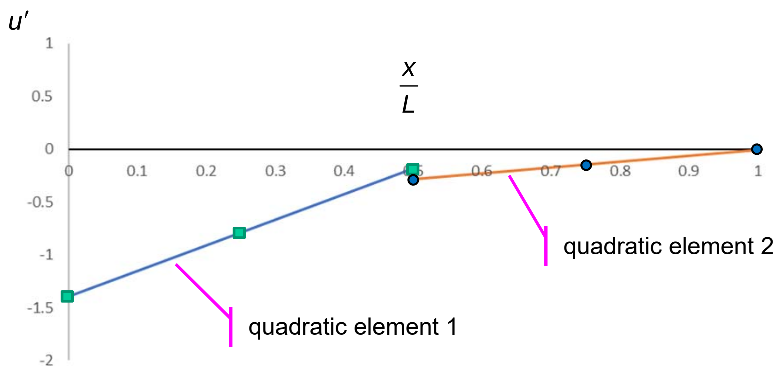
$$u'_{e_2} = N'_i u_3 + N'_{i+1} u_4 + N'_{i+2} u_5$$

$$= \Big[0.211779(4\xi - 3) + 0.103910(-8\xi + 4) + 0.065315(4\xi - 1) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using **two quadratic** element.



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The essential boundary condition is satisfied automatically by enforcing the corresponding constraint.

The natural boundary condition is only satisfied in the limit as the number of nodes and elements is increased.

With $\psi = 1$, the nondimensional form of the boundary condition at $s = 1$ is: $u'(1) + \psi u(1) = 0$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

$$u'(1) + \psi u(1) = 0$$

$$\underbrace{\langle 0.211780 \quad 0.103911 \quad 0.065315 \rangle}_{\{u\}^T} \left[\underbrace{\begin{Bmatrix} 4(1)-3 \\ -8(1)+4 \\ 4(1)-1 \end{Bmatrix}}_{\{N'(1)\}} + \underbrace{\begin{Bmatrix} 2(1)^2 - 3(1) + 1 \\ -4(1)^2 + 4(1) \\ 2(1)^2 - (1) \end{Bmatrix}}_{\{N(1)\}} \right] = 0.0495 \neq 0$$

This calculation shows the 2-element solution is a significant improve over the 1 element solution at satisfying the natural boundary condition at s = 1.

Recall, for the one element solution: $u'(1) + \psi u(1) = 0.5340 \neq 0$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **four quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N'N'^T}\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} N'_i \\ N'_{i+1} \\ N'_{i+2} \end{Bmatrix} \langle N'_i \quad N'_{i+1} \quad N'_{i+2} \rangle\, d\xi$$

$$= \frac{1}{l_e}\int_0^1 \begin{Bmatrix} 4\xi - 3 \\ -8\xi + 4 \\ 4\xi - 1 \end{Bmatrix} \langle 4\xi - 3 \quad -8\xi + 4 \quad 4\xi - 1 \rangle\, d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **four quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N'N'^T}\, d\xi \qquad l_e = \frac{1}{4}$$

$$= 4\int_0^1 \begin{bmatrix} (4\xi-3)^2 & (-8\xi+4)(4\xi-3) & (4\xi-1)(4\xi-3) \\ (-8\xi+4)(4\xi-3) & (-8\xi+4)^2 & (-8\xi+4)(4\xi-1) \\ (4\xi-1)(4\xi-3) & (-8\xi+4)(4\xi-1) & (4\xi-1)^2 \end{bmatrix} d\xi$$

$$= \frac{1}{3}\begin{bmatrix} 28 & -32 & 4 \\ -32 & 64 & -32 \\ 4 & -32 & 28 \end{bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **four quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{q_e} = \int_0^1 \mathbf{N}q(x)\mathbf{N^T}\, l_e d\xi$$

$$= \int_0^1 \begin{Bmatrix} N_i \\ N_{i+1} \\ N_{i+2} \end{Bmatrix} (10) \left\langle N_i \quad N_{i+1} \quad N_{i+2} \right\rangle l_e\, d\xi$$

$$= 10\int_0^1 \begin{Bmatrix} 2\xi^2-3\xi+1 \\ -4\xi^2+4\xi \\ 2\xi^2-\xi \end{Bmatrix} \left\langle 2\xi^2-3\xi+1 \quad -4\xi^2+4\xi \quad 2\xi^2-\xi \right\rangle l_e\, d\xi$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Let repeat the variational formulation using **four quadratic elements**, the integrals $\mathbf{p_e}$ and $\mathbf{q_e}$ are defined as:

$$\mathbf{q_e} = \int_0^1 \mathbf{N} q(x) \mathbf{N^T} \, l_e d\xi \qquad l_e = \frac{1}{4}$$

$$= \frac{5}{2}\int_0^1 \begin{bmatrix} (2\xi^2 - 3\xi + 1)^2 & (2\xi^2 - 3\xi + 1)(-4\xi^2 + 4\xi) & (2\xi^2 - 3\xi + 1)(2\xi^2 - \xi) \\ (-4\xi^2 + 4\xi)(2\xi^2 - 3\xi + 1) & (-4\xi^2 + 4\xi)^2 & (-4\xi^2 + 4\xi)(2\xi^2 - \xi) \\ (2\xi^2 - \xi)(2\xi^2 - 3\xi + 1) & (2\xi^2 - \xi)(-4\xi^2 + 4\xi) & (2\xi^2 - \xi)^2 \end{bmatrix} d\xi$$

$$= \frac{1}{12}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

With $A = \alpha = B = 0$, and $\beta = \psi$, $bt_G = 0$ and

$$\mathbf{BT_G} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{bt_G} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

$$\mathbf{K_G} = \sum_e (\mathbf{p_e} + \mathbf{q_e}) + \mathbf{BT_G}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The unconstrained global equations can be expressed in augmented form as: $\mathbf{K_G u_G} = \mathbf{F_G}$

Element 1 Element 2 Element 3 Element 4

$$\frac{1}{12}\begin{bmatrix} 116 & -126 & 15 & 0 & 0 & 0 & 0 & 0 & 0 \\ -126 & 272 & -126 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & -126 & 232 & -126 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & -126 & 272 & -126 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & -126 & 232 & -126 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 126 & 272 & -126 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & -126 & 232 & -126 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -126 & 272 & -126 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15 & -126 & 128 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The essential boundary condition $u(0) = 1$ is enforced as the constraint $u_1 = 1$, which leads to:

$$\frac{1}{12}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 272 & -126 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -126 & 232 & -126 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & -126 & 272 & -126 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & -126 & 232 & -126 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 126 & 272 & -126 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & -126 & 232 & -126 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & -126 & 272 & -126 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15 & -126 & 128 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 10.5 \\ -1.25 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Solution - The equations are ready to be solved.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1.000000 \\ 0.674155 \\ 0.455318 \\ 0.308276 \\ 0.210167 \end{Bmatrix} \qquad \begin{Bmatrix} u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 0.145197 \\ 0.103274 \\ 0.077654 \\ 0.064320 \end{Bmatrix}$$

The variation of u over the element 1 is described as:

$$u_{e_1} = N_i u_1 + N_{i+1} u_2 + N_{i+2} u_3$$

$$= \Big[\left(2\xi^2 - 3\xi + 1\right) + 0.674155\left(-4\xi^2 + 4\xi\right) + 0.455318\left(2\xi^2 - \xi\right) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Solution - The equations are ready to be solved.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1.000000 \\ 0.674155 \\ 0.455318 \\ 0.308276 \\ 0.210167 \end{Bmatrix} \qquad \begin{Bmatrix} u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 0.145197 \\ 0.103274 \\ 0.077654 \\ 0.064320 \end{Bmatrix}$$

The variation of u over the element 2 is described as:

$$u_{e_2} = N_i u_3 + N_{i+1} u_4 + N_{i+2} u_5$$

$$= \Big[0.455318\left(2\xi^2 - 3\xi + 1\right) + 0.308276\left(-4\xi^2 + 4\xi\right) + 0.210167\left(2\xi^2 - \xi\right) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Solution</u> - The equations are ready to be solved.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1.000000 \\ 0.674155 \\ 0.455318 \\ 0.308276 \\ 0.210167 \end{Bmatrix} \qquad \begin{Bmatrix} u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 0.145197 \\ 0.103274 \\ 0.077654 \\ 0.064320 \end{Bmatrix}$$

The variation of *u* over the element 3 is described as:

$$u_{e_3} = N_i u_5 + N_{i+1} u_6 + N_{i+2} u_7$$

$$= \Big[0.210167\left(2\xi^2 - 3\xi + 1\right) + 0.145197\left(-4\xi^2 + 4\xi\right) + 0.103274\left(2\xi^2 - \xi\right) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Solution</u> - The equations are ready to be solved.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 1.000000 \\ 0.674155 \\ 0.455318 \\ 0.308276 \\ 0.210167 \end{Bmatrix} \qquad \begin{Bmatrix} u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 0.145197 \\ 0.103274 \\ 0.077654 \\ 0.064320 \end{Bmatrix}$$

The variation of *u* over the element 4 is described as:

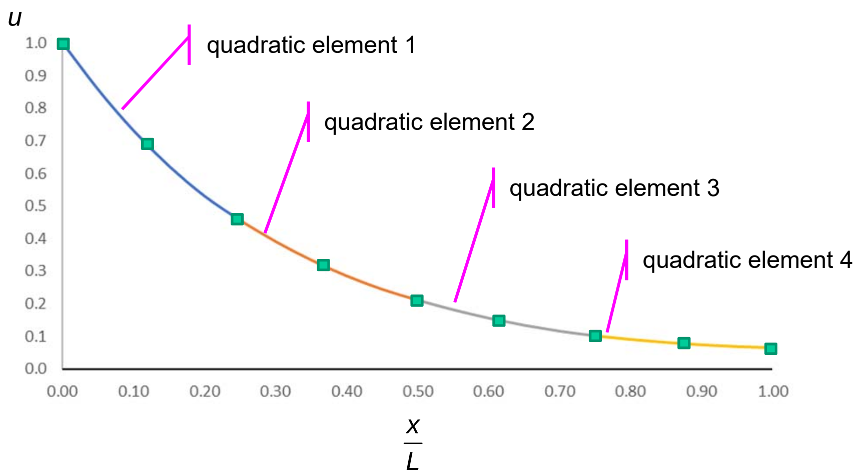
$$u_{e_4} = N_i u_7 + N_{i+1} u_8 + N_{i+2} u_9$$

$$= \Big[0.103274\left(2\xi^2 - 3\xi + 1\right) + 0.077654\left(-4\xi^2 + 4\xi\right) + 0.064320\left(2\xi^2 - \xi\right) \Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using **four quadratic elements**.



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Computation of Derived Variables - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u'* using the quadratic elemental interpolation functions:

$$u'_{e_1} = N'_i u_1 + N'_{i+1} u_2 + N'_{i+2} u_3$$

$$= \left[(4\xi - 3) + 0.674155(-8\xi + 4) + 0.455318(4\xi - 1) \right]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Computation of Derived Variables</u> - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u'* using the quadratic elemental interpolation functions:

$$u'_{e_2} = N'_i u_3 + N'_{i+1} u_4 + N'_{i+2} u_5$$

$$= \Big[0.455318(4\xi - 3) + 0.308276(-8\xi + 4) + 0.210167(4\xi - 1)\Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

<u>Computation of Derived Variables</u> - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u'* using the quadratic elemental interpolation functions:

$$u'_{e_3} = N'_i u_5 + N'_{i+1} u_6 + N'_{i+2} u_7$$

$$= \Big[0.210167(4\xi - 3) + 0.145197(-8\xi + 4) + 0.103274(4\xi - 1)\Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Computation of Derived Variables - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u'* using the quadratic elemental interpolation functions:

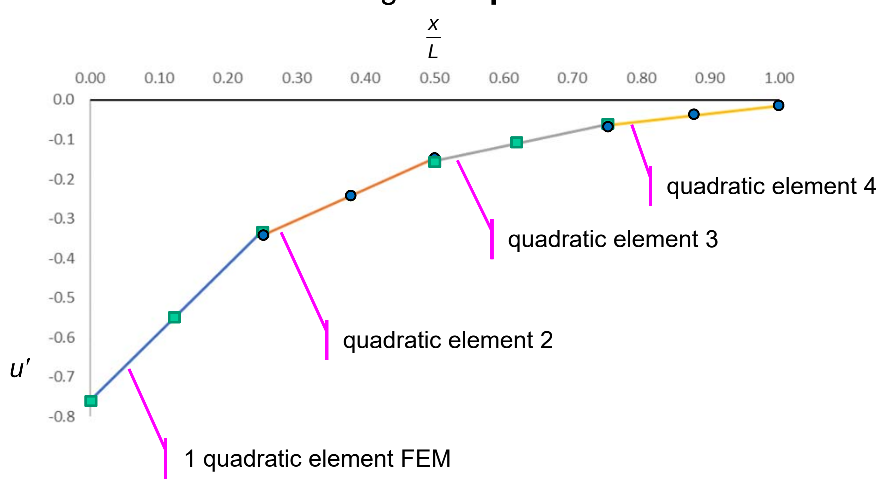
$$u'_{e_4} = N'_i u_7 + N'_{i+1} u_8 + N'_{i+2} u_9$$

$$= \Big[0.103274(4\xi - 3) + 0.077635(-8\xi + 4) + 0.064320(4\xi - 1)\Big]$$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

Variational formulation using **four quadratic elements**.



1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

The essential boundary condition is satisfied automatically by enforcing the corresponding constraint.

The natural boundary condition is only satisfied in the limit as the number of nodes and elements is increased.

With ψ = 1, the nondimensional form of the boundary condition at s = 1 is: $u'(1) + \psi u(1) = 0$

1-D FEM - Higher Order Interpolation Functions

One-Dimensional Heat Conduction with Convection

$$u'(1) + \psi u(1) = 0$$

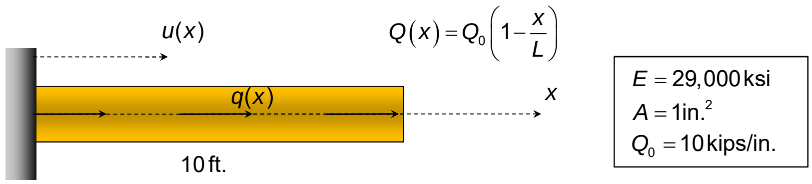
$$\underbrace{\langle 0.103274 \quad 0.077635 \quad 0.064320 \rangle}_{\{u\}^T} \left[\underbrace{\begin{Bmatrix} 4(1) - 3 \\ -8(1) + 4 \\ 4(1) - 1 \end{Bmatrix}}_{\{N'(1)\}} + \underbrace{\begin{Bmatrix} 2(1)^2 - 3(1) + 1 \\ -4(1)^2 + 4(1) \\ 2(1)^2 - (1) \end{Bmatrix}}_{\{N(1)\}} \right] = 0.0071 \neq 0$$

This calculation shows the 4-element solution is a significant improve over the 2 element solution at satisfying the natural boundary condition at s = 1.

Recall, for the two element solution: $u'(1) + \psi u(1) = 0.0495 \neq 0$

1-D FEM - Higher Order Interpolation Functions

Example - Consider the problem of the axial deformation of a prismatic bar we worked previously.



The boundary value problem for this case is:

$$(AEu')' + Q(x) = 0 \qquad 0 \leq x \leq L$$

the boundary conditions are: $\begin{cases} u(0) = 0 \\ AEu'(L) = 0 \end{cases}$

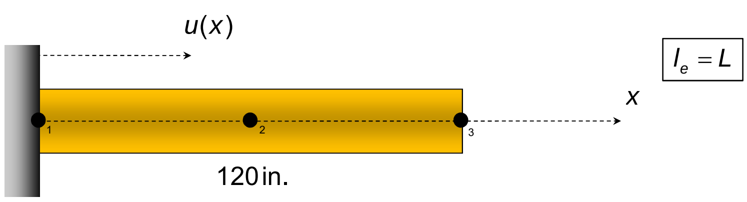
1-D FEM - Higher Order Interpolation Functions

Example - The Sturm-Liouville form of this equation requires that $p = AE$, $q = 0$, $f = Q(x)$, and $A = B = \alpha = \beta = 0$.

The corresponding functional is:

$$Z(u) = \int_0^L \left(\frac{AE(u')^2}{2} - Q(x)u \right) dx$$

Discretization - The domain will be divided into three nodes.



1-D FEM - Higher Order Interpolation Functions

Interpolation - We will use a **quadratic element.**

In developing a Ritz FEM model, the solution was represented in the form of a set of admissible functions:

$$u_e = N_i u_i + N_{i+1} u_{i+1} + N_{i+2} u_{i+2}$$

$$x_e = N_i x_i + N_{i+1} x_{i+1} + N_{i+2} x_{i+2}$$

Element Formulation - The approximation of the energy functional may be written in the following form:

$$Z(u) = \sum_e \left(\frac{Z_{pe}}{2} - Z_{fe} \right) dx$$

1-D FEM - Higher Order Interpolation Functions

Element Formulation - Where the integrals Z_{pe} and Z_{fe} are defined as:

$$Z_{pe} = \int_{x_i}^{x_{i+2}} u' A E u' \, dx \qquad\qquad Z_{fe} = \int_{x_i}^{x_{i+2}} Q(x) u \, dx$$

Element Formulation - Substituting the coordinate transformation for x in the integrals Z_{pe}, and Z_{fe} result in

$$Z_{pe} = \frac{1}{l_e} \int_0^1 u' A E u' \, d\xi \qquad Z_{fe} = \int_0^1 Q(x_i + \xi l_e) u \, l_e d\xi$$

1-D FEM - Higher Order Interpolation Functions

Element Formulation - Now we replace u' with the linear elemental approximation using the shape functions in the elemental coordinate ξ.

$$Z_{pe} \approx \mathbf{u_e^T}\left(\frac{1}{l_e}\int_0^1 \mathbf{N}' AE \mathbf{N}'^{\mathbf{T}}\, d\xi\right)\mathbf{u_e} \quad = \quad \mathbf{u_e^T p_e u_e}$$

$$\mathbf{p_e} = \frac{1}{l_e}\int_0^1 \mathbf{N}' AE\, \mathbf{N}'^{\mathbf{T}}\, d\xi$$

$$Z_{fe} \approx \mathbf{u_e^T}\left(\int_0^1 \mathbf{N} Q(x) l_e\, d\xi\right)\mathbf{u_e} \quad = \quad \mathbf{u_e^T f_e}$$

$$\mathbf{f_e} = \int_0^1 \mathbf{N} Q(x) l_e d\xi$$

1-D FEM - Higher Order Interpolation Functions

Element Formulation - Therefore, the integrals $\mathbf{p_e}$ and $\mathbf{f_e}$ may be written as:

$$\mathbf{p_e} = \frac{AE}{l_e}\int_0^1 \begin{bmatrix} (4\xi-3)^2 & (-8\xi+4)(4\xi-3) & (4\xi-1)(4\xi-3) \\ (-8\xi+4)(4\xi-3) & (-8\xi+4)^2 & (-8\xi+4)(4\xi-1) \\ (4\xi-1)(4\xi-3) & (-8\xi+4)(4\xi-1) & (4\xi-1)^2 \end{bmatrix} d\xi$$

$$\mathbf{f_e} = Q_0 \int_0^1 \begin{Bmatrix} 2\xi^2 - 3\xi + 1 \\ -4\xi^2 + 4\xi \\ 2\xi^2 - \xi \end{Bmatrix} \left(1 - \frac{x_i + \xi l_e}{L}\right) l_e d\xi$$

1-D FEM - Higher Order Interpolation Functions

Assembly - The functional $Z(u)$, through the discretization and interpolation procedures has been converted into an approximate function $Z(u_1, u_2, u_3, \ldots, u_{N+1})$, which may be written as:

$$Z\left(u_1, u_2, u_3, \ldots, u_{N+1}\right) = \left(\frac{AE}{L}\right)\frac{\sum \mathbf{u_e}^{\mathbf{T}}\mathbf{k_e}\mathbf{u_e}}{2} - \sum \mathbf{u_e}^{\mathbf{T}}\mathbf{f_e}$$

$$\frac{\partial Z}{\partial \mathbf{u_G}} = 0 \qquad \rightarrow \qquad \boxed{\mathbf{K_G}\mathbf{u_G} = \mathbf{F_G}}$$

1-D FEM - Higher Order Interpolation Functions

Assembly - Consider the $\mathbf{k_e}$ term for the quadratic element.

$$\mathbf{p_e} = \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

Assembly - The right-hand side terms involving the loading function for each element are:

$$\mathbf{f_e} = \begin{Bmatrix} -\frac{Q_0 l_e}{6L}\left(x_i - L\right) \\ -\frac{Q_0 l_e}{6L}\left(4x_i + 2l_e - 4L\right) \\ -\frac{Q_0 l_e}{6L}\left(x_i + l_e - L\right) \end{Bmatrix} = \frac{Q_0 L}{6}\begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix} \qquad \begin{matrix} l_e = L \\ x_i = 0 \end{matrix}$$

1-D FEM - Higher Order Interpolation Functions

Assembly - Compiling these terms into the global system equations gives:

$$\mathbf{K_G u_G} = \mathbf{F_G} \quad \Rightarrow \quad \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{Q_0 L}{6}\begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix}$$

Constraints - The global system "stiffness" matrix constrained by the boundary condition $u(0) = 0$ is:

$$\frac{AE}{3L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & -8 \\ 0 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{Q_0 L}{6}\begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Solution - The equations are ready to be solved.

$$u_1 = 0 \qquad u_2 = 0.14583\frac{Q_0 L^2}{AE} \qquad u_3 = 0.16667\frac{Q_0 L^2}{AE}$$

Solution - Substituting the numerical values for Q_0, L, A, and E into the displacement expressions gives:

$$u_1 = 0 \qquad u_2 = 0.7241\,in \qquad u_3 = 0.8276\,in$$

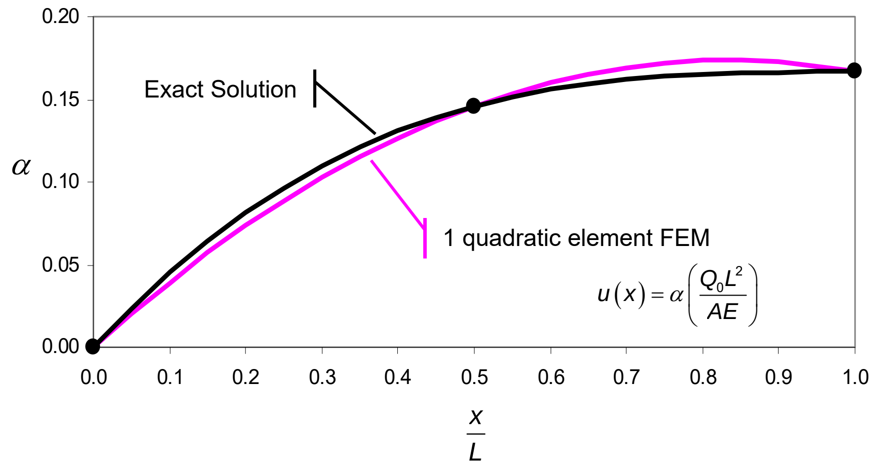
Solution - The variation of u over the element is described as:

$$u_e = N_i u_i + N_{i+1}u_{i+1} + N_{i+2}u_{i+2}$$

$$u = \frac{Q_0 L^2}{AE}\left[0.14583\left(-4\xi^2 + 4\xi\right) + 0.16667\left(2\xi^2 - \xi\right)\right]$$

1-D FEM - Higher Order Interpolation Functions

<u>FEM Solution</u> - Comparison of 1 quadratic element FEM formulation with exact solution:



1-D FEM - Higher Order Interpolation Functions

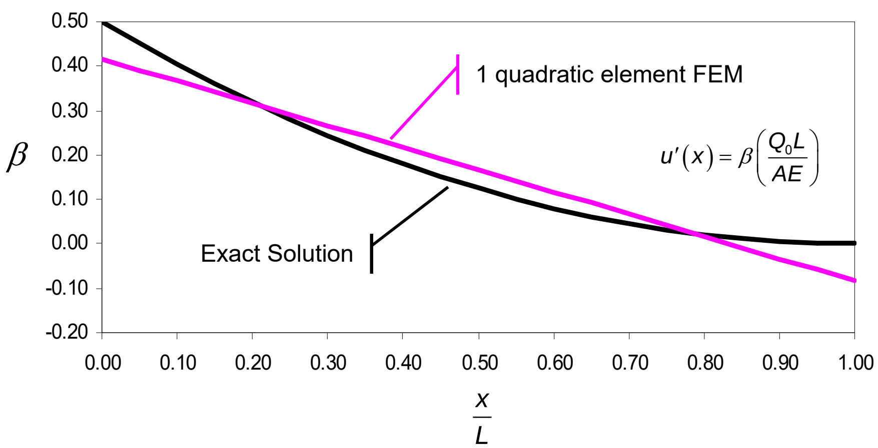
<u>Computation of Derived Variables</u> - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u*' the quadratic elemental interpolation functions:

$$u'_e = N'_i u_i + N'_{i+1} u_{i+1} + N'_{i+2} u_{i+2}$$

$$u' = \frac{Q_0 L}{AE}\left[0.14583(-8\xi + 4) + 0.16667(4\xi - 1)\right]$$

1-D FEM - Higher Order Interpolation Functions

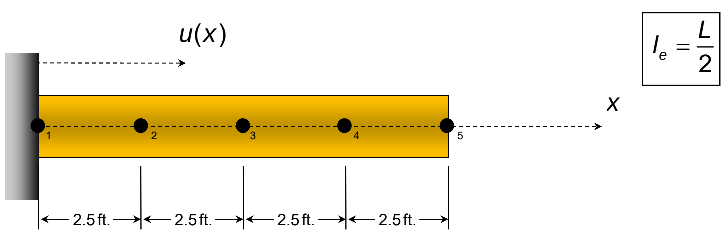
FEM Solution - Comparison of 1 quadratic element FEM formulation with exact solution:



1-D FEM - Higher Order Interpolation Functions

Example - Repeat the previously problem using two quadratic elements.

Discretization - The domain will be described by five nodes.



Interpolation - We will use a **quadratic element.**

$$u_e = N_i u_i + N_{i+1} u_{i+1} + N_{i+2} u_{i+2}$$

$$x_e = N_i x_i + N_{i+1} x_{i+1} + N_{i+2} x_{i+2}$$

1-D FEM - Higher Order Interpolation Functions

<u>Element Formulation</u> - For each quadratic element the terms $\mathbf{p_e}$ and $\mathbf{f_e}$ are:

$$\mathbf{p_e} = \frac{AE}{l_e}\int_0^1 \begin{bmatrix} (4\xi-3)^2 & (-8\xi+4)(4\xi-3) & (4\xi-1)(4\xi-3) \\ (-8\xi+4)(4\xi-3) & (-8\xi+4)^2 & (-8\xi+4)(4\xi-1) \\ (4\xi-1)(4\xi-3) & (-8\xi+4)(4\xi-1) & (4\xi-1)^2 \end{bmatrix} d\xi$$

$$\mathbf{f_e} = Q_0 \int_0^1 \begin{Bmatrix} 2\xi^2 - 3\xi + 1 \\ -4\xi^2 + 4\xi \\ 2\xi^2 - \xi \end{Bmatrix} \left(1 - \frac{x_i + \xi l_e}{L}\right) l_e d\xi$$

1-D FEM - Higher Order Interpolation Functions

<u>Assembly</u>

For element #1: $x_i = 0$ For element #2 : $x_i = 5$ ft.

$$\mathbf{p}_1 = \frac{2AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad \mathbf{p}_2 = \frac{2AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

<u>Assembly</u> - The right-hand side terms involving the loading function for each element are:

$$\mathbf{f_e} = \begin{Bmatrix} -\frac{Q_0 l_e}{6L}(x_i - L) \\ -\frac{Q_0 l_e}{6L}(4x_i + 2l_e - 4L) \\ -\frac{Q_0 l_e}{6L}(x_i + l_e - L) \end{Bmatrix} \qquad \mathbf{f}_1 = \frac{Q_0 L}{24}\begin{Bmatrix} 2 \\ 6 \\ 1 \end{Bmatrix} \qquad \mathbf{f}_2 = \frac{Q_0 L}{24}\begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Assembly Compiling these terms into the global system equations gives:

Element 1

$$\mathbf{K_G u_G} = \mathbf{F_G} \rightarrow \frac{2AE}{3L}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \frac{Q_0 L}{24}\begin{Bmatrix} 2 \\ 6 \\ 2 \\ 2 \\ 0 \end{Bmatrix}$$

Element 2

Element 1

Element 2

Constraints - The global system "stiffness" matrix constrained by the boundary condition $u(0) = 0$ is:

$$\frac{2AE}{3L}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 16 & -8 & 0 & 0 \\ 0 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \frac{Q_0 L}{24}\begin{Bmatrix} 0 \\ 6 \\ 2 \\ 2 \\ 0 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Solution - The equations are ready to be solved.

$$u_2 = 0.0964\frac{Q_0 L^2}{AE} \quad u_3 = 0.1458\frac{Q_0 L^2}{AE} \quad u_4 = 0.1641\frac{Q_0 L^2}{AE} \quad u_5 = 0.1667\frac{Q_0 L^2}{AE}$$

Solution - The variation of u over an element is described as:

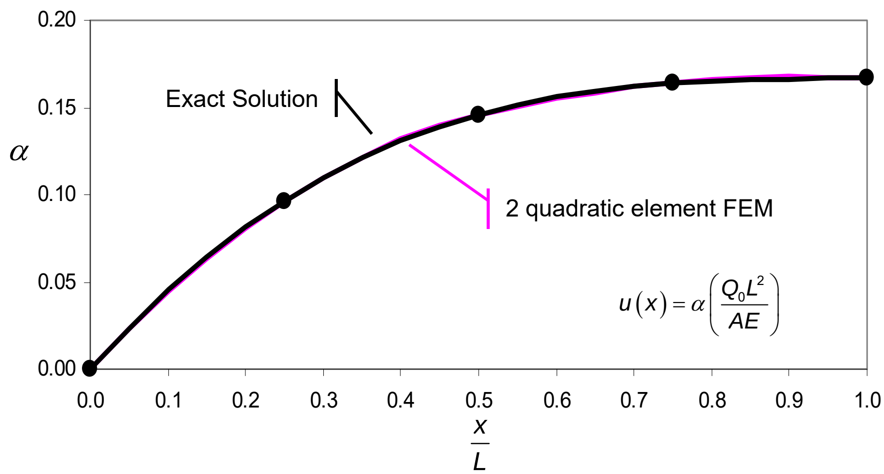
$$u_e = N_i u_i + N_{i+1} u_{i+1} + N_{i+2} u_{i+2}$$

$$u_{element\,1} = \frac{Q_0 L^2}{AE}\left[0.0964\left(-4\xi^2 + 4\xi\right) + 0.1458\left(2\xi^2 - \xi\right)\right]$$

$$u_{element\,2} = \frac{Q_0 L^2}{AE}\left[0.1458\left(2\xi^2 - 3\xi + 1\right) + 0.1641\left(-4\xi^2 + 4\xi\right) + 0.1667\left(2\xi^2 - \xi\right)\right]$$

1-D FEM - Higher Order Interpolation Functions

FEM Solution - Comparison of a 2 quadratic element FEM formulation with exact solution:



1-D FEM - Higher Order Interpolation Functions

Computation of Derived Variables - Since we used a quadratic interpolation in the variational formulation we can calculate an approximate value of *u*' the quadratic elemental interpolation functions:

$$u'_e = N'_i u_i + N'_{i+1} u_{i+1} + N'_{i+2} u_{i+2}$$

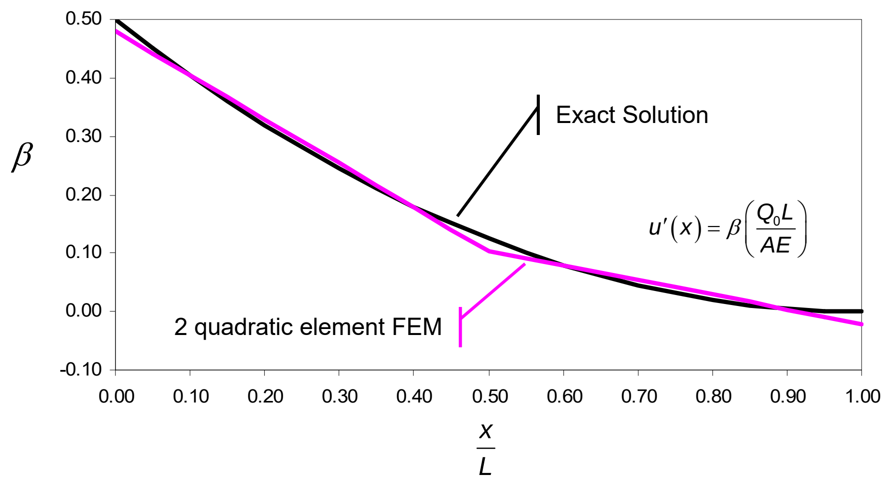
$$u'_{element1} = \frac{2Q_0 L}{AE}\left[0.0964(-8\xi+4)+0.1458(4\xi-1)\right]$$

$$u'_{element2} = \frac{2Q_0 L}{AE}\left[0.1458(4\xi-3)+0.1641(-8\xi+4)+0.1667(4\xi-1)\right]$$

$$u'_{element1}(\xi=1) = \frac{2Q_0 L}{AE}[0.0518] \neq u'_{element2}(\xi=0) = \frac{2Q_0 L}{AE}[0.0523]$$

1-D FEM - Higher Order Interpolation Functions

<u>FEM Solution</u> - Comparison of a 2 quadratic element FEM formulation with exact solution:



1-D FEM - Higher Order Interpolation Functions

PROBLEM #14 – Show by hand calculations each of the following:

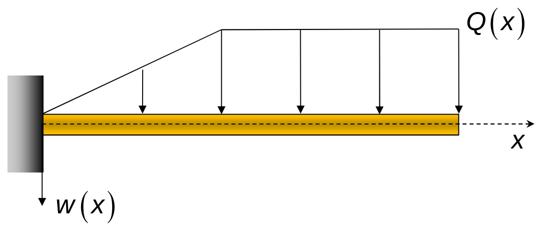
a) for a quadratic element with equally spaced nodes show that: $dx = l_e d\xi$ where $l_e = (x_{i+2} - x_i)$;

b) for a general linear approximations for $p(x)$, $q(x)$, and $f(x)$ derive the terms for the $\mathbf{p_e}$, $\mathbf{q_e}$, and $\mathbf{f_e}$ matrices.

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

One important problem in engineering mechanics is the analysis of beams subjected to a transverse load. A typical beam bending problem may be posed as:



$$\left(EIw''\right)'' = Q\left(x\right) \qquad 0 \le x \le L$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - Since the governing differential equation is a fourth-order relationship, we need four boundary conditions.

Physical, we generally known *two* conditions, in terms of the deflection, the slope, the bending moment, and/or the shear force at each end of the beam.

The corresponding functional, representing the potential energy, is:

$$Z(u) = \int_0^L \left(\frac{EI\left(w''\right)^2}{2} - Q\left(x\right)w \right) dx$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - The requirements of the interpolation function used in the finite element formulation are:

(1) continuity at the interelement boundaries, and
(2) the ability to approximate the solution as at least a constant function that has a constant first derivative.

Functions that are required to be continuous are commonly referred to as C^0 functions, with the superscript referring to the zeroth derivative.

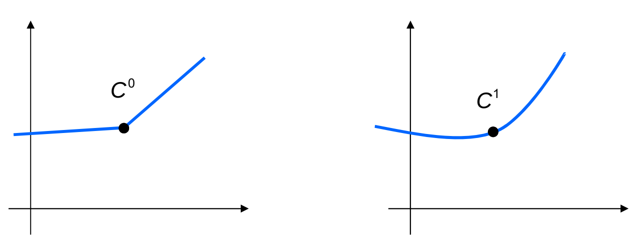
$$Z(u) = \int_0^L \left(\frac{EI(w'')^2}{2} - Q(x)w \right) dx$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - In a fourth-order FEM problem the interpolation functions are require to be at least cubic polynomials.

Physically, this will allow the deflection, the slope, and the bending moment to be C^0 continuous functions.

In addition, the deflection and the slope will have continuous first derivatives; this is referred to as C^1 continuity.

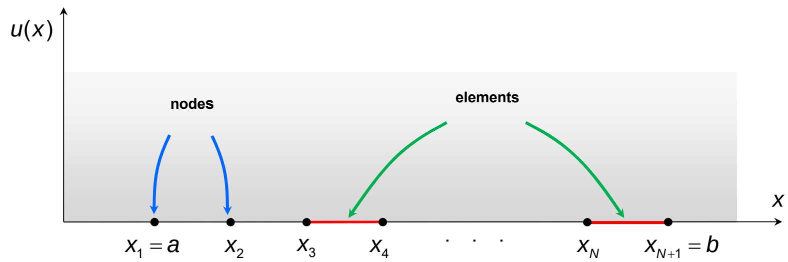


1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Cubic Interpolation - A cubic curve is uniquely defined by four consecutive points or by two consecutive points and two derivatives.

Therefore, for a beam element we will consider a set of elemental shape functions constituting a cubic interpolation over a two-node element.



1-D FEM - Higher Order Interpolation Functions

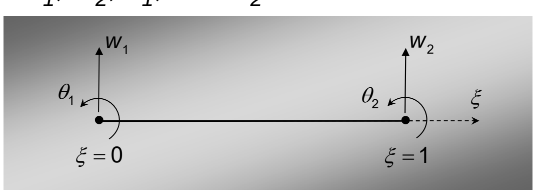
Transverse Deflections of Beams - The cubic variation of the unknown function over an element may be written in global coordinates as:

$$w_e = c_1 + c_2x + c_3x^2 + c_4x^3$$

Matching the values of the deflection and the slope at each end of the element require w_e to be:

$$w_e(0) = w_1 \qquad w_e'(0) = \theta_1$$
$$w_e(1) = w_2 \qquad w_e'(1) = \theta_2$$

where w_1, w_2, θ_1, and θ_2 are shown below:



1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - The cubic variation of the unknown function over an element may be written in ***local*** coordinates as: $w_e = N_1 w_1 + N_2\theta_1 + N_3 w_2 + N_4\theta_2$

$$N_1 = 2\xi^3 - 3\xi^2 + 1 \qquad N_2 = \left(\xi^3 - 2\xi^2 + \xi\right) l_e$$

$$N_3 = -2\xi^3 + 3\xi^2 \qquad N_4 = \left(\xi^3 - \xi^2\right) l_e$$

The set of shape functions are called **Hermite** (*er MEET*) or **cubic interpolating polynomials**.

In matrix form we may write the variation of *w* over an element as:

$$w_e = \mathbf{N}^{\mathbf{T}} \mathbf{w_e} \qquad \mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \qquad \mathbf{w_e} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

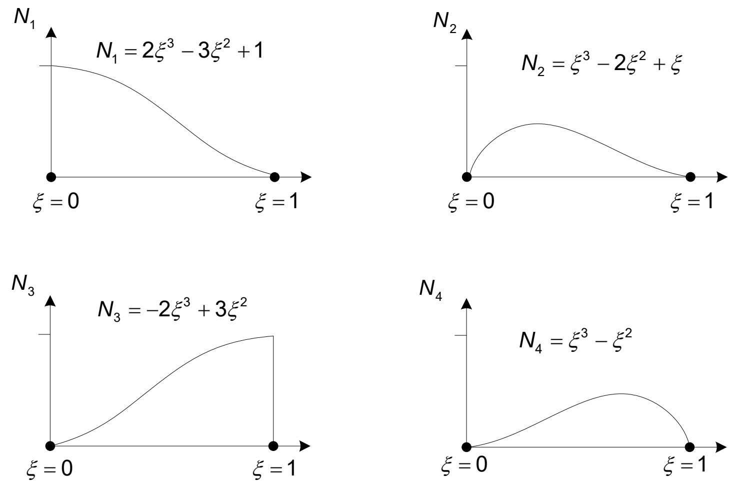
1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - The Hermite interpolation functions may be pictured as:

Charles Hermite (*er MEET*) (December 24, 1822 – January 14, 1901) was a French mathematician who did research on number theory, quadratic forms, invariant theory, orthogonal polynomials, elliptic functions, and algebra. Hermite polynomials, Hermite interpolation, Hermite normal form, Hermitian operators, and cubic Hermite splines are named in his honor.

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - The Hermite interpolation functions may be pictured as:



1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams - In a manner identical to the linear elemental transformation we discussed in an early section, the transformation from global coordinates, *x*, to element coordinates, ξ may be written as:

$$x_e = x_i + \xi\left(x_{i+1} - x_i\right)$$

Differentiating *x* with respect to ξ gives:

$$dx = \frac{d}{d\xi}\left[\left(1-\xi\right)x_i + \xi x_{i+1}\right]d\xi = \left(x_{i+1} - x_i\right)d\xi$$

$$dx = l_e d\xi \qquad l_e = \text{the length of the element}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - The potential energy of a transversely loaded beam is:

$$Z(w) = \int_0^L \left(\frac{EI\left(w''\right)^2}{2} - Q(x)w \right) dx$$

The potential may be expressed as a sum of the potential energies over all the elements:

$$Z(w) = \sum_e \int_{x_i}^{x_{i+1}} \left(\frac{EI\left(w''\right)^2}{2} - Q(x)w \right) dx$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - which we will express as:

$$Z(w) = \sum_e U_e + \sum_e \Omega_e$$

where U_e represents the potential energy of a beam element and Ω_e is the external energy applied to the beam element. Therefore, the terms U_e and Ω_e are:

$$2U_e = \int_{x_i}^{x_{i+1}} EI\left(w''\right)^2 dx = \int_0^1 EI\left(w''\right)^2 l_e \, d\xi$$

$$-\Omega_e = \int_{x_i}^{x_{i+1}} Q(x)w \, dx = \int_0^1 Q(x)w \, l_e \, d\xi$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - Substituting the Hermite shape function into the above expression for U_e gives:

$$2U_e = \int_0^1 EI\left(w''\right)^2 l_e\, d\xi$$

$$= \frac{1}{l_e^4}\int_0^1 w_e^T N''\left(EI\right)N''^T w_e\, l_e\, d\xi = \mathbf{w_e^T k_e w_e}$$

where $\mathbf{k_e}$ is:

$$\mathbf{k_e} = \frac{1}{l_e^3}\int_0^1 N''\left(EI\right)N''^T\, d\xi$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - If the properties of the material, *EI*, are constant over an element, then $\mathbf{k_e}$ becomes:

$$\mathbf{k_e} = \frac{EI}{l_e^3}\int_0^1 \begin{Bmatrix} 12\xi - 6 \\ (6\xi - 4)l_e \\ -12\xi + 6 \\ (6\xi - 2)l_e \end{Bmatrix} \left\langle 12\xi - 6 \quad (6\xi - 4)l_e \quad -12\xi + 6 \quad (6\xi - 2)l_e \right\rangle d\xi$$

$$\mathbf{k_e} = \frac{EI}{l_e^3}\int_0^1 \begin{Bmatrix} (12\xi-6)^2 & (12\xi-6)(6\xi-4)l_e & -(12\xi-6)^2 & (12\xi-6)(6\xi-2)l_e \\ (12\xi-6)(6\xi-4)l_e & \left[(6\xi-4)l_e\right]^2 & -(12\xi-6)(6\xi-4)l_e & (6\xi-2)(6\xi-4)l_e^2 \\ -(12\xi-6)^2 & -(12\xi-6)(6\xi-4)l_e & (12\xi-6)^2 & -(12\xi-6)(6\xi-2)l_e \\ (12\xi-6)(6\xi-2)l_e & (6\xi-2)(6\xi-4)l_e^2 & -(12\xi-6)(6\xi-2)l_e & \left[(6\xi-2)l_e\right]^2 \end{Bmatrix} d\xi$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - If the properties of the material, EI, are constant over an element, then $\mathbf{k_e}$ becomes:

$$\mathbf{k_e} = \frac{EI}{l_e^3}\left[\begin{array}{cc|cc} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ \hline -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{array}\right]$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - Substituting the Hermite shape function into the expression for Ω_e gives:

$$-\Omega_e = \int_0^1 Q(x) w \, l_e \, d\xi$$

$$= \int_0^1 w_e^T N Q(x) l_e \, d\xi = \mathbf{w_e^T q_e}$$

Assuming Q is a linear function of ξ:

$$Q(\xi) = (1-\xi)Q_i + \xi Q_{i+1}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - Substituting the Hermite shape function and the linear variation of Q into Ω_e gives:

$$\Omega_e = \int_0^1 \begin{Bmatrix} 2\xi^3 - 3\xi^2 + 1 \\ \left(\xi^3 - 2\xi^2 + \xi\right) l_e \\ -2\xi^3 + 3\xi^2 \\ \left(\xi^3 - \xi^2\right) l_e \end{Bmatrix} \left[(1-\xi)Q_i + \xi Q_{i+1}\right] l_e \, d\xi$$

$$\Omega_e = \frac{l_e}{60} \begin{Bmatrix} 21Q_i + 9Q_{i+1} \\ \left(3Q_i + 2Q_{i+1}\right) l_e \\ 9Q_i + 21Q_{i+1} \\ \left(-2Q_i - 3Q_{i+1}\right) l_e \end{Bmatrix}$$

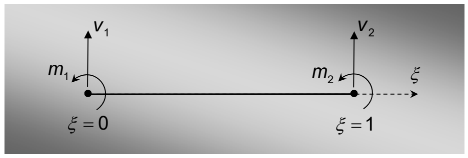
1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Element Formulation - Substituting the Hermite shape function and the linear variation of Q into Ω_e gives:

$$\Omega_{\mathbf{e}} = \frac{l_e}{20} \begin{Bmatrix} 7Q_i + 3Q_{i+1} \\ \left(Q_i + \frac{2}{3}Q_{i+1}\right) l_e \\ 3Q_i + 7Q_{i+1} \\ \left(-\frac{2}{3}Q_i - Q_{i+1}\right) l_e \end{Bmatrix} = \begin{Bmatrix} v_1 \\ m_1 \\ v_2 \\ m_2 \end{Bmatrix}$$

where v_1, v_2, m_1, and m_2 are shown below:



1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Assembly - Assembly of the elemental stiffness matrices is similar to assembly procedure we have previously discussed.

For example, for an element indicated by *a*'s and a second element by *b*'s the global system matrix is:

Element 1

$$\mathbf{K_G} = \sum_e \mathbf{k_e} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33}+b_{11} & a_{34}+b_{12} & b_{13} & b_{14} \\ a_{41} & a_{42} & a_{43}+b_{21} & a_{44}+b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

Element 2

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Assembly - The right-hand side becomes:

$$F_e = \sum_e \Omega_e = \begin{Bmatrix} (v_i)_1 \\ (m_i)_1 \\ (v_{i+1})_1 + (v_i)_2 \\ (m_{i+1})_1 + (m_i)_2 \\ (v_{i+1})_2 \\ (m_{i+1})_2 \end{Bmatrix}$$

Element 1

Element 2

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Derived Variables - The derived variables for this problem are the higher-order derivatives *w''* and *w'''*.

They are related to the force variables, the moment *M* and the shear *Z* given by the following expressions:

$$M = EIw'' \qquad V = -EIw'''$$

The moments *M* and the shear force *Z* at the ends of the elements are given by the matrix equation:

$$\mathbf{k_e w_e} = \mathbf{F_e} \qquad \rightarrow \qquad \mathbf{k_e} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Derived Variables - The value of the moment and shear may be computed at any location within an element from:

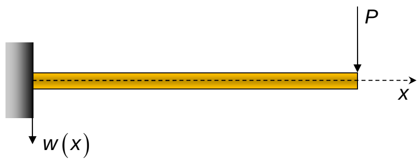
$$M = EIw'' = EI\,\mathbf{w_e}^{\mathbf{T}}\mathbf{N}'' = \langle w_1 \quad \theta_1 \quad w_2 \quad \theta_2 \rangle \begin{Bmatrix} 12\xi - 6 \\ (6\xi - 4)l_e \\ -12\xi + 6 \\ (6\xi - 2)l_e \end{Bmatrix}$$

$$V = -EIw''' = EI\,\mathbf{w_e}^{\mathbf{T}}\mathbf{N}''' = \langle w_1 \quad \theta_1 \quad w_2 \quad \theta_2 \rangle \begin{Bmatrix} 12 \\ 6l_e \\ -12 \\ 6l_e \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Example - Consider the bending of a cantilever beam loaded with a concentrated force at the end. Develop a variational FEM model using a single beam element.



$$\left(EIw''\right)'' = Q(x) \qquad 0 \le x \le L$$

For a single element the system equations are:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ m_1 \\ v_2 \\ m_2 \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Example - Applying the boundary conditions, $w_1 = \theta_1 = 0$, and the transverse loading results in:

$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 12 & -6L \\ 0 & 0 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \\ 0 \end{Bmatrix}$$

The solution to these equations is:

$$w_2 = -\frac{PL^3}{3EI} \qquad \theta_2 = -\frac{PL^2}{2EI}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

<u>Example</u> - The variation of the deflection over the element is:

$$w_e = N_3 w_2 + N_4 \theta_2 = -\frac{PL^3}{6EI}\left(3\xi^2 - \xi^3\right)$$

$$w_e' = \theta = N_3' w_2 + N_4' \theta_2 = -\frac{PL^2}{2EI}\left(2\xi - \xi^2\right)$$

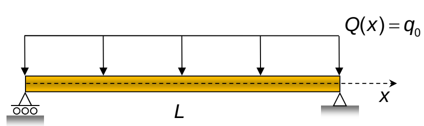
The above functions for the deflection and the slope of the cantilever beam subjected to a concentrated force are exact. The derived variables may be computed as:

$$M = EIw'' = -PL\left(1-\xi\right) \qquad V = -EIw''' = -P$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

<u>Example</u> - Consider the bending of a simply supported beam loaded with a uniform load. Develop a variational FEM model using a single beam element.



$$\left(EIw''\right)'' = Q\left(x\right) \qquad 0 \le x \le L$$

The terms associated with the loading function are:

$$\Omega_e = \frac{l_e}{20}\begin{Bmatrix} 7Q_i + 3Q_{i+1} \\ \left(Q_i + \frac{2}{3}Q_{i+1}\right)l_e \\ 3Q_i + 7Q_{i+1} \\ \left(-\frac{2}{3}Q_i - Q_{i+1}\right)l_e \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} -6 \\ -L \\ -6 \\ L \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Example - For a single element the system equations are:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} -6 \\ -L \\ -6 \\ L \end{Bmatrix}$$

After applying the boundary conditions, $w_1 = w_2 = 0$, and the transverse loading the system equations become:

$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4L^2 & 0 & 2L^2 \\ 0 & 0 & 1 & 0 \\ 0 & 2L^2 & 0 & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 0 \\ -L \\ 0 \\ L \end{Bmatrix}$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

Example - The solution to these equations is:

$$\theta_1 = -\frac{q_0 L^3}{24EI} \qquad \theta_2 = \frac{q_0 L^3}{24EI}$$

The variation of the deflection over the element is:

$$w_e = N_2\theta_1 + N_4\theta_2 = \frac{q_0 L^4}{24EI}\left(\xi^2 - \xi\right)$$

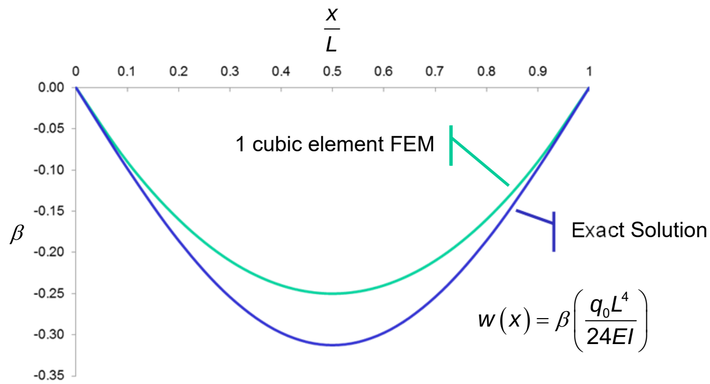
$$w_e' = N_2'\theta_1 + N_4'\theta_2 = \frac{q_0 L^3}{24EI}\left(1 - 2\xi\right)$$

1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

<u>Example</u> - The deflection has a maximum value at $x=L/2$ ($\xi = ½$) of:

$$w = -\frac{4q_0L^4}{384EI} \qquad w_{exact} = -\frac{5q_0L^4}{384EI}$$

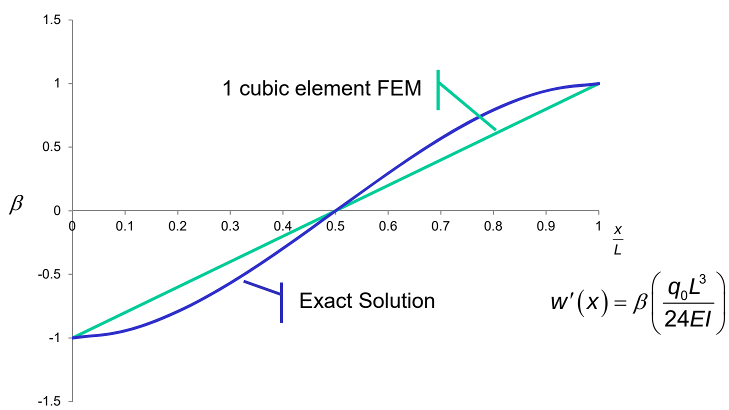


1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

<u>Example</u> - The slope is:

$$\theta_1 = -\frac{q_0L^3}{24EI} \qquad \theta_2 = \frac{q_0L^3}{24EI}$$



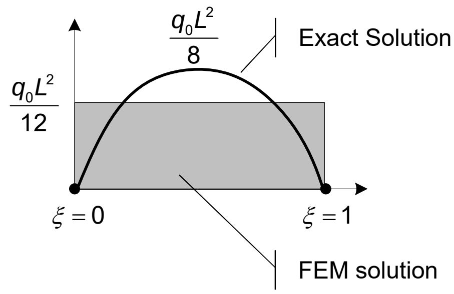
1-D FEM - Higher Order Interpolation Functions

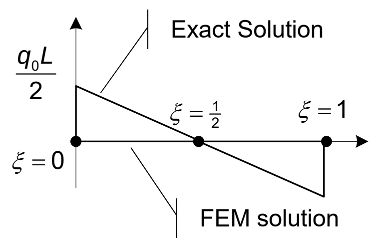
Transverse Deflections of Beams

The slopes at the end of the beam are: $w' = \pm\frac{q_0 L^3}{24EI}$ which are the exact values.

The derived variables may be computed as:

$$M = EIw'' = \frac{q_0 L^2}{12} \qquad V = -EIw''' = 0$$





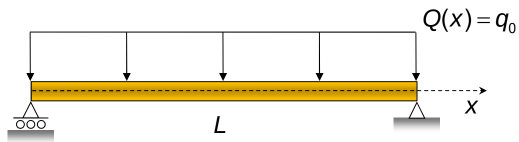
1-D FEM - Higher Order Interpolation Functions

Transverse Deflections of Beams

PROBLEM #15 - Consider the bending of a simply supported beam loaded with a uniform load.

Develop a variational FEM model using two cubic elements.

Compare your results with the single element solution in the notes and the exact solution.



$\boxed{(EIw'')'' = Q(x)}$ $\quad 0 \le x \le L$

End of
Chapter 2c