ONE-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Introduction

Typically, when engineers and scientists investigate the behavior of a solid deformable body, the flow of heat, the motion of a fluid, or the vibration of a system, the focus of the initial study is on a small differential region in the domain of the physical problem.

The differential element may be a free body diagram, a control mass, or a control volume on which the basic physical behavior of the system such as a balance of momentum, a balance of energy, or a balance of mass is applied.

ONE-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Introduction

In this section we will discuss, explore, and develop finite element approximations to one-dimensional boundary value and eigenvalue problems.

There are two basic approaches used to develop finite element estimations of physical and mathematical problems.

One approach deals directly with the differential equation in terms of the so-called ***weak formulation***.

The resulting set of equations is obtained by combining the weak formulation and ***Galerkin's*** method.

ONE-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Introduction

In the second approach, the total potential energy of the physical system, represented as an integral over the domain and over the boundary of the region, is determined.

The system equations are developed by applying the ***Ritz*** method.

For the problem discussed in this section, both approaches produce identical finite element models.

The General Problem

Many physical problems may be represented mathematically by the same general class of boundary value problems defined by the following differential equation:

$$(pu')' - qu + \lambda\rho u + f = 0 \qquad a < x < b$$

with two boundary conditions of the form:

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

where the functions q, ρ, and f are piecewise continuous functions of x, the function p has a continuous derivative, and that p, q, and ρ are positive on the interval $a < x < b$.

The General Problem

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In the general equation, λ is a parameter and α, β, A and B are constants.

This differential equation and the boundary conditions are called a ***regular Sturm-Liouville system***.

The General Problem

Jacques Charles François Sturm (September 29, 1803 – December 15, 1855) was a French mathematician of German extraction.

Joseph Liouville (March 24, 1809 – September 8, 1882) was a French mathematician.

Jacques Charles François Sturm

Joseph Liouville

The General Problem

The general form of the Sturm-Liouville problem may be rewritten as:

$$a_0(x)u'' + a_1(x)u' + a_2(x)u = a_3(x)$$

This equation may be returned to the standard form by multiplying both sides by the following integration factor:

$$\mu = \frac{1}{a_0} exp\left(\int \frac{a_1}{a_0} dt \right)$$

and defining the following functions:

$$\mu a_0 = p \qquad \mu a_2 = \lambda\rho - q \qquad \mu a_3 = -f$$

The General Problem

The general purpose of the integration factor is to collapse the first terms into a single term.

This means the coefficient of *u*' must be the derivative of the coefficient of *u*''.

In general, the value of $\lambda\rho u$ strongly influences the character of the solution.

The solution may be determined by the sum of two linearly independent equations, the homogeneous differential equation combined with a particular solution.

If you are interested in these methods, consult a differential equations textbook or check out the section in your book.

APPROXIMATION METHODS - WEIGHTED RESIDUALS

One of the approaches mentioned previously is known as ***Galerkin's*** method and may be considered one of the so-called ***methods of weighted residuals (MWR)***.

In this section we will discuss several MWR techniques used to approximate typical Sturm-Liouville problems

$$(pu')' - qu + \lambda \rho u + f = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Classical MWR techniques search for continuous functions that solve the differential equation over the interval [a, b].

In contrast, the Galerkin method uses piecewise continuous functions defined over intervals defined by elements rather than continuous functions over the entire interval.

$$(pu')' - qu + \lambda \rho u + f = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

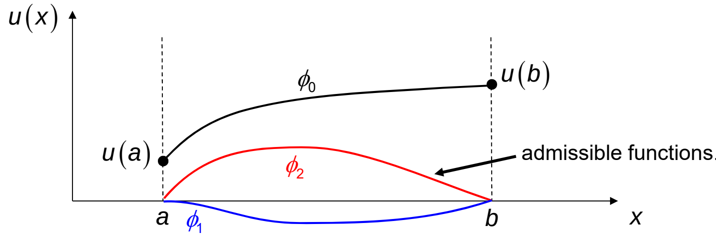
$$p(b)u'(b) + \beta u(b) = B$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

The first step in the classic MWR is to assume an approximate solution of the form:

$$u(x) = \phi_0(x) + \sum_{n=1}^{N} a_n \phi_n(x)$$

where $\phi_0(x)$ is chosen to satisfy the boundary conditions of the problem, and each of the $\phi_n(x)$ is chosen so as to satisfy all the corresponding homogeneous boundary conditions. The $\phi_n(x)$ are called ***admissible functions***.



APPROXIMATION METHODS - WEIGHTED RESIDUALS

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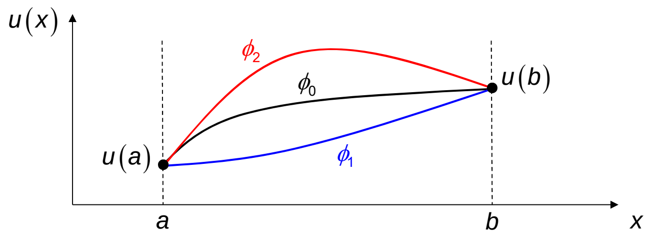


APPROXIMATION METHODS - WEIGHTED RESIDUALS

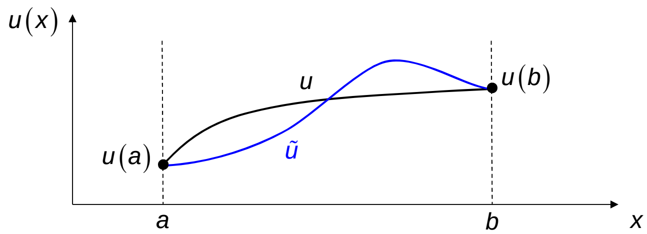
Substituting this expression into the differential equation results in:

$$L(u) = (pu')' - qu + \lambda\rho u + f = 0 \qquad a < x < b$$

$$= (p\phi_0')' + (\lambda\rho - q)\phi_0 + \Sigma\left[(p\phi_n')' + (\lambda\rho - q)\phi_n\right]a_n + f$$

$$= E_N(x, \mathbf{a})$$

where $E_N(x, \mathbf{a})$ is called the ***residual*** or ***error*** of the solution.



APPROXIMATION METHODS - WEIGHTED RESIDUALS

The method of weighted residuals requires the error or the residual to be orthogonal to a set of weight functions $w_j(x)$ according to:

$$\int_a^b w_j(x) E_N dx = 0 \qquad j = 1, 2, \ldots, N$$

This equation is satisfied for each of the N independent weight functions $w_j(x)$.

The result is a set of N linear equations in $\boldsymbol{a}$ unknowns.

The difference between MWR techniques is the form of the weighting function.

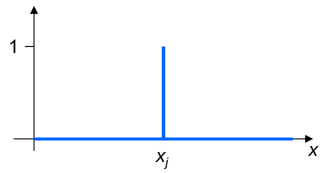
APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>The collocation method</u> - In this technique, the weighting functions are:

$$w_j(x) = \delta(x, x_j) \qquad j = 1, 2, \ldots, N$$

where $\delta(x, x_j)$ is the Dirac delta function:

$$\delta(x, x_j) = 0 \qquad x \neq x_j$$
$$\delta(x, x_j) = 1 \qquad x = x_j$$



One interesting property of the delta function is that for a continuous function $f(x)$.

$$\int_a^b f(x)\delta(x, x_j)\,dx = f(x_j) \qquad a < x_j < b$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>The collocation method</u> - In this technique, the weighting functions are:

$$w_j(x) = \delta(x, x_j) \qquad j = 1, 2, \ldots, N$$

Setting the weighting function equal to the delta function results in a MWR approximation of:

$$\int_a^b E_N(x, \mathbf{a})\delta(x, x_j)\,dx = E_N(x_j, \mathbf{a}) = 0 \qquad j = 1, 2, \ldots, N$$

Therefore, the residual error E_N is evaluated at N interval points and set equal to zero. The result is a set of N linear equations, in the unknown coefficients $\boldsymbol{a}$.

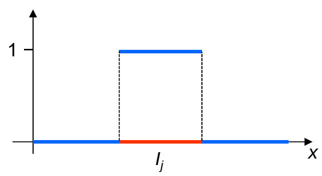
APPROXIMATION METHODS - WEIGHTED RESIDUALS

The subdomain method - In this method, the problem domain is divided into a set of N subintervals defined as I_j. The integral of the residual error is set equal to zero over each subdomain.

$$\int_{I_j} E_N(x,\mathbf{a})dx = 0 \qquad j = 1,2,\ \ldots\ ,N$$

In this case, the weighting function $w(x, x_j)$ is of:

$$w_j(x) = 1 \qquad x \in I_j$$
$$w_j(x) = 0 \qquad x \notin I_j$$



The resulting subdomains cannot overlap or be defined in such a way as to leave some part of the interval from a to b unaccounted.

APPROXIMATION METHODS - WEIGHTED RESIDUALS

The least squares method - This method is similar to the method used in regression techniques. The residual error is squared and values of the coefficients a_N are determined which minimize the error.

$$\int_a^b E_N(x,\mathbf{a})^2 dx = 0$$

The "best" values of a_N are found by:

$$\int_a^b E_N(x,\mathbf{a})\frac{\partial E_N(x,\mathbf{a})}{\partial a_j}dx = 0 \qquad j = 1,2,\ \ldots\ ,N$$

The weighting function $w(x_j)$ is: $w(x,x_j) = \dfrac{\partial E_N(x,\mathbf{a})}{\partial a_j}$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

The Galerkin method - In the Galerkin method the weighting functions $w_j(x)$ are the admissible functions $\phi_j(x)$. The resulting MWR statement is:

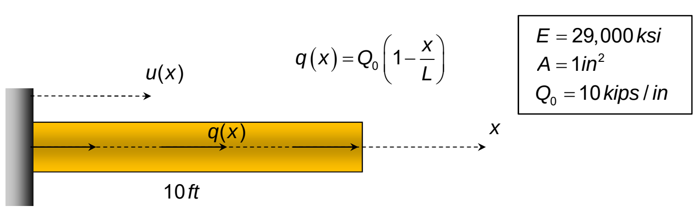
$$\int_a^b E_N(x,\mathbf{a})\phi_j(x)\,dx = 0 \qquad j = 1, 2, \ldots, N$$

This relationship states that error $E_N(x, \mathbf{a})$ is orthogonal to each of the admissible functions $\phi_j(x)$. Orthogonal functions have many relationships. Several important definitions are:

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \qquad \int_a^b (\phi_m(x))^2\,dx = 1$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Example - Consider the problem of the axial deformation of a prismatic bar we worked previously.



The boundary value problem for this case is:

$$(AEu')' + q(x) = 0 \qquad 0 \leq x \leq L$$

the boundary conditions are: $\begin{cases} u(0) = 0 \\ AEu'(L) = P \end{cases}$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

For the given functions and parameters the differential equation and boundary conditions are:

$$AEu'' + q(x) = 0 \qquad u(0) = 0 \qquad AEu'(L) = 0$$

The approximate solution is given as:

$$u \approx \sum_{1}^{N} a_n sin\left(\frac{2n-1}{2L}\pi x\right)$$

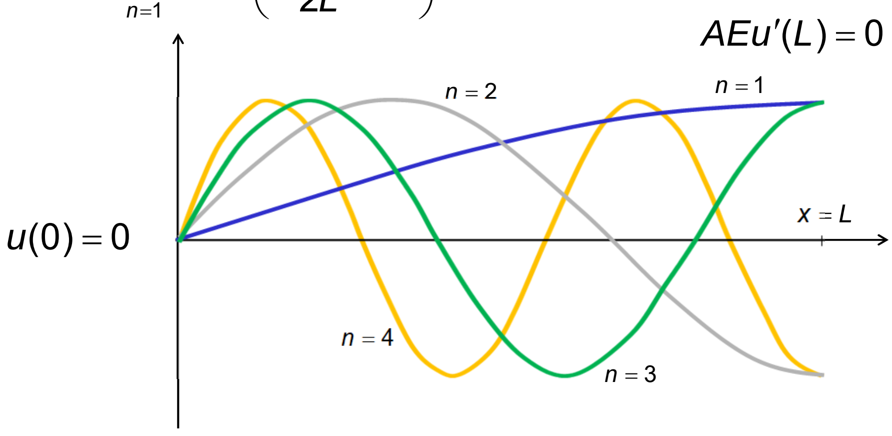
Note this function satisfies the boundary conditions of the problem and is therefore an admissible function.

$$u(0) = 0 \qquad u'(L) = \sum_{1}^{N} a_n \cos\left(\frac{2n-1}{2}\pi\right) = 0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

The first four admissible functions of the approximation are:

$$u \approx \sum_{n=1}^{4} a_n sin\left(\frac{2n-1}{2L}\pi x\right)$$



APPROXIMATION METHODS - WEIGHTED RESIDUALS

The residual error term may be calculated as:

$$E_N(x,\mathbf{a}) = AEu'' + q(x) \qquad u \approx \sum_{1}^{N} a_n sin\left(\frac{2n-1}{2L}\pi x\right)$$

$$= -AE\sum_{n=1}^{N} a_n (m\pi)^2 \sin(m\pi x) + q(x)$$

where: $m = \dfrac{2n-1}{2L}$

In these examples, we will set N equal to 3.

APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>Collocation Method</u> - Choose three locations where the residual error will be assumed to be zero.

Generally, these locations are equally spaced. Therefore, the values of x_j are $L/4$, $L/2$, and $3L/4$

$$E_3\left(\frac{L}{4},\mathbf{a}\right) = 0 \qquad E_3\left(\frac{L}{2},\mathbf{a}\right) = 0 \qquad E_3\left(\frac{3L}{4},\mathbf{a}\right) = 0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Collocation Method - Three equations in a_N may be written as:

$$E_3\left(\frac{L}{4},\mathbf{a}\right)=q\left(\frac{L}{4}\right)-\frac{AE}{L^2}\sum_{n=1}^{3}a_n\left(\frac{2n-1}{2}\pi\right)^2 sin\left(\frac{2n-1}{2}\left(\frac{\pi}{4}\right)\right)$$

$$=-\frac{AE}{L^2}\left(0.9442a_1+20.5162a_2+56.9895a_3\right)+0.75Q_0$$

$$E_3\left(\frac{L}{2},\mathbf{a}\right)=-\frac{AE}{L^2}\left(1.7447a_1+15.7024a_2-43.6179a_3\right)+0.5Q_0$$

$$E_3\left(\frac{3L}{4},\mathbf{a}\right)=-\frac{AE}{L^2}\left(2.2796a_1-8.4981a_2-23.6058a_3\right)+0.25Q_0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Collocation Method - These equations may be expressed in matrix form as:

$$\frac{AE}{L^2}\begin{bmatrix}0.9442 & 20.5162 & 56.9895\\ 1.7447 & 15.7024 & -43.6179\\ 2.2796 & -8.4981 & -23.6058\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\\ a_3\end{Bmatrix}=Q_0\begin{Bmatrix}0.75\\ 0.50\\ 0.25\end{Bmatrix}$$

Solving these equations for a_N gives:

$$a_1=0.2099\frac{Q_0L^2}{AE}\qquad a_2=0.0177\frac{Q_0L^2}{AE}\qquad a_3=0.0033\frac{Q_0L^2}{AE}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Collocation Method - The three term approximation is:

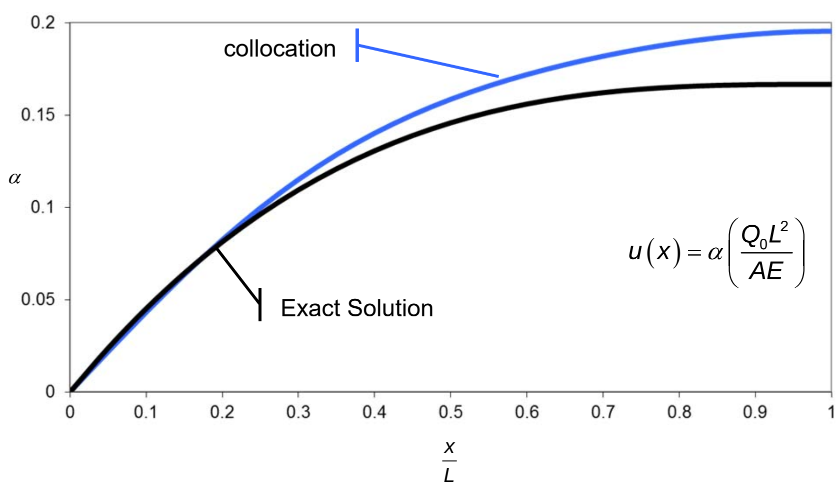
$$\boxed{\frac{Q_0L^2}{AE}\left(0.2099sin\left(\frac{\pi x}{2L}\right)+0.0177sin\left(\frac{3\pi x}{2L}\right)+0.0033sin\left(\frac{5\pi x}{2L}\right)\right)}$$

The exact solution may be determined from the following expression:

$$\begin{aligned} AEu(x) &= -\int^{x}\int^{x'} q(x)dx'dx \\ &= -\int^{x}\int^{x'} Q_0\left(1-\frac{x}{L}\right)dx'dx \\ &= Q_0L^2\left[\frac{1}{6}\left(\frac{x}{L}\right)^3-\frac{1}{2}\left(\frac{x}{L}\right)^2+\frac{1}{2}\left(\frac{x}{L}\right)\right] \end{aligned}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

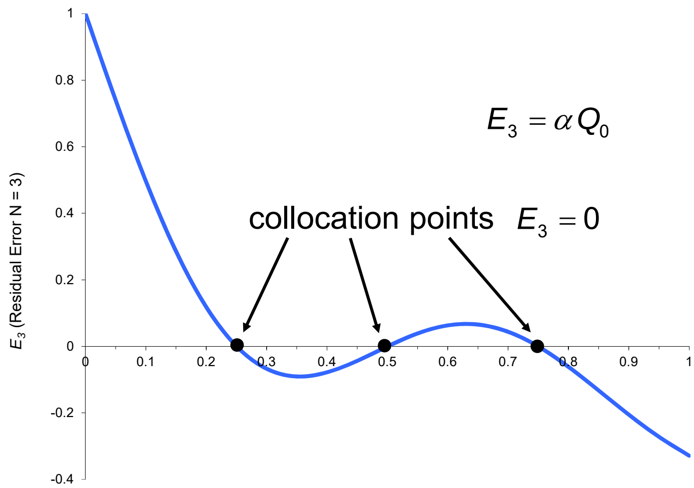
Collocation Method - The three term approximation is:



Comparison of various MWR techniques

APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>Collocation Method</u> - The error E_3 is:



APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>Subdomain Method</u> - Choose three equally-spaced intervals [0, L/3], [L/3, 2L/3], and [2L/3, L].

The resulting three equations are:

$$\int_0^{L/3} E_3 dx = 0 \qquad \int_{L/3}^{2L/3} E_3 dx = 0 \qquad \int_{2L/3}^{L} E_3 dx = 0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Subdomain Method - Three equations in a_N may be written as:

$$\int_0^{L/3} E_3 dx = \int_0^{L/3} q(x)dx - \int_0^{L/3} \frac{AE}{L^2} \sum_{n=1}^{3} a_n \left(\frac{2n-1}{2}\pi\right)^2 sin(m\pi x)dx$$

$$= -\frac{AE}{L}\left(0.2104a_1 + 4.7123a_2 + 14.6557a_3\right) + 0.2778Q_0L$$

$$\int_{L/3}^{2L/3} E_3 dx = -\frac{AE}{L}\left(0.5749a_1 + 4.7123a_2 + 10.7287a_3\right) + 0.1666Q_0L$$

$$\int_{2L/3}^{L} E_3 dx = -\frac{AE}{L}\left(0.7852a_1 + 4.7123a_2 + 3.9269a_3\right) + 0.0556Q_0L$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Subdomain Method - These equations may be expressed in matrix form as:

$$\frac{AE}{L^2}\begin{bmatrix} 0.2104 & 4.7123 & 14.6557 \\ 0.5749 & 4.7123 & -10.7287 \\ 0.7853 & -4.7123 & 3.9269 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = Q_0 \begin{Bmatrix} 0.2778 \\ 0.1666 \\ 0.0556 \end{Bmatrix}$$

Solving these equations for a_N gives:

$$a_1 = 0.1996\frac{Q_0L^2}{AE} \qquad a_2 = 0.0275\frac{Q_0L^2}{AE} \qquad a_3 = 0.0072\frac{Q_0L^2}{AE}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Subdomain Method - The three term approximation is:

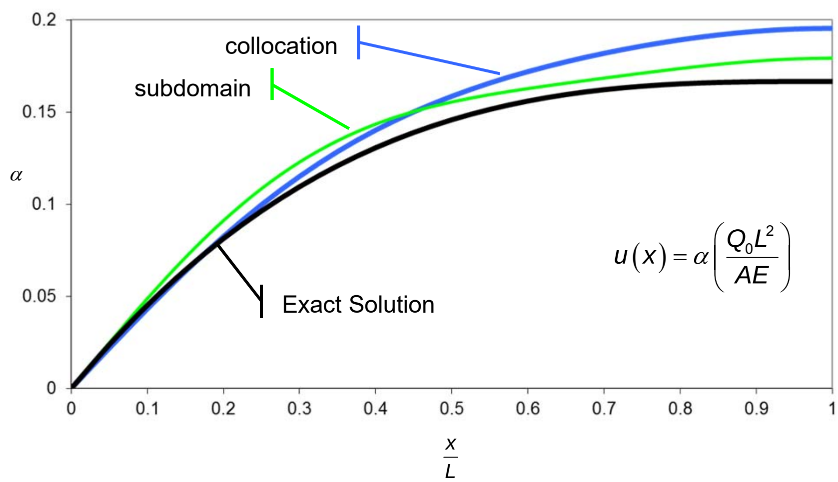
$$\boxed{\frac{Q_0L^2}{AE}\left(0.1996sin\left(\frac{\pi x}{2L}\right)+0.0275sin\left(\frac{3\pi x}{2L}\right)+0.0072sin\left(\frac{5\pi x}{2L}\right)\right)}$$

Collocation Method - The three term approximation is:

$$\boxed{\frac{Q_0L^2}{AE}\left(0.2099sin\left(\frac{\pi x}{2L}\right)+0.0177sin\left(\frac{3\pi x}{2L}\right)+0.0033sin\left(\frac{5\pi x}{2L}\right)\right)}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

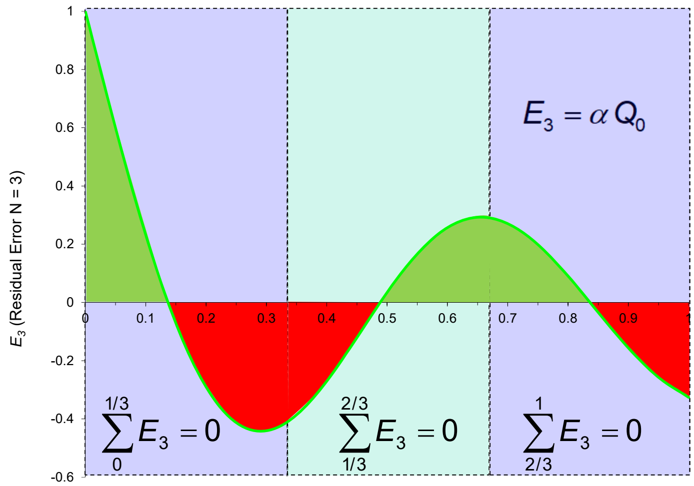
Subdomain Method - The three term approximation is:



Comparison of various MWR techniques

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Subdomain Method - The error E_3 is:



APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - The weighting function for this method is:

$$w_n(x) = \frac{\partial E_3}{\partial a_n} = \left| AE(m\pi)^2 \sin(m\pi x) \right|_n$$

In each case the weighting functions have a few constants that may be eliminated since the residual statement is set equal to zero.

There the final form of the $w_n(x)$ is:

$$w_n(x) = \sin(m\pi x) \qquad m = \frac{2n-1}{2L}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - Three equations in a_N may be written as:

$$\int_0^L E_3 w_n\left(x\right)dx = 0$$

$$= \int_0^L \left(q(x) - \frac{AE}{L^2}\sum_{n=1}^{3} a_n \left(\frac{2n-1}{2}\pi\right)^2 sin\left(m\pi x\right)\right) w_n\left(x\right)dx$$

n = 1

$$\int_0^L E_3 w_1\left(x\right)dx = \int_0^L \left(q(x) - \frac{AE}{L^2}\sum_{n=1}^{3} a_n \left(m\pi\right)^2 sin\left(m\pi x\right)\right) sin\left(\frac{\pi x}{2L}\right)dx = 0$$

$$= -AE\frac{\pi^2}{8L}a_1 + \frac{2LQ_0}{\pi}\left(1 - \frac{2}{\pi}\right) = 0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - Three equations in a_N may be written as:

$$\int_0^L E_3 w_n\left(x\right)dx = 0$$

$$= \int_0^L \left(q(x) - \frac{AE}{L^2}\sum_{n=1}^{3} a_n \left(\frac{2n-1}{2}\pi\right)^2 sin\left(m\pi x\right)\right) w_n\left(x\right)dx$$

n = 2

$$\int_0^L E_3 w_2\left(x\right)dx = \int_0^L \left(q(x) - \frac{AE}{L^2}\sum_{n=1}^{3} a_n \left(m\pi\right)^2 sin\left(m\pi x\right)\right) sin\left(\frac{3\pi x}{2L}\right)dx = 0$$

$$= -AE\frac{9\pi^2}{8L}a_2 + \frac{2LQ_0}{\pi}\left(\frac{1}{3} - \frac{2}{9\pi}\right) = 0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - Three equations in a_N may be written as:

$$\int_0^L E_3 w_n(x)\,dx = 0$$

$$= \int_0^L \left(q(x) - \frac{AE}{L^2} \sum_{n=1}^{3} a_n \left(\frac{2n-1}{2}\pi \right)^2 sin(m\pi x) \right) w_n(x)\,dx$$

---

n = 3

$$\int_0^L E_3 w_3(x)\,dx = \int_0^L \left(q(x) - \frac{AE}{L^2} \sum_{n=1}^{3} a_n (m\pi)^2 sin(m\pi x) \right) sin\left(\frac{5\pi x}{2L}\right) dx = 0$$

$$= -AE\,\frac{25\pi^2}{8L} a_3 + \frac{2LQ_0}{\pi}\left(\frac{1}{5} - \frac{2}{25\pi}\right) = 0$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - These equations may be expressed in matrix form as:

$$\frac{AE\pi^2}{8L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = LQ_0 \begin{Bmatrix} 0.2313 \\ 0.2572 \\ 0.1111 \end{Bmatrix}$$

Solving these equations for a_N gives:

$$a_1 = 0.1875\frac{Q_0 L^2}{AE} \quad a_2 = 0.0232\frac{Q_0 L^2}{AE} \quad a_3 = 0.0036\frac{Q_0 L^2}{AE}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - The three term approximation is:

$$\boxed{\frac{Q_0L^2}{AE}\left(0.1875sin\left(\frac{\pi x}{2L}\right)+0.0232sin\left(\frac{3\pi x}{2L}\right)+0.0036sin\left(\frac{5\pi x}{2L}\right)\right)}$$

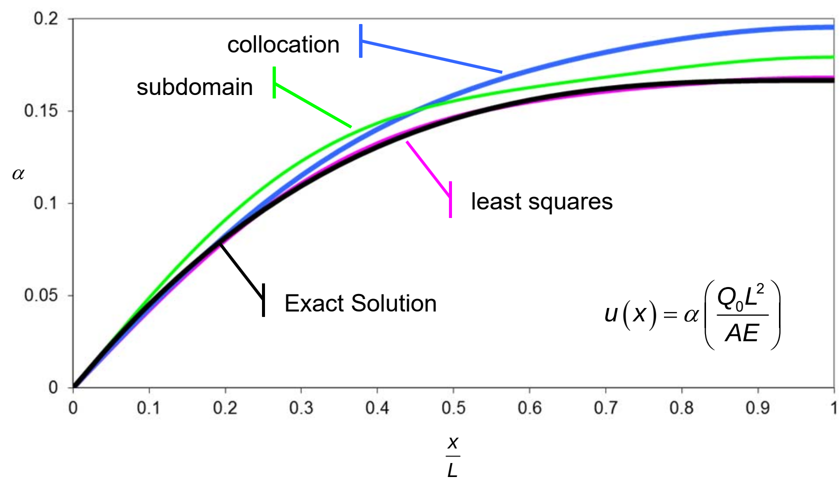
Subdomain Method - The three term approximation is:

$$\boxed{\frac{Q_0L^2}{AE}\left(0.1996sin\left(\frac{\pi x}{2L}\right)+0.0275sin\left(\frac{3\pi x}{2L}\right)+0.0072sin\left(\frac{5\pi x}{2L}\right)\right)}$$

Collocation Method - The three term approximation is:

$$\boxed{\frac{Q_0L^2}{AE}\left(0.2099sin\left(\frac{\pi x}{2L}\right)+0.0177sin\left(\frac{3\pi x}{2L}\right)+0.0033sin\left(\frac{5\pi x}{2L}\right)\right)}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

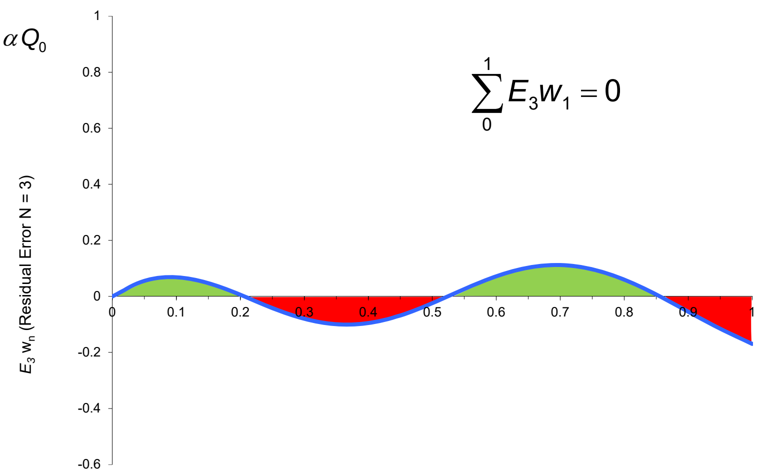


Comparison of various MWR techniques

APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>Least Squares Method</u> - The error E_3 is:

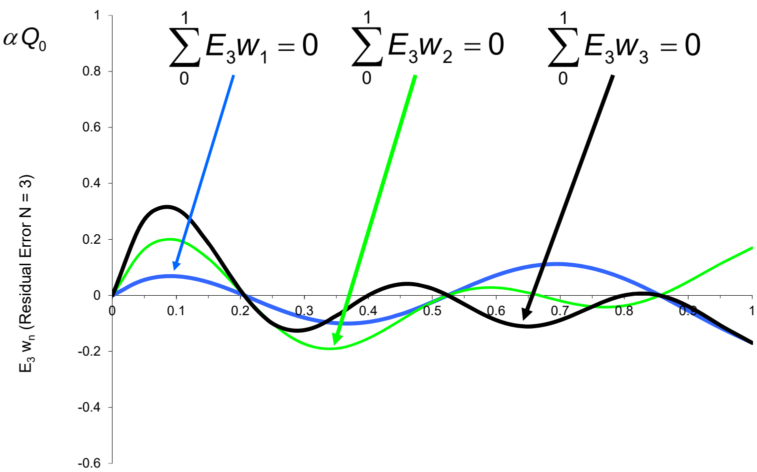
$E_3 w_n = \alpha Q_0$



APPROXIMATION METHODS - WEIGHTED RESIDUALS

<u>Least Squares Method</u> - The error E_3 is:

$E_3 w_n = \alpha Q_0$



APPROXIMATION METHODS - WEIGHTED RESIDUALS

Galerkin Method - The weighting function for this method is the same as the admissible function.

In this case, the weighting function is the same as in the least squares method.

$$w_n(x) = \frac{\partial E_3}{\partial a_n} = \left| AE(m\pi)^2 \sin(m\pi x) \right|_n$$

$$w_n(x) = \sin(m\pi x) \qquad\qquad m = \frac{2n-1}{2L}$$

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Galerkin Method - The resulting equations are determined from the following relationships:

$$\int_0^L E_3 \sin\left(\frac{\pi x}{2L}\right) dx = 0$$

$$\int_0^L E_3 \sin\left(\frac{3\pi x}{2L}\right) dx = 0$$

$$\int_0^L E_3 \sin\left(\frac{5\pi x}{2L}\right) dx = 0$$

Galerkin Method - The Galerkin approximation is the same as the least squares approximation.

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Galerkin/Least Squares Method

$$\frac{Q_0L^2}{AE}\left(0.1875sin\left(\frac{\pi x}{2L}\right)+0.0232sin\left(\frac{3\pi x}{2L}\right)+0.0036sin\left(\frac{5\pi x}{2L}\right)\right)$$

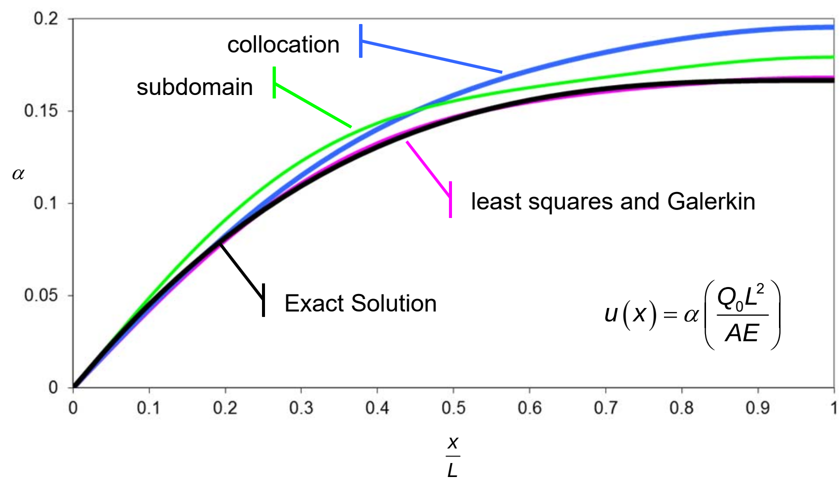
Subdomain Method

$$\frac{Q_0L^2}{AE}\left(0.1996sin\left(\frac{\pi x}{2L}\right)+0.0275sin\left(\frac{3\pi x}{2L}\right)+0.0072sin\left(\frac{5\pi x}{2L}\right)\right)$$

Collocation Method

$$\frac{Q_0L^2}{AE}\left(0.2099sin\left(\frac{\pi x}{2L}\right)+0.0177sin\left(\frac{3\pi x}{2L}\right)+0.0033sin\left(\frac{5\pi x}{2L}\right)\right)$$

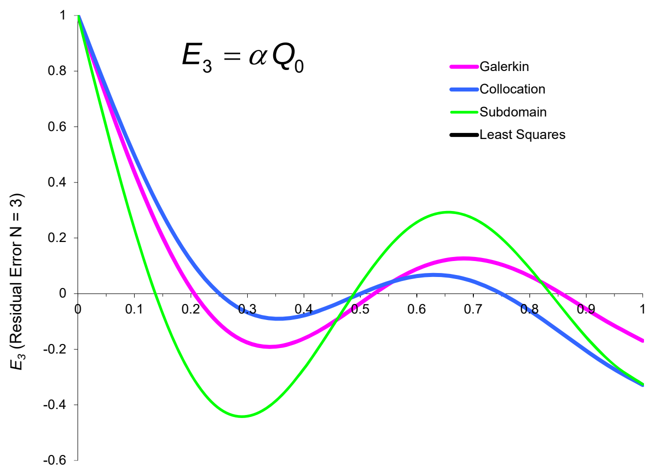
APPROXIMATION METHODS - WEIGHTED RESIDUALS



Comparison of various MWR techniques

APPROXIMATION METHODS - WEIGHTED RESIDUALS

Least Squares Method - The error E_3 is:



APPROXIMATION METHODS - WEIGHTED RESIDUALS

PROBLEM #8 - For the following differential equations construct a three term approximation using the a) collocation method, b) subdomain method, c) least squares method, and d) Galerkin method.

$$\boxed{u'' + x^2 u + x = 0} \qquad 0 \le x \le 1 \qquad \begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases}$$

Assume an approximate solution of the form:

$$u = \sum_{1}^{N} a_n \sin\left(n\pi x\right)$$

Plot the final results and comment on any differences in the various methods compared to the exact solution.

APPROXIMATE METHODS - VARIATION METHODS

The weighted residual methods we discussed previously attacked the differential equation directly to generate an approximation.

In this section, we will use the **Ritz** method to generate an approximation to the differential equation based on variational principles.

Consider the **weak formulation** used in the method of weighted residuals for the general Sturm-Liouville boundary value problem:

$$(pu')' - qu + \lambda\rho u + f = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

$$p(b)u'(b) + \beta u(b) = B$$

APPROXIMATE METHODS - VARIATION METHODS

First, consider the case where the $\lambda\rho u$ term is absent.

Multiple the weak formulation by a test function ν(x):

$$\int_a^b \left[(pu')' - qu + f\right]\nu(x)dx = 0$$

where $\nu(x)$ is a suitable admissible function which satisfies the homogeneous form of any boundary conditions of the dependent variable *u*.

The term “weak” used in describing this formulation is based on the fact that we require the differential equation to be satisfied in an average sense over a small interval *a* < *x* < *b* and not at every point in the interval.

APPROXIMATE METHODS - VARIATION METHODS

Notice that the differential equation has a second derivative of u and that the test function ν (admissible function) is required only to be continuous. To eliminate this inconsistency, integrate the first term by parts:

$$\int_a^b \left[(pu')'\right]\nu(x)dx = \nu pu'\Big|_a^b - \int_a^b (\nu' pu')dx$$

The resulting weak formulation is:

$$\nu pu'\Big|_a^b - \int_a^b (\nu' pu' - \nu qu + \nu f)dx$$

$$\nu(b)p(b)u'(b) - \nu(a)p(a)u'(a) - \int_a^b (\nu' pu' - \nu qu + \nu f)dx$$

APPROXIMATE METHODS - VARIATION METHODS

From the boundary conditions given for the general Sturm-Liouville problem:

$$p(b)u'(b) = B - \beta u(b) \qquad p(a)u'(a) = -A + \alpha u(a)$$

This eliminates the derivative terms at the boundary. The weak statement becomes:

$$\int_a^b (p\nu' u' - q\nu u)dx + \beta\nu(b)u(b) + \alpha\nu(a)u(a)$$

$$= \int_a^b (\nu f)dx + B\nu(b) + A\nu(a)$$

APPROXIMATE METHODS - VARIATION METHODS

The dependent variable is now required to have only a first derivative and the continuity of the u and v are the same.

The left-hand side of the above expression is an example of a bilinear function and may be written as:

$$B(v,u) = \int_a^b \left(pv'u' - qvu\right)dx + \beta v(b)u(b) + \alpha v(a)u(a)$$

The right-hand side of the weak formulation is a linear function and may written as:

$$r(v) = \int_a^b \left(vf\right)dx + Bv(b) + Av(a)$$

APPROXIMATE METHODS - VARIATION METHODS

The weak formulation may be written as: $B\left(v,u\right) = r\left(v\right)$

It can be shown that $B(v,u) = B(u,v)$, therefore there exist a functional of the form:

$$Z\left(u\right) = \frac{B\left(u,u\right)}{2} - r\left(u\right)$$

If this functional is required to be **stationary** then it yields the original boundary value problem. In our case, the Sturm-Liouville problem can be written as:

$$Z(u) = \frac{1}{2}\int_a^b \left[p\left(u'\right)^2 - qu^2\right]dx + \frac{\beta u(b)^2}{2} + \frac{\alpha u(a)^2}{2}$$

$$-\int_a^b \left(uf\right)dx + Bu(b) + Au(a)$$

APPROXIMATE METHODS - VARIATION METHODS

In order to determine the stationary value that yield the statement of the boundary value problem, some basic ideas of the **calculus of variations** are needed.

Basic Calculus of Variations - Unlike plain calculus which deals with the change in a function, say *f*(*x*), calculus of variations deals with a **functional** which are dependent on functions as well as variables.

For example, a functional may be written as: $F\left(x,u,u'\right)$

APPROXIMATE METHODS - VARIATION METHODS

In our discussion, the functional in question is:

$$Z(u) = \int_a^b F\left(x,u,u'\right)$$

with boundary conditions $u(a) = u_a$ and $u(b) = u_b$. Such a problem is referred to as a **fixed endpoint problem**. This is one of the simplest problems in calculus of variations.

First, assume a possible solution *u* (admissible function) and compute the value of Z(*u*).

Next, vary the solution a small amount, say an infinitesimal amount and recompute the value of the functional.

If the change in the functional is very small then you have the solution to the differential equation.

APPROXIMATE METHODS - VARIATION METHODS

This is not a convenient way for solving these types of problems.

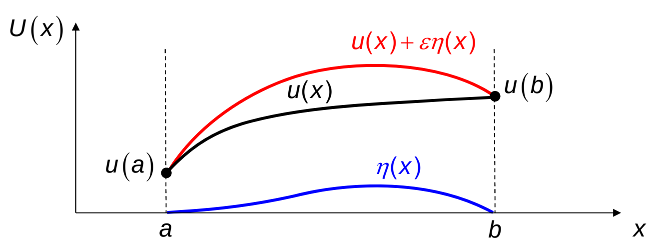
The process seems to be a hopelessly involved sequence of trials and experiences.

Calculus of variations allows a way to set up a form that determines all possible function(s) which render the functional stationary.

APPROXIMATE METHODS - VARIATION METHODS

Let's define a family of curve $U(x) = u(x) + \varepsilon\eta(x)$ which varies from the solution $u(x)$ by a term $\varepsilon\eta(x)$.

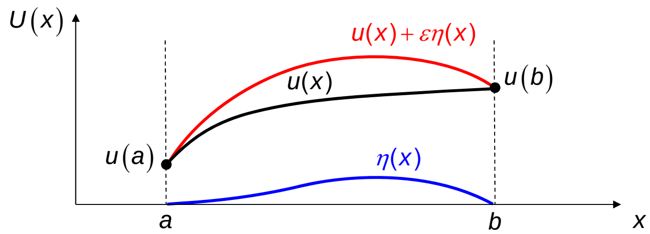
The value of ε is considered small and the function $\eta(x)$ is an arbitrary function which vanishes at the ends of the interval $a < x < b$, $\eta(a) = \eta(b) = 0$.



APPROXIMATE METHODS - VARIATION METHODS

Therefore the original functional may be written as:

$$Z(u) = \int_a^b F\left(x,\, u + \varepsilon\eta,\, u' + \varepsilon\eta'\right) dx$$



APPROXIMATE METHODS - VARIATION METHODS

Requiring the functional Z(u) to be stationary with respect to ε as ε approaches zero states that:

$$\lim_{\varepsilon\to 0} \frac{dZ}{d\varepsilon} = Z'(0) = 0$$

Computing $dZ/d\varepsilon$ results in:

$$\frac{dZ(u)}{d\varepsilon} = \frac{d}{d\varepsilon}\int_a^b F\left(x,\, u + \varepsilon\eta,\, u' + \varepsilon\eta'\right) dx = \int_a^b \frac{dF}{d\varepsilon} dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial\left(u + \varepsilon\eta\right)} \frac{d\left(u + \varepsilon\eta\right)}{d\varepsilon} + \frac{\partial F}{\partial\left(u' + \varepsilon\eta'\right)} \frac{d\left(u' + \varepsilon\eta'\right)}{d\varepsilon} \right) dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial\left(u + \varepsilon\eta\right)} \eta + \frac{\partial F}{\partial\left(u' + \varepsilon\eta'\right)} \eta' \right) dx$$

$$Z'(0) = \int_a^b \left(\frac{\partial F}{\partial u} \eta + \frac{\partial F}{\partial u'} \eta' \right) dx = 0$$

APPROXIMATE METHODS - VARIATION METHODS

Integrating the second term by parts results in:

$$\frac{\partial F}{\partial u'}\eta\Bigg|_a^b - \int_a^b \left(\frac{d\left(\partial F / \partial u'\right)}{dx} - \frac{\partial F}{\partial u} \right) \eta \, dx = 0$$

Since η(a) is zero, the equation reduces to:

$$\frac{\partial F}{\partial u'}\eta\Bigg|^b - \int_a^b \left(\frac{d\left(\partial F / \partial u'\right)}{dx} - \frac{\partial F}{\partial u} \right) \eta \, dx = 0$$

If we choose η (b) to be equal to zero:

$$-\int_a^b \left(\frac{d\left(\partial F / \partial u'\right)}{dx} - \frac{\partial F}{\partial u} \right) \eta \, dx = 0$$

APPROXIMATE METHODS - VARIATION METHODS

Since the η function is an admissible function the integrand should vanish:

$$\frac{d\left(\partial F / \partial u'\right)}{dx} - \frac{\partial F}{\partial u} = 0$$

This second order differential equation is the **Euler equation** whose solutions are called **extermals**.

The solution to the original problem, the function which gives the stationary value, is an extermal with the two boundary conditions $u(a) = u_a$ and $u(b) = u_b$.

APPROXIMATE METHODS - VARIATION METHODS

If we choose an arbitrary η that does not vanish at $x = b$ then the condition in the above equation leads to:

$$\left.\frac{\partial F}{\partial u}\eta\right|^{b}$$

This term is called a **natural boundary condition**.

This type of condition is usually associated with a derivative condition at the boundary.

If a value of u is specified at a boundary, then the condition is called an **essential** or **forced boundary condition**.

APPROXIMATE METHODS - VARIATION METHODS

Example - Consider the axial deformation of the rod we have previously discussed. The potential energy functional is:

$$Z(u) = \int_0^L \left(\frac{AE\left(u'\right)^2}{2} - qu \right) dx \qquad u(0) = 0$$

Since u at $x = L$ is not prescribed, then the boundary condition at $x = L$ will be a natural boundary condition.

Generate the Euler equations and corresponding boundary conditions.

APPROXIMATE METHODS - VARIATION METHODS

Requiring the functional Z to be stationary with respect to ε as ε approaches zero states that:

$$lim_{\varepsilon\to 0}\frac{dZ}{d\varepsilon}=Z'(0)=0$$

therefore

$$Z'(0)=\int_a^b\left(\frac{\partial F}{\partial u}\eta+\frac{\partial F}{\partial u'}\eta'\right)dx=0$$

where the functional F is:

$$F=\frac{AE(u')^2}{2}-qu$$

APPROXIMATE METHODS - VARIATION METHODS

Integrating the second term by parts results in:

$$\frac{\partial F}{\partial u}\eta\Big|_a^b-\int_a^b\left(\frac{d(\delta F/\partial u')}{dx}-\frac{\partial F}{\partial u}\right)\eta\,dx=0$$

The forced boundary condition of $u(0) = 0$ requires $\eta(a) = 0$, therefore:

$$\frac{\partial F}{\partial u}\eta\Big|^b-\int_a^b\left(\frac{d(\delta F/\partial u')}{dx}-\frac{\partial F}{\partial u}\right)\eta\,dx=0$$

Since η is arbitrary at $x = L$ then it may not vanish at $x = b$, leads to the conclusion that:

$$\eta\frac{\partial F}{du'}\Big|^b=0\qquad\to\qquad\frac{\partial F}{\partial u'}\Big|^b=0$$

APPROXIMATE METHODS - VARIATION METHODS

The Euler equation is the integrand of the functional:

$$\frac{d\left(\partial F / \partial u'\right)}{dx} - \frac{\partial F}{\partial u} = 0$$

which for this problem becomes:

$$\frac{d\left(AEu'\right)}{dx} + q = 0 \quad \rightarrow \quad \left(AEu'\right)' + q = 0$$

$$\text{with} \quad AEu'\left(L\right) = 0$$

APPROXIMATE METHODS - VARIATION METHODS

PROBLEM #9 - Generate the Euler equations and forced boundary conditions for the following functionals:

$$Z(u) = \int_1^2 \left(\frac{x\left(u'\right)^2}{2} - \frac{u^2}{2x} - \frac{u}{x} \right) dx \qquad u(2) = 1$$

$$Z(u) = \int_0^L \left(\frac{AE\left(u'\right)^2}{2} - \frac{ku^2}{2x} - qu \right) dx \qquad u(0) = 0$$

APPROXIMATE METHODS - VARIATION METHODS

APPROXIMATE METHOD OF RITZ

In the previous section we found that if the energy functional for a given problem exists, the governing differential equation may be obtained by requiring the energy functional to be stationary.

Instead of finding the governing differential equation from the energy functional we could use the relationship directly to approximate the solution.

One of the most powerful techniques of obtaining an approximate solution to the boundary value problem is the method of **Ritz**.

APPROXIMATE METHODS - METHOD OF RITZ

Consider the following functional:

$$Z(u) = \int_a^b F\left(x, u, u'\right) dx \qquad\qquad u(a) = u_a$$

with an approximate solution of the form:

$$u = \phi_0\left(x\right) + \sum_{n=1}^{N} c_n \phi_n\left(x\right)$$

The nonhomogeneous boundary conditions are satisfied by $\phi_0(x)$ and the homogeneous boundary conditions are satisfied by each admissible function $\phi_n(x)$.

APPROXIMATE METHODS - METHOD OF RITZ

This approximation is identical to the weighted residual methods we discussed earlier except for one important difference, the admissible functions for MWR are required to satisfy all boundary conditions, whereas the admissible function for the Ritz method are required to satisfy only the essential or forced boundary conditions.

The approximation of *u* is substituted into the functional *Z*.

The stationary value of the functional is found by requiring:

$$\frac{\partial Z}{\partial c_i} = 0 \qquad i = 1, 2, \ldots, N$$

APPROXIMATE METHODS - METHOD OF RITZ

This relationship gives us a set of *N* algebraic equations in the unknowns c_i.

The Sturm-Liouville problem:

$$pu'' - qu + \lambda\rho u + f = 0 \qquad a < x < b$$

$$-p(a)u'(a) + \alpha u(a) = A$$

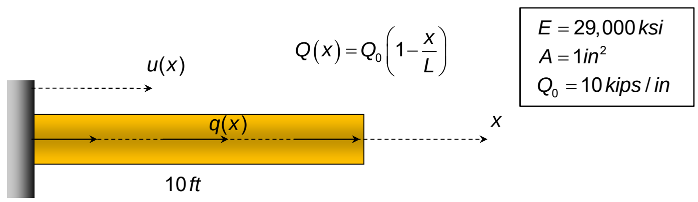
$$p(b)u'(b) + \beta u(b) = B$$

with the corresponding functional:

$$Z(u) = \int_a^b \left[\frac{p(u')^2 - qu^2}{2} - uf \right] dx + \frac{\alpha u(a)^2}{2} + \frac{\beta u(b)^2}{2} + Au(a) + Bu(b)$$

APPROXIMATE METHODS - METHOD OF RITZ

Example - Consider the problem of the axial deformation of a prismatic bar we worked previously.



The boundary value problem for this case is:

$$(AEu')' + Q(x) = 0 \qquad 0 \le x \le L$$

the boundary conditions are: $\begin{cases} u(0) = 0 \\ AEu'(L) = 0 \end{cases}$

APPROXIMATE METHODS - METHOD OF RITZ

The Sturm-Liouville form of this equation requires that $p = AE$, $q = 0$, $f = Q(x)$, and A = B = $\alpha = \beta = 0$. The corresponding functional is:

$$Z(u) = \int_0^L \left(\frac{AE(u')^2}{2} - Qu \right) dx$$

Let's assume an approximate solution as:

$$u = \sum_{n=1}^{N} a_n \sin\left(\frac{2n-1}{2L}\pi x\right) \qquad m = \frac{2n-1}{2L}$$

APPROXIMATE METHODS - METHOD OF RITZ

Substituting the approximation into the functional $Z(u)$ gives:

$$Z(u) = \int_0^L \left[\frac{AE}{2} \left(\sum_{n=1}^{N} a_n m\pi \cos(m\pi x) \right)^2 - Q_0 \left(1 - \frac{x}{L}\right) \left(\sum_{n=1}^{N} a_n \sin(m\pi x) \right) \right] dx$$

Requiring the functional to be stationary with respect to each a_i gives:

$$\frac{\partial Z(u)}{\partial a_i} = 0 = \int_0^L \left[AE \left(\sum_{n=1}^{N} a_n m\pi \cos(m\pi x) \right) m_i \pi \cos(m_i \pi x) \right] dx$$

$$- \int_0^L \left[Q_0 \left(1 - \frac{x}{L}\right) \sin(m_i \pi x) \right] dx = 0 \qquad m_i = \frac{2i-1}{2L}$$

APPROXIMATE METHODS - METHOD OF RITZ

The preceding equation may be written as:

$$AE \sum_{n=1}^{N} a_n \int_0^L m_i m \pi^2 \cos(m_i \pi x) \cos(m\pi x) dx$$

$$= Q_0 \int_0^L \left(1 - \frac{x}{L}\right) \sin(m_i \pi x) dx \qquad m_i = \frac{2i-1}{2L}$$

APPROXIMATE METHODS - METHOD OF RITZ

Taking N = 3, a three term Ritz solution may be formed. For example, when i = 1 the following equation is generated:

$$\frac{AE}{L}\sum_{1}^{N} a_n \int_0^L \frac{m\pi^2}{2}\cos\left(\frac{\pi x}{2L}\right)\cos(m\pi x)\,dx - Q_0\int_0^L \left(1-\frac{x}{L}\right)\sin\left(\frac{\pi x}{2L}\right)dx$$

When i = 2

$$\frac{AE}{L}\sum_{1}^{N} a_n \int_0^L \frac{3m\pi^2}{2}\cos\left(\frac{3\pi x}{2L}\right)\cos(m\pi x)\,dx - Q_0\int_0^L \left(1-\frac{x}{L}\right)\sin\left(\frac{3\pi x}{2L}\right)dx$$

When i = 3

$$\frac{AE}{L}\sum_{1}^{N} a_n \int_0^L \frac{5m\pi^2}{2}\cos\left(\frac{5\pi x}{2L}\right)\cos(m\pi x)\,dx - Q_0\int_0^L \left(1-\frac{x}{L}\right)\sin\left(\frac{5\pi x}{2L}\right)dx$$

APPROXIMATE METHODS - METHOD OF RITZ

After integration, the following set of equations may be formed:

$$\frac{AE\pi^2}{8L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = LQ_0\begin{Bmatrix} 0.2313 \\ 0.2572 \\ 0.1111 \end{Bmatrix}$$

Solving these equations for a_N gives:

$$a_1 = 0.1875\frac{Q_0L^2}{AE} \quad a_2 = 0.0232\frac{Q_0L^2}{AE} \quad a_3 = 0.0036\frac{Q_0L^2}{AE}$$

APPROXIMATE METHODS - METHOD OF RITZ

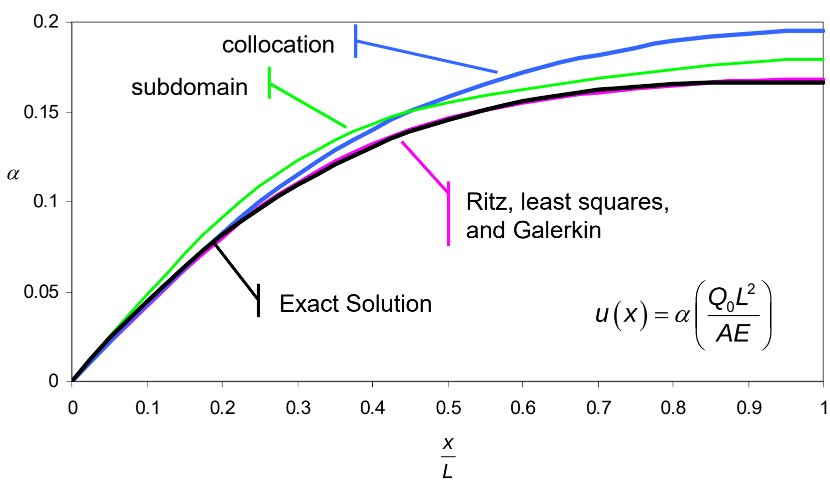
Method of Ritz - The three term approximation is:

$$\boxed{\frac{Q_0L^2}{AE}\left(0.1875 sin\left(\frac{\pi x}{2L}\right)+0.0232 sin\left(\frac{3\pi x}{2L}\right)+0.0036 sin\left(\frac{5\pi x}{2L}\right)\right)}$$

Least Squares/Galerkin Method - The three term approximation is:

$$\boxed{\frac{Q_0L^2}{AE}\left(0.1875 sin\left(\frac{\pi x}{2L}\right)+0.0232 sin\left(\frac{3\pi x}{2L}\right)+0.0036 sin\left(\frac{5\pi x}{2L}\right)\right)}$$

APPROXIMATE METHODS - METHOD OF RITZ



Comparison of various MWR and variational techniques

APPROXIMATE METHODS - METHOD OF RITZ

PROBLEM #10 - For the following functional use a three term Ritz approximation.

$$Z(u) = \int_0^1 \left(\frac{(u')^2 - x^2u^2}{2} - xu \right) dx \qquad \begin{cases} u(0) = 0 \\ u(1) = 0 \end{cases}$$

Assume an approximate solution of the form:

$$u(x) = \sum_{n=1}^{N} a_n sin\left(n\pi x\right)$$

Plot the final results and comment on any differences in the variational method and the results from Problem #8.

End of Chapter 2a