# A First Course in Finite Elements

### *Introduction*

- The ***finite element method*** has become a powerful tool for the numerical solution of a wide range of engineering problems.
- Applications range from deformation and stress analysis of automotive, aircraft, building, and bridge structures to field analysis of heat flux, fluid flow, magnetic flux, seepage, and other flow problems.

# A First Course in Finite Elements

### *Introduction*

- With the advances in computer technology and CAD systems, complex problems can be modeled with relative ease.
- Several alternative configurations can be tried out on a computer before the first prototype is built.
- All of this suggests that we need to keep pace with these developments by understanding the basic theory, modeling techniques, and computational aspects of the finite element method.

# *A First Course in Finite Elements*

***Introduction***

- In this method of analysis, a complex region defining a continuum is discretized into simple geometric shapes called **finite elements**.

- The material properties and the governing relationships are considered over these elements and expressed in terms of unknown values at element corners.

- An assembly process, duly considering the boundary conditions, results in a set of equations.

- Solution of these equations gives us the approximate behavior of the continuum.

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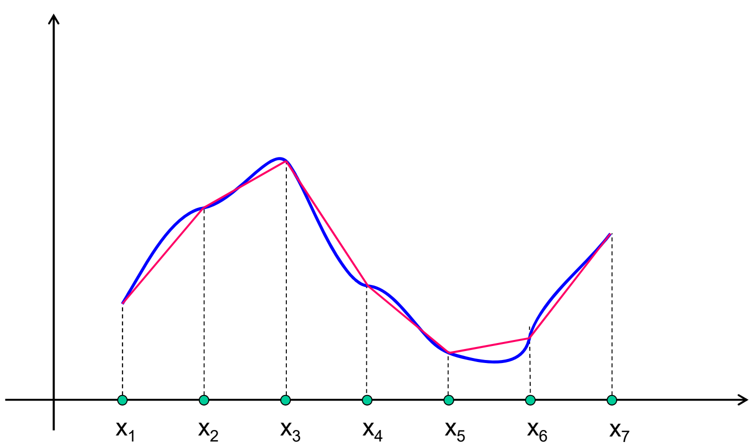


Piecewise linear function in one dimensions.

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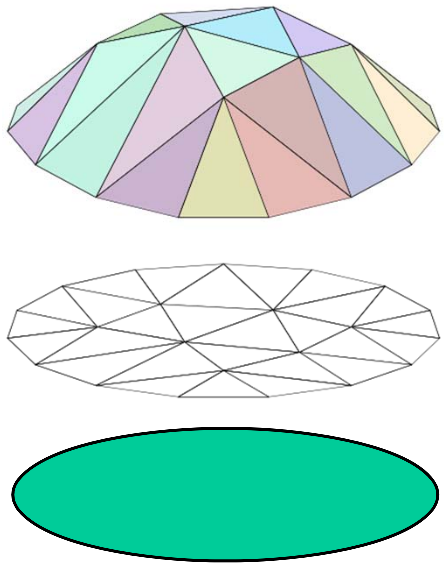

Piecewise linear function in two dimensions.

Discretization of two dimensional domain

Original two dimensional domain

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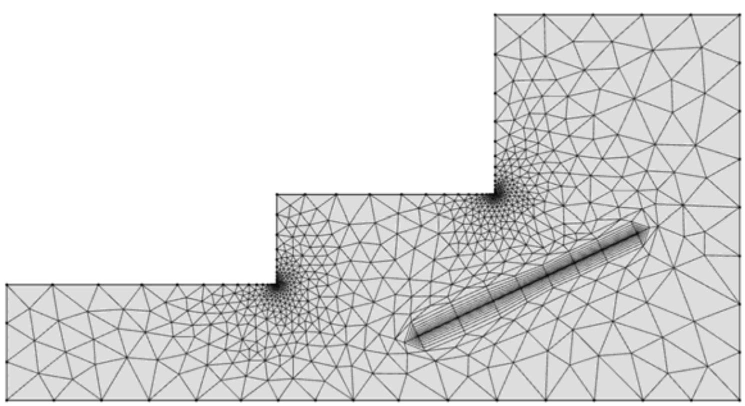

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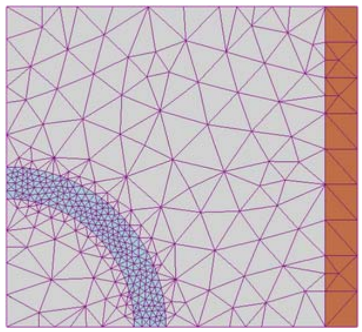
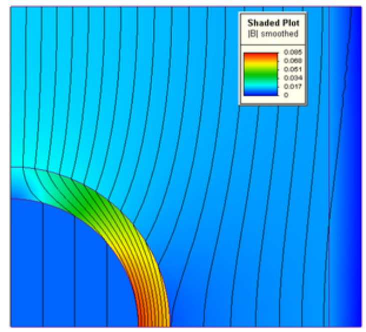
A magnetic problem using FEM software



Colors indicate that the analyst has set material properties for each zone, in this case a conducting wire coil in orange; a ferromagnetic component (perhaps iron) in light blue; and air in grey.

FEM solution to the problem




The color represents the amplitude of the magnetic flux density, as indicated by the scale in the inset legend, red being high amplitude.

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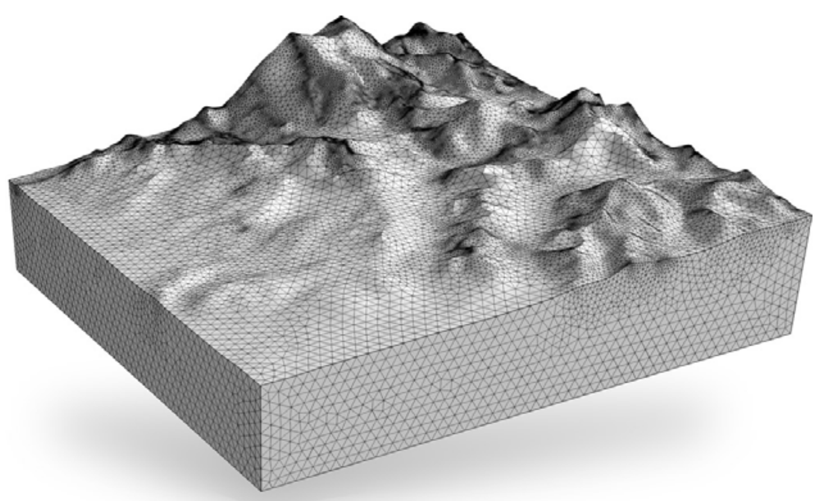

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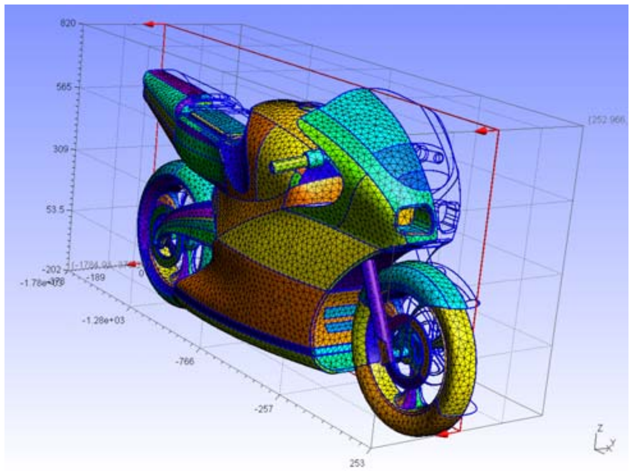


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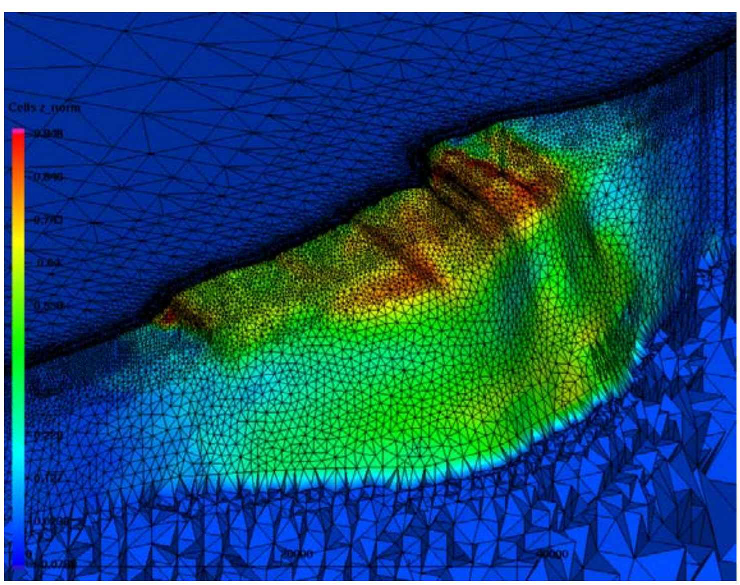

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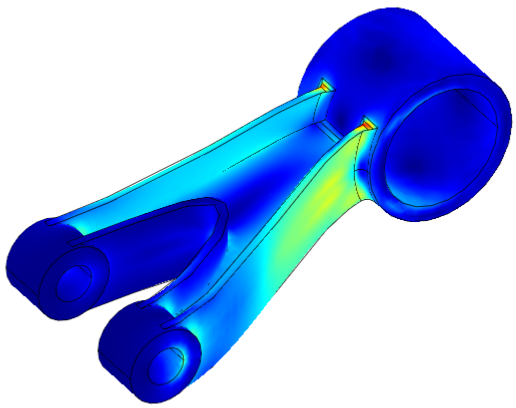

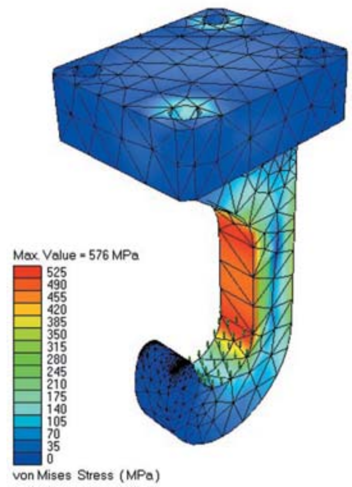


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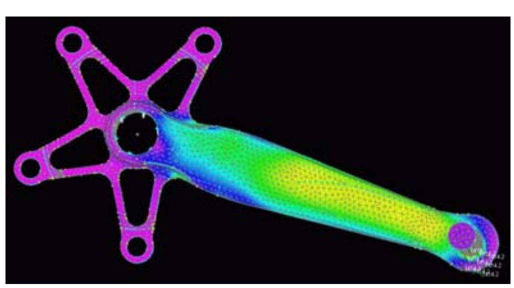

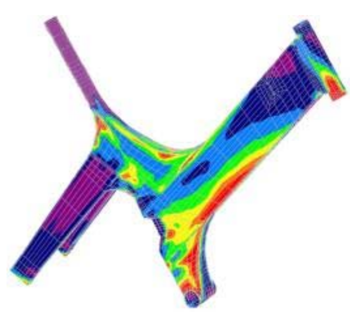

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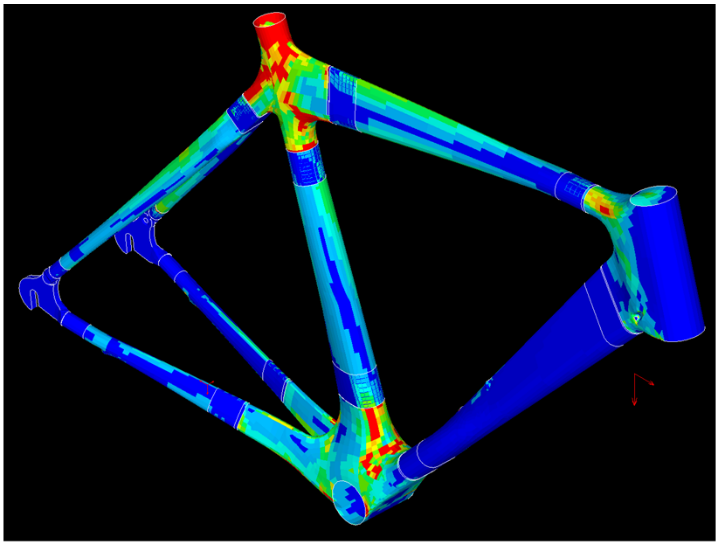

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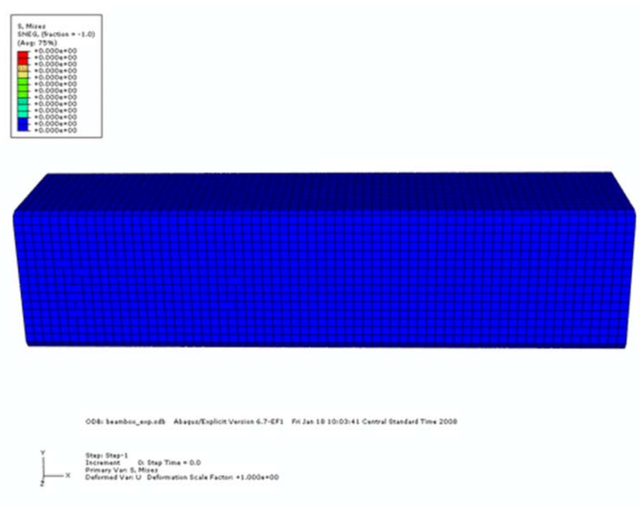

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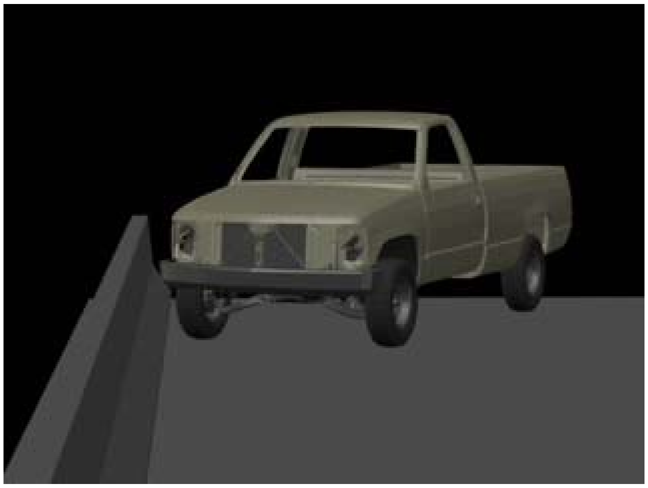

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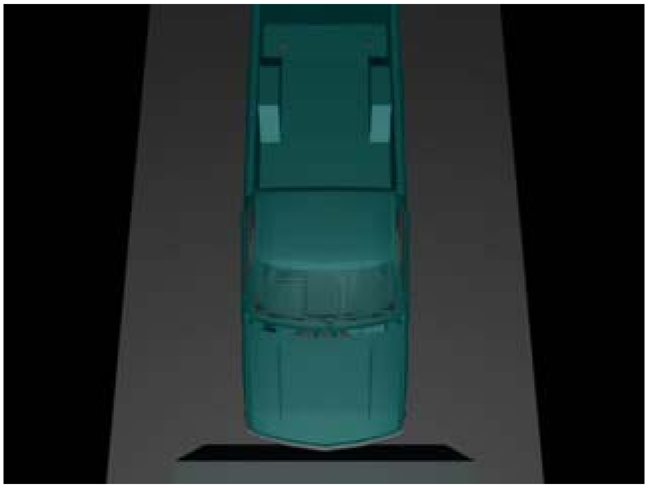

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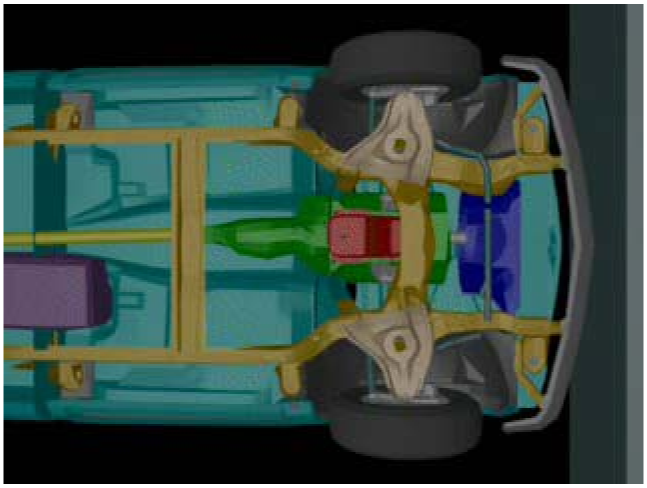

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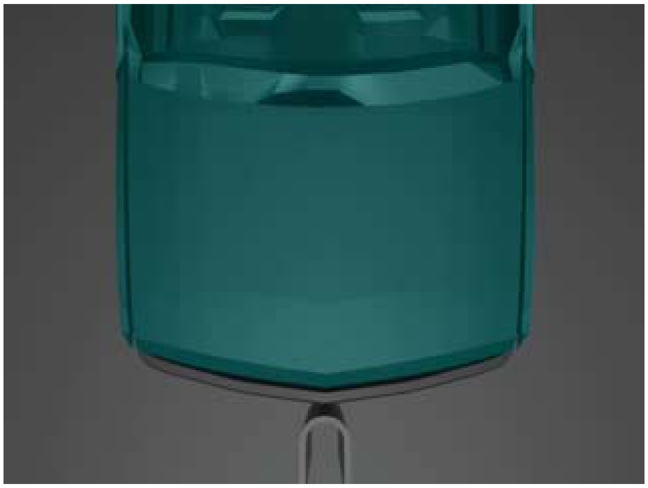
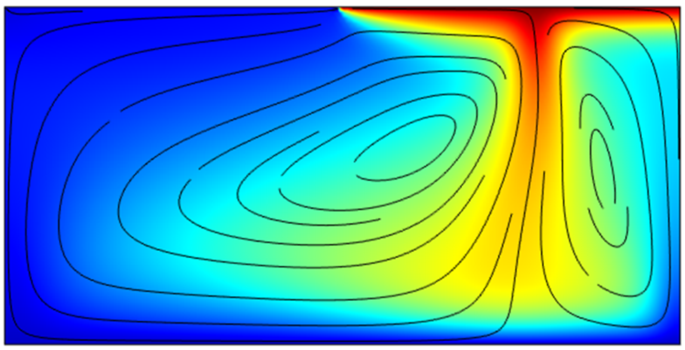


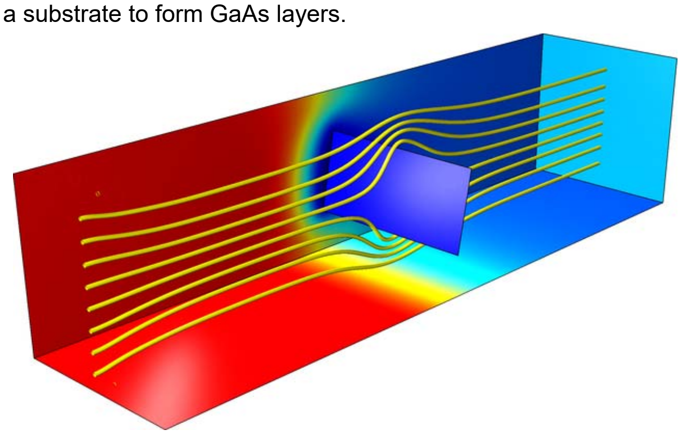
# A First Course in Finite Elements

This example duplicates a benchmark problem for time-dependent buoyant flow in porous media. Known as the Elder problem, it follows a laboratory experiment to study thermal convection. This model examines the Elder problem for concentrations through a 2-way coupling of two physics interfaces: Darcy's Law and Solute Transport



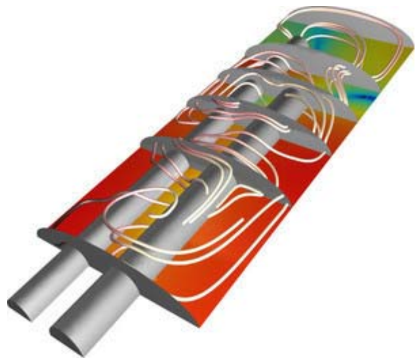
# A First Course in Finite Elements

Chemical vapor deposition (CVD) allows a thin film to be grown on a substrate through molecules and molecular fragments adsorbing and reacting on a surface. This example illustrates the modeling of such a CVD reactor where triethyl-gallium first decomposes, and the reaction products along with arsine ($AsH_3$) adsorb and react on a substrate to form GaAs layers.



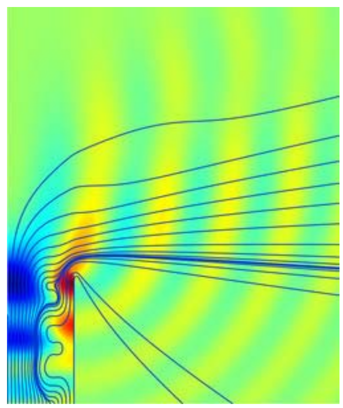
## A First Course in Finite Elements

A typical automotive exhaust system is a hybrid construction consisting of a combination of reflective and dissipative muffler elements. The reflective parts are normally tuned to remove dominating low-frequency engine harmonics while the dissipative parts are designed to take care of higher-frequency noise. The muffler analyzed in this model, is an example of a complex hybrid muffler in which the dissipative element is created completely by flow through perforated pipes and plates.



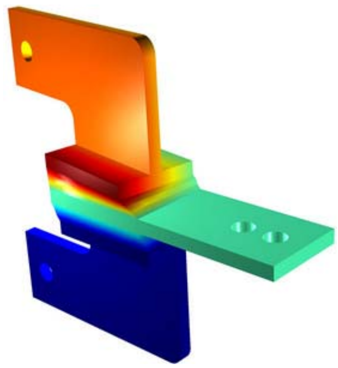
## A First Course in Finite Elements

This example models the radiation of fan noise from the annular duct of a turbofan aeroengine. When the jet stream exits the duct, a vortex sheet appears along the extension of the duct wall due to the surrounding air moving at a lower speed. The near field on both sides of the vortex sheet is calculated.



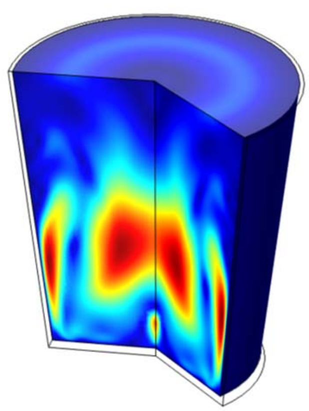
## A First Course in Finite Elements

Damping elements involving layers of viscoelastic materials are often used for reduction of seismic and wind induced vibrations in buildings and other tall structures. The common feature is that the frequency of the forced vibrations is low. This model studies a forced response of a typical viscoelastic damper. The analysis involves two cases: a frequency response analysis and a time-dependent analysis.



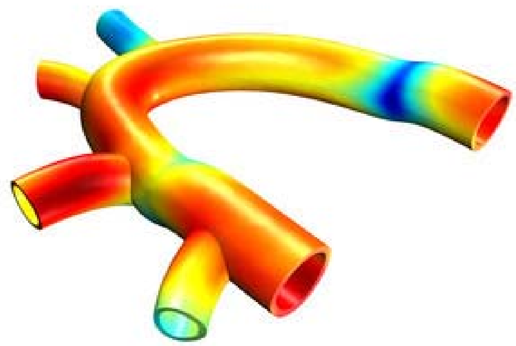
## A First Course in Finite Elements

This model treats the free convection and heat transfer of a glass of cold water heated to room temperature. Initially, the glass and the water are at 5 °C and are then put on a table in a room at 25 °C. The nonisothermal flow is coupled to heat transfer using the Heat Transfer module.



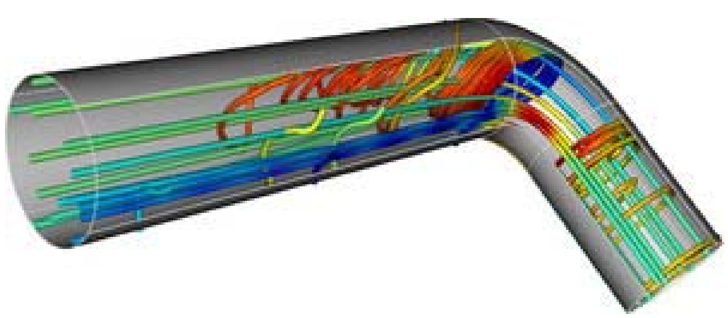
## *A First Course in Finite Elements*

The complete analysis consists of two distinct but coupled procedures: a fluid-dynamics analysis with the calculation of the velocity field and pressure distribution in the blood (variable in time and in space) and the mechanical analysis with the deformation of the tissue and artery. The material is assumed to be nonlinear and a hyperelastic model is used.



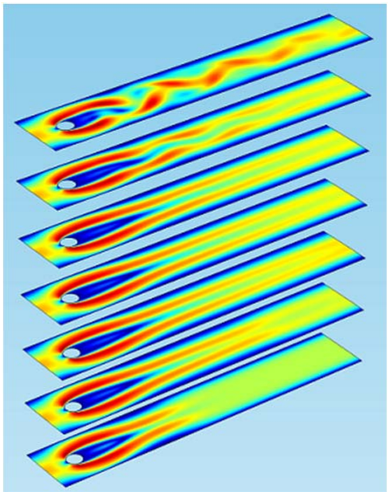
## *A First Course in Finite Elements*

This model studies the fluid flow through a bending pipe in 3D for the Reynolds number 300,000. Because of the high Reynolds number, the k-epsilon turbulence model is used. Calculations with and without corner smoothing are performed. The results are compared with experimental data.



## A First Course in Finite Elements

This model simulates the time-dependent flow past a cylinder. The velocity field magnitude at different time steps is shown.



## A First Course in Finite Elements

### *Historical Background*

- Basic ideas of the finite element method originated from advances in aircraft structural analysis.
- In 1941, Hrenikoff presented a solution of elasticity problems using the “frame work method.”
- Courant’s paper, which used piecewise polynomial interpolation over triangular subregions to model torsion problems, appeared in 1943.
- Turner et al. (1956) derived stiffness matrices for truss, beam, and other elements.
- The term “finite element” was first coined and used by Clough in 1960.

## A First Course in Finite Elements

**_Historical Background_**

- In the early 1960s, engineers used the method for approximate solution of problems in stress analysis, fluid flow, heat transfer, and other areas.
- A book by Argyris in 1955 on energy theorems and matrix methods laid a foundation for further developments in finite element studies.
- The first book on finite elements by Zienkiewicz and Chung was published in 1967.
- In the late 1960s and early 1970s, finite element analysis was applied to nonlinear problems and large deformations.

## A First Course in Finite Elements

**_Historical Background_**

- Mathematical foundations were laid in the 1970s.
- New element development, convergence studies, and other related areas fall in this category.
- Today, developments in high-performance-computers and availability of powerful microcomputers have brought this method within reach of students and engineers working in small industries.

# A First Course in Finite Elements

***Historical Background***

- Until the early 1950s, matrix methods and the associated finite element method were not readily adaptable for solving complicated problems because of the large number of algebraic equations that resulted.

- With the advent of the modern computers, the solution of thousands of equations in a matter of seconds became possible.

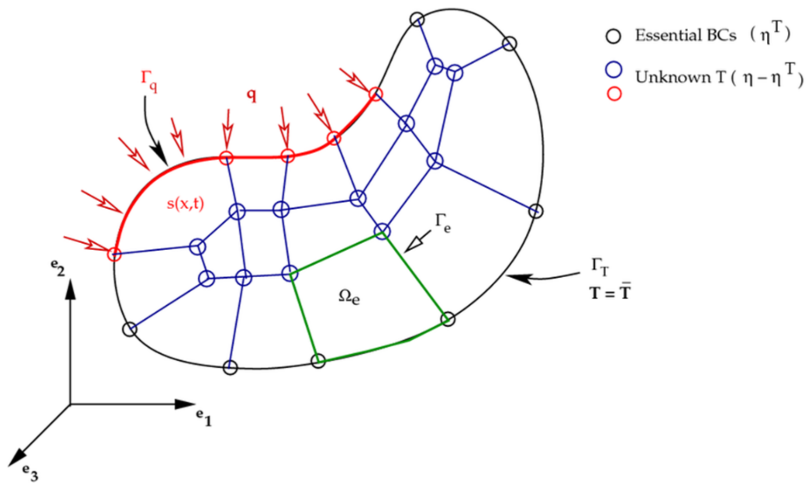
# A First Course in Finite Elements

**Basic Ingredients - Discrete Problems**

The basic steps or building blocks of any application of FEM to a mathematical or physical problem are:

1. Discretization
2. Interpolation
3. Elemental Description or Formulation
4. Assembly
5. Constraints
6. Solution
7. Computation of Derived Variables

**Step 1 - Discretize and Select Element Types**




**Step 1 - Discretize and Select Element Types**

Step 1 involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type.

The total number of elements used and their variation in size and type within a given body are primarily matters of engineering judgment.

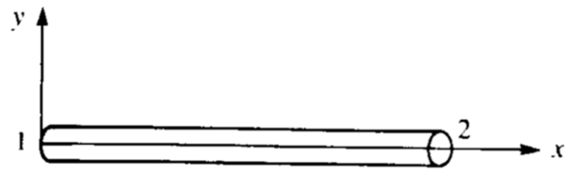
The elements must be made small enough to give usable results and yet large enough to reduce computational effort.

Small elements (and possibly higher-order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur, whereas large elements can be used where results are relatively constant.

**Step 1 - Discretize and Select Element Types**

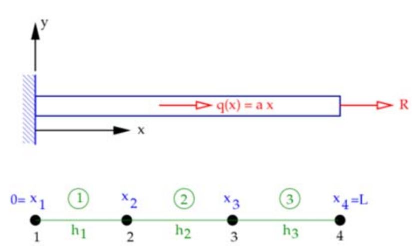
Primary ***line elements*** consist of bar (or truss) and ***beam elements***.

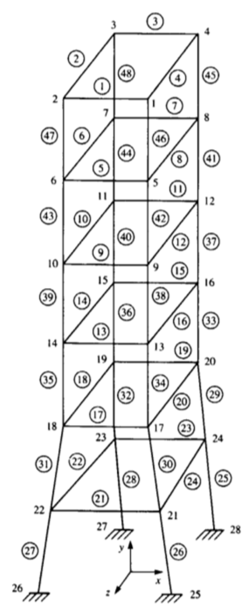
They have a cross-sectional area but are usually represented by line segments.




The simplest line element (called a ***linear element***) has two nodes, one at each end, although higher-order elements having three nodes or more (called ***quadratic***, ***cubic***, etc. ***elements***) also exist.

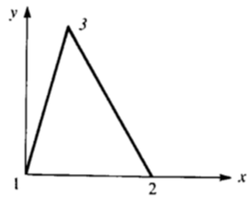
**Step 1 - Discretize and Select Element Types**

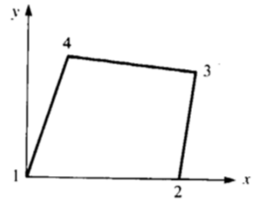


**Step 1 - Discretize and Select Element Types**

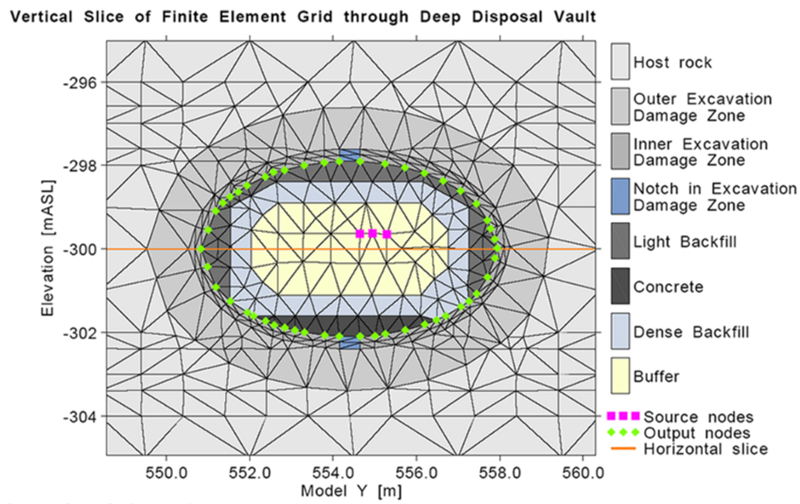
The basic two-dimensional (or plane) elements are loaded by forces in their own plane (plane stress or plane strain conditions). They are triangular or quadrilateral elements.




The simplest two-dimensional elements have corner nodes only (linear elements) with straight sides or boundaries although there are also higher-order elements, typically with mid-side nodes (called ***quadratic elements***) and curved sides.

**Step 1 - Discretize and Select Element Types**



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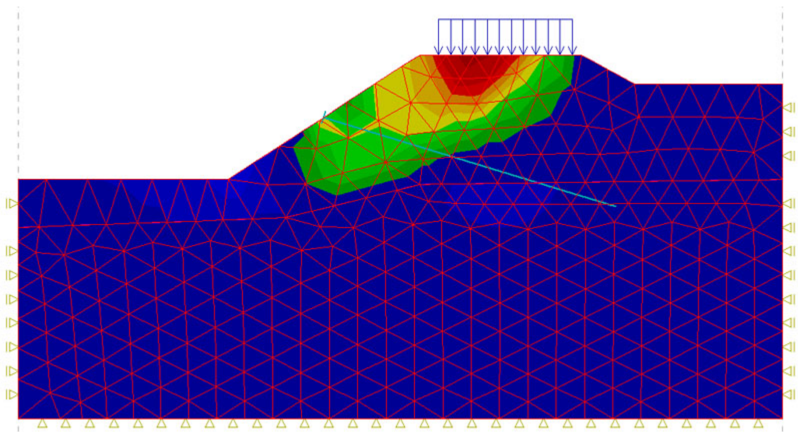
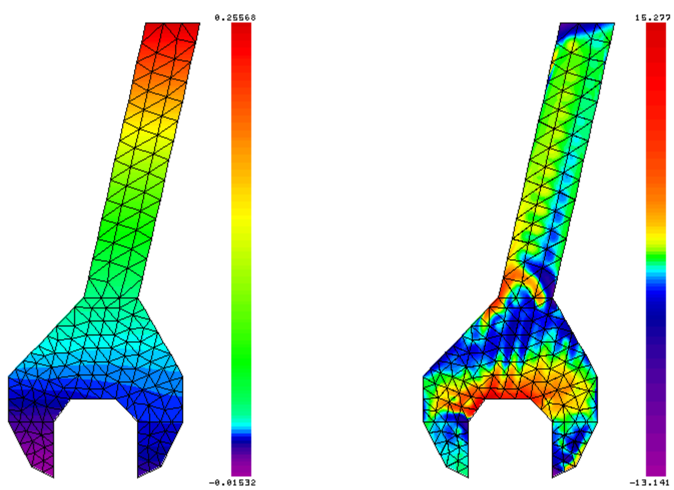


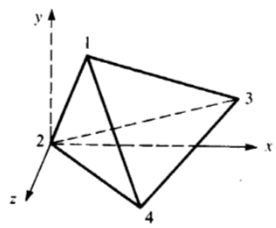
**Step 1 - Discretize and Select Element Types**

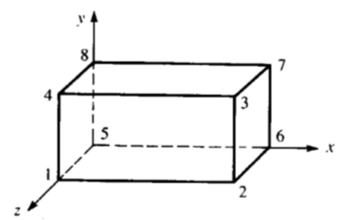


**Step 1 - Discretize and Select Element Types**

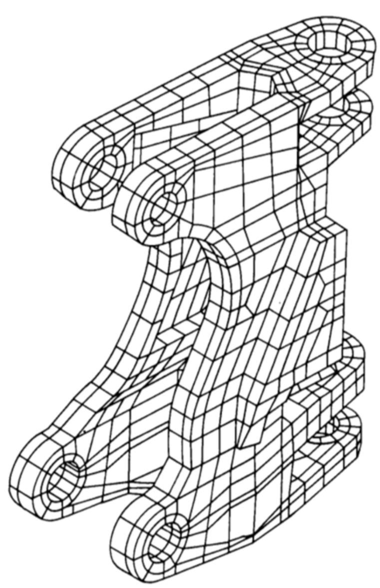
The most common three-dimensional elements are tetrahedral and hexahedral (or ***brick***) ***elements***; they are used when it becomes necessary to perform a three-dimensional stress analysis.

The basic three dimensional elements have corner nodes only and straight sides, whereas higher-order elements with mid-edge nodes (and possible mid-face nodes) have curved surfaces for their sides



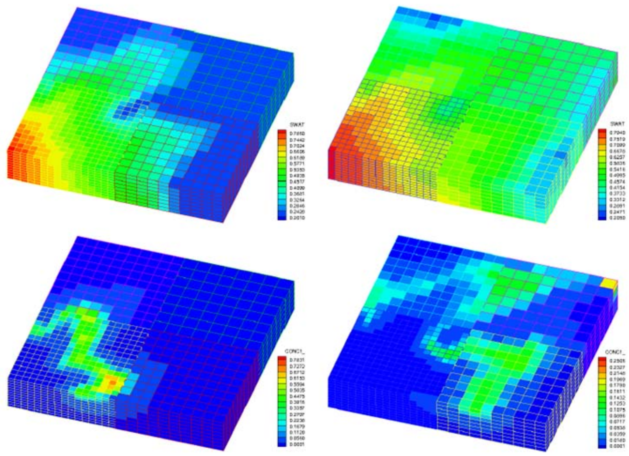




**Step 1 - Discretize and Select Element Types**

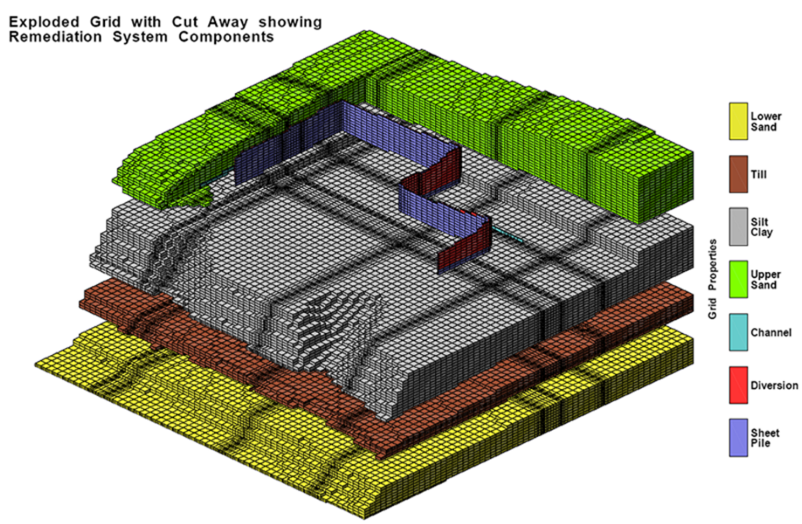


## Step 1 - Discretize and Select Element Types

Water phase saturations (top figures) and $CO_2$ concentration (bottom figures) profiles at 50 and 100 days. The $CO_2$ moves in complicated and unexpected ways.



## Step 1 - Discretize and Select Element Types



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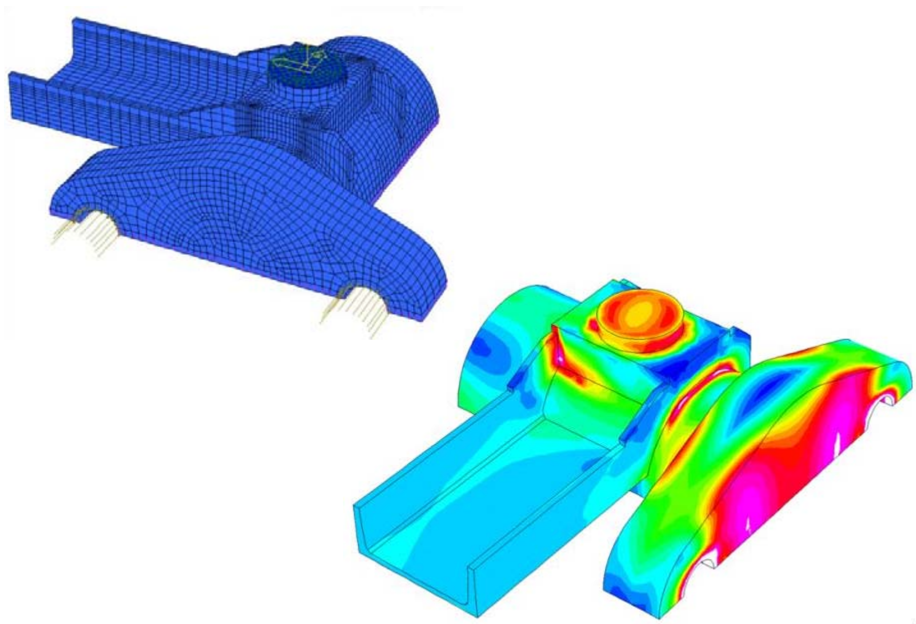
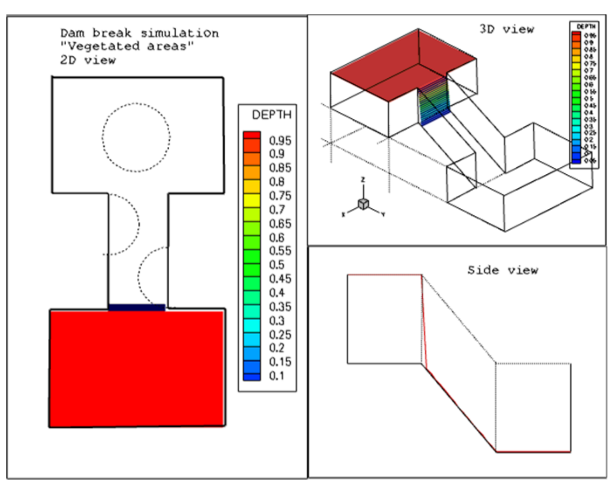
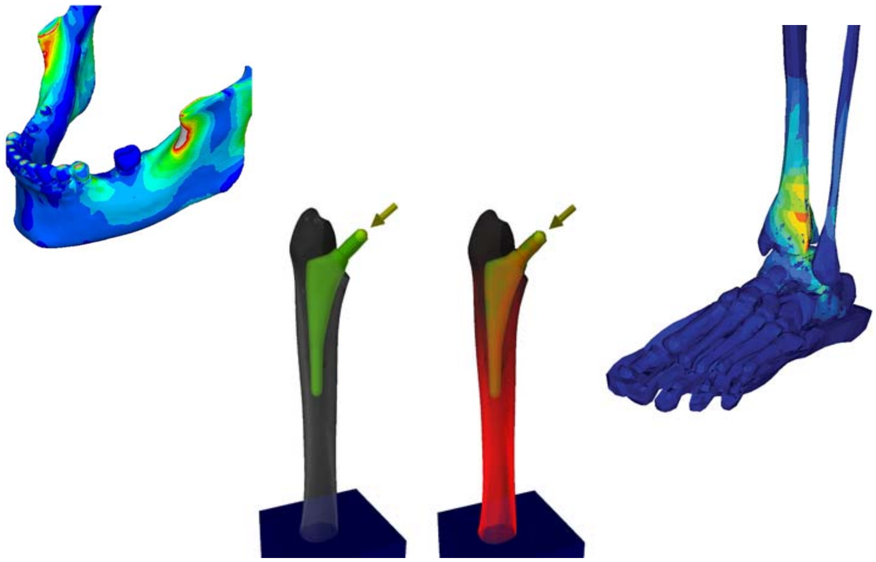


## Step 1 - Discretize and Select Element Types

As a consequence of our changing climate, large efforts have been made to understand the social risks of storm surges (hypothesized to increase in frequency in warmer climate scenarios) and sea level rise in coastal areas. Of particular interest is the role that wetlands and coastal marshes play in storm surges and flooding events.



**Step 1 - Discretize and Select Element Types**



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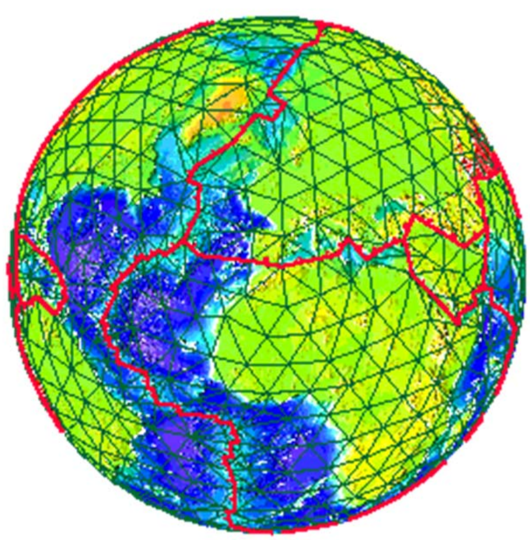
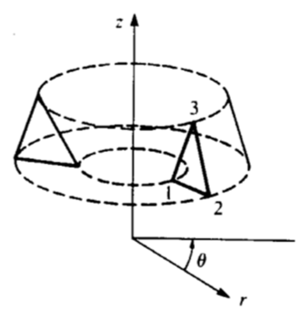


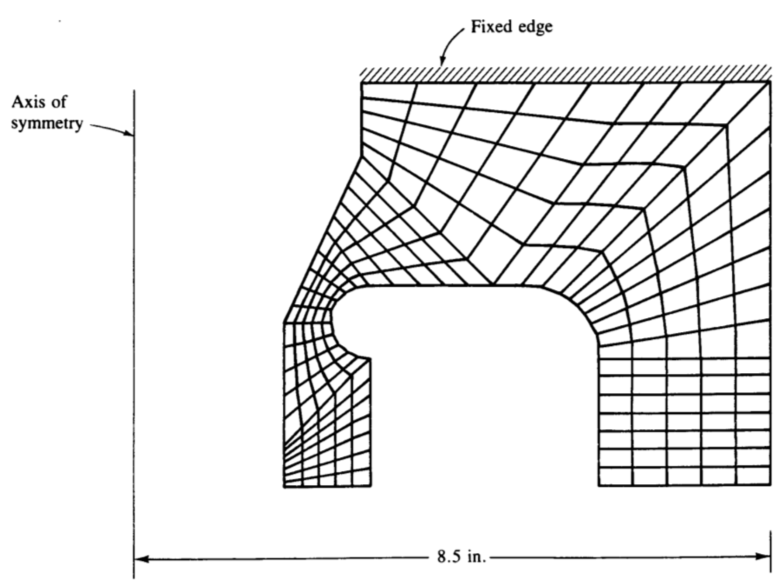
**Step 1 - Discretize and Select Element Types**

The ***axisymmetric element*** is developed by rotating a triangle or quadrilateral about a fixed axis located in the plane of the element through 360°.

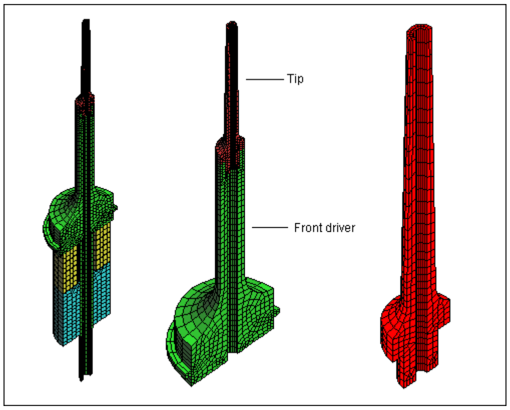
This element can be used when the geometry and loading of the problem are axisymmetric.

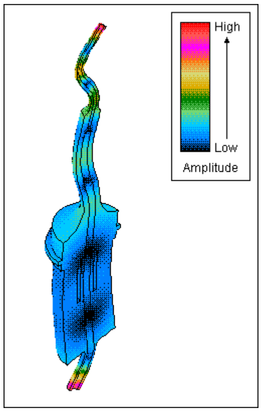



**Step 1 - Discretize and Select Element Types**



**Step 1 - Discretize and Select Element Types**

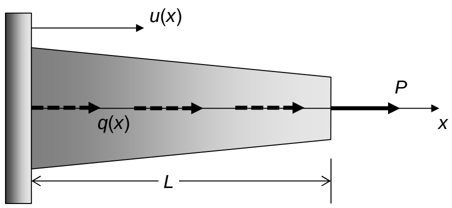



**Step 1 - Discretize and Select Element Types**

Consider the problem of the axial deformation of a linearly elastic bar under an axial load $P$ at $x = L$ and distributed external load $q(x)$.

The cross-sectional area, $A(x)$, the modulus of elasticity, $E$, and the mass density, $\rho\,(x)$, are given.

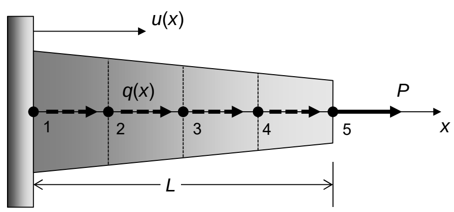



$P$ = external load
$q(x)$ = distributed load
$u(x)$ = axial displacement

**Step 1 - Discretize and Select Element Types**

Let's assume that the variation of the loads, $P(x)$ and $q(x)$, and the cross-sectional area, $A(x)$, are complicated and the exact solution to the above equation cannot be found.

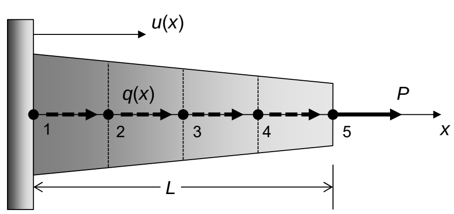
The basic concept of FEM is to cut the problem up into a series of simpler discrete problems and relate the parts to each other to model the continuous material. A possible example of a discrete model of the bar is:




**Step 1 - Discretize and Select Element Types**

Discrete means essentially that we are willing to accept a model that will yield information about the dependent variables at a finite number of points, referred to as **nodes**, within the interval $0 \leq x \leq L$.

Each node is assigned a displacement $u_i$, i = 1 to 5. The problem has been converted from a continuous model of infinite degrees of freedom to one with a finite number of degrees of freedom, in this case $n = 5$.

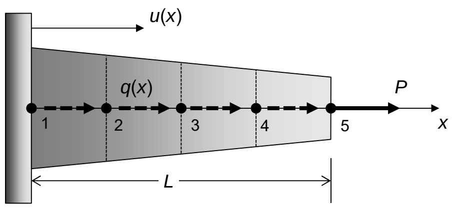


**Step 1 - Discretize and Select Element Types**

The elastic effects of the discrete parts of the bar may be represented as **elements**.

In our problem, the elongation of an axial bar under an axial load is represented by:

$$e = \frac{Pl}{A_{avg}E} \qquad P = \left(\frac{A_{avg}E}{l}\right)e = ke$$

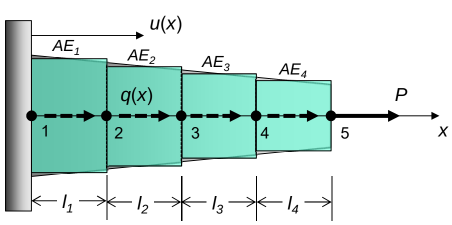



**Step 1 - Discretize and Select Element Types**

Therefore, an elastic bar of length $l$ is equivalent to a simple linear spring.

The stiffness associated with each "element" will be a different value since $A_{avg}$ varies from node to node. Let's approximate the stiffness, $k$, by taking:

$$A_{avg} = \frac{A_i + A_{i+1}}{2} \qquad k = \frac{(A_i + A_{i+1})E}{2l_i}$$



**Step 1 - Discretize and Select Element Types**

Equivalent systems of springs connecting each set of nodes are referred to as **elements**.

An element generally describes some basic physical property of the system. In the case of the axial bar, the relationship between force and displacement is:

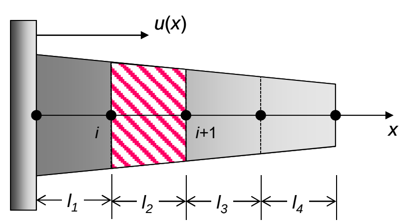
$$F_i = k_i e_i = k_i \left(u_{i+1} - u_i\right)$$

Another important physical parameter associated with the element is the mass.

There are several ways to distribute the mass. Keeping the concept of the element we have developed so far, let’s consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.

**Step 1 - Discretize and Select Element Types**

Keeping the concept of the element we have developed so far, let’s consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.

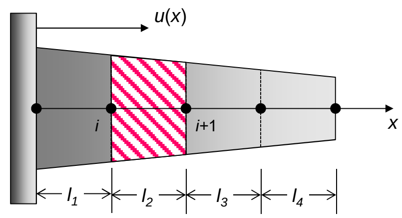



One method of distributing the mass is to average the mass over the element and divide it equally between the two nodes defining the element. The average mass intensity is:

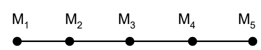
$$m^* = \frac{\rho(x_i)A(x_i) + \rho(x_{i+1})A(x_{i+1})}{2} = \frac{m_i + m_{i+1}}{2}$$

**Step 1 - Discretize and Select Element Types**

Keeping the concept of the element we have developed so far, let's consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.




Therefore the discrete lumped mass system is:




$$M_1 = \frac{(m_1 + m_2)l_1}{4}$$

$$M_2 = \frac{(m_1 + m_2)l_1}{4} + \frac{(m_2 + m_3)l_2}{4}$$

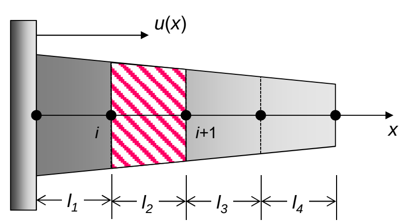
$$M_3 = \frac{(m_2 + m_3)l_2}{4} + \frac{(m_3 + m_4)l_3}{4}$$

$$M_4 = \frac{(m_3 + m_4)l_3}{4} + \frac{(m_4 + m_5)l_4}{4}$$

$$M_5 = \frac{(m_4 + m_5)l_4}{4}$$

**Step 1 - Discretize and Select Element Types**

Keeping the concept of the element we have developed so far, let's consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.

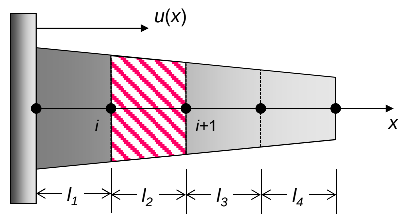



The sum of the masses should approximately satisfy the following relationship:

$$\sum M_i = \int_0^L \rho(x)A(x)dx$$

## Step 1 - Discretize and Select Element Types

Keeping the concept of the element we have developed so far, let's consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.

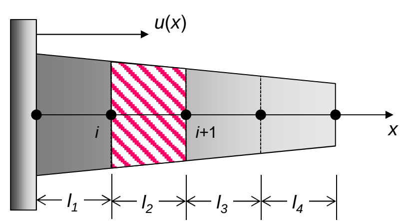



Identical to the lumping technique used for mass, we will take the average of the loading intensity:

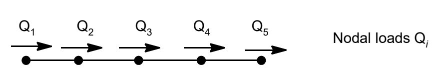
$$q^* = \frac{q(x_i) + q(x_{i+1})}{2} = \frac{q_i + q_{i+1}}{2}$$

## Step 1 - Discretize and Select Element Types

Keeping the concept of the element we have developed so far, let's consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.




Therefore the discrete lumped loading is:

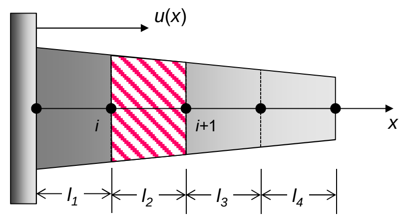



$$Q_1 = \frac{(q_1 + q_2)l_1}{4} \qquad Q_3 = \frac{(q_2 + q_3)l_2}{4} + \frac{(q_3 + q_4)l_3}{4} \qquad Q_5 = \frac{(q_4 + q_5)l_4}{4}$$

$$Q_2 = \frac{(q_1 + q_2)l_1}{4} + \frac{(q_2 + q_3)l_2}{4} \qquad Q_4 = \frac{(q_3 + q_4)l_3}{4} + \frac{(q_4 + q_5)l_4}{4}$$

**Step 1 - Discretize and Select Element Types**

Keeping the concept of the element we have developed so far, let's consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.

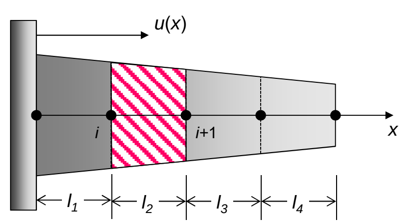



The sum of the nodal loads should approximately satisfy the following relationship:

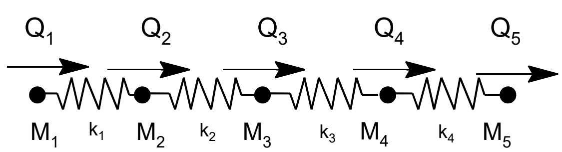
$$\sum Q_i = \int_0^L q(x)dx$$

**Step 1 - Discretize and Select Element Types**

Keeping the concept of the element we have developed so far, let's consider the mass of the portion of the bar between nodes *i* and *i*+1 defining element *i*.




The final discrete model for this system with springs, masses, and loads would be:



**Step 2 - Select a Displacement Function**

This completes the process of converting the continuous system into what is hoped to be a equivalent discrete system.

The **discretization** should be implicit in the representation of the mass, elastic properties, and loads.

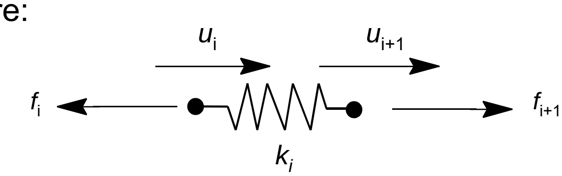
Whether the axial model is continuous or discrete, equilibrium of the system (Newton's second law) must be satisfied.

The remaining steps of **assembly**, **constraints**, **solution**, and **computation of derived variables** can be best illustrated in an example.

**Equilibrium of a Spring Mass System - Vectorial Approach**

Consider a typical spring-mass system, where each spring $k_i$ is assumed to behave in a linear way ( $F = kx$ ) and the loads $P_i$ are applied slowly to the system so that the problem is static.

The nodal displacements and the corresponding internal forces for an element are:




$$f_i = k_i \left( u_i - u_{i+1} \right) \qquad f_{i+1} = k_i \left( u_{i+1} - u_i \right)$$

$$\begin{pmatrix} f_i \\ f_{i+1} \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} u_i \\ u_{i+1} \end{pmatrix} \quad \text{or} \quad f_e = k_e u_e$$

**Equilibrium of a Spring Mass System - Vectorial Approach**

$$f_e = k_e u_e$$

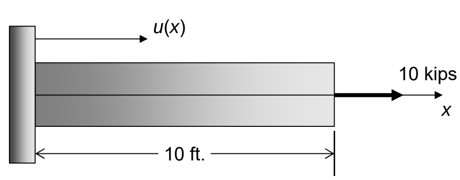
where $\boldsymbol{k_e}$ is called the **element stiffness matrix**, $\boldsymbol{f_e}$ is the **element force**, and $\boldsymbol{u_e}$ is the **element displacement vector**.

This equation is a statement of the spring relationship $F = kx$ on the *elemental level*.

The individual $\boldsymbol{k_e}$ can be **assembled** into the **global stiffness matrix** which represents the physical nature of the entire system.

**Introduction**

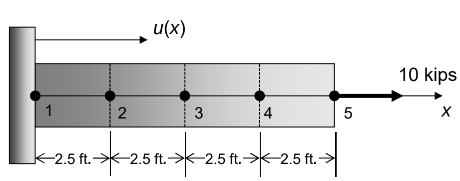
**Example** - Consider a uniform square bar under a distributed loading. Use five equally-spaced nodes to discretize the following problem. Solve for the displacement at each node.




$E = 29{,}000\,ksi$

$A = 1\text{ in.}^2$

The discretization of the bar is:




$$f_e = k_e u_e$$

$$\begin{pmatrix} f_i \\ f_{i+1} \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} u_i \\ u_{i+1} \end{pmatrix}$$

## Introduction

Since the area of the bar does not vary, the value of stiffness for each element is constant:

$$k = \frac{\left(A_i + A_{i+1}\right)E}{2l_i} = \frac{\left(1+1\right)\text{in.}^2\left(29{,}000\,ksi\right)}{2\left(2.5\,\text{ft.}\right)\left(12\right)^{\text{in}}/_{\text{ft.}}} = 966.667\,kips\,/\,in.$$

The equilibrium equations are:

$$\begin{pmatrix} f_i \\ f_{i+1} \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} u_i \\ u_{i+1} \end{pmatrix} \quad \text{or} \quad f_e = k_e u_e$$

Element 1: $$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Element 2: $$\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

Element 3: $$\begin{pmatrix} f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$$

Element 4: $$\begin{pmatrix} f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \end{pmatrix}$$

## Introduction

These equations can be written in matrix form as: $\boldsymbol{K_G u_G} = \boldsymbol{P_G}$

$$\boldsymbol{K_G} = \begin{bmatrix} k_1 & -k_1 & & & \\ -k_1 & \left(k_1 + k_2\right) & -k_2 & & \\ & -k_2 & \left(k_2 + k_3\right) & -k_3 & \\ & & -k_3 & \left(k_3 + k_4\right) & -k_4 \\ & & & -k_4 & k_4 \end{bmatrix}$$

$$\boldsymbol{u_G}^T = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{bmatrix}$$

$$\boldsymbol{P_G}^T = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & P_5 \end{bmatrix}$$

where $\boldsymbol{K_G}$ is called the **global stiffness matrix**, $\boldsymbol{u_G}$ is the **global displacement vector**, and $\boldsymbol{P_G}$ is the **global load vector**.

## Introduction

A careful inspection of the global equilibrium equations reveals that each elemental stiffness matrix, $\boldsymbol{k}_{ei,}$ is present in the global stiffness matrix.

Therefore the global stiffness matrix can be written as:

$$k_{G1} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad k_{G2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_{G3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad k_{G4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

## Introduction

These equations can be written in matrix form as:

$$k \begin{bmatrix} 1 & \overset{\text{Element 1}}{-1} & & & \\ -1 & 2 & \overset{\text{Element 2}}{-1} & & \\ & -1 & 2 & \overset{\text{Element 3}}{-1} & \\ & & -1 & 2 & \overset{\text{Element 4}}{-1} \\ & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix}$$

Applying the values for the geometry, material properties, and the boundary conditions given for this problem result in:

$$k \begin{bmatrix} 1 & 0 & & & \\ 0 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{Bmatrix} \longrightarrow u_1 = 0$$

## Introduction

The solution of these equations is:

$$u_1 = 0 \qquad u_2 = \frac{10}{k} \qquad u_3 = \frac{20}{k} \qquad u_4 = \frac{30}{k} \qquad u_5 = \frac{40}{k}$$

Substituting for *k* the numerical values for the displacement are:

$$u_1 = 0 \qquad u_2 = 0.0103 \text{ in.} \qquad u_3 = 0.0207 \text{ in.}$$
$$u_4 = 0.0310 \text{ in.} \qquad u_5 = 0.0414 \text{ in.}$$

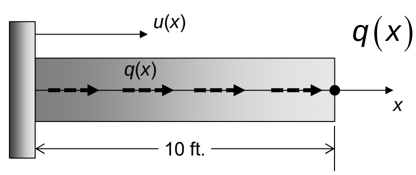
The exact solution may be determined from the following expression:

$$u(x) = \int \frac{P}{EA} dx = \frac{Px}{EA} = \frac{(10k)x}{(1 \text{ in.}^2)(29{,}000\, ksi)}$$

$$u_1 = 0 \qquad u_2 = 0.0103 \text{ in.} \qquad u_3 = 0.0207 \text{ in.}$$
$$u_4 = 0.0310 \text{ in.} \qquad u_5 = 0.0414 \text{ in.}$$

## Introduction

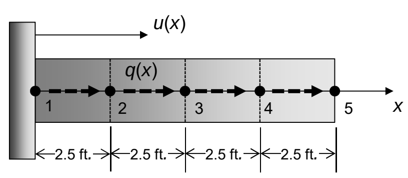
**Example** - Consider a uniform square bar under a distributed loading. Use five equally-spaced nodes to discretize the following problem. Solve for the displacement at each node.




$$q(x) = P_0\left(1 - \frac{x}{L}\right)$$

$$E = 29{,}000\, ksi$$
$$A = 1 in^2$$
$$P_0 = 10\, kips / in$$

The discretization of the bar is:




$$f_e = k_e u_e$$

$$\begin{Bmatrix} f_i \\ f_{i+1} \end{Bmatrix} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$

## Introduction

To handle the distributed load, we will lump the loads into each node. The average loading intensity is computed as:

$$q^* = \frac{q(x_i) + q(x_{i+1})}{2} = \frac{q_i + q_{i+1}}{2}$$

The sum of the nodal loads should approximately satisfy the following relationship:

$$\sum Q_i = \int_o^L q(x)dx = \frac{P_0 L}{2}$$

## Introduction

The individual values for the distributed lumped loads are:

$$Q_1 = \frac{(q_1 + q_2) l_1}{4}$$

$$\boxed{q(x) = P_0\left(1 - \frac{x}{L}\right)}$$

$$q_1(x = 0) = P_0$$

$$q_2(x = 2.5\,\text{ft.}) = P_0\left(1 - \frac{2.5}{10}\right) = 0.75P_0$$

$$Q_1 = \frac{(q_1 + q_2) l_1}{4} = \frac{P_0(1.75)}{4}\left(\frac{L}{4}\right) = \frac{7P_0 L}{64} = \frac{28P_0 L}{256}$$

## Introduction

The individual values for the distributed lumped loads are:

$$q(x) = P_0\left(1 - \frac{x}{L}\right)$$

$$Q_1 = \frac{(q_1 + q_2)l_1}{4} = \frac{28P_0L}{256}$$

$$Q_2 = \frac{(q_1 + q_2)l_1}{4} + \frac{(q_2 + q_3)l_2}{4} = \frac{48P_0L}{256}$$

$$Q_3 = \frac{(q_2 + q_3)l_2}{4} + \frac{(q_3 + q_4)l_3}{4} = \frac{32P_0L}{256}$$

$$Q_4 = \frac{(q_3 + q_4)l_3}{4} + \frac{(q_4 + q_5)l_4}{4} = \frac{16P_0L}{256}$$

$$Q_5 = \frac{(q_4 + q_5)l_4}{4} = \frac{4P_0L}{256}$$

## Introduction

Applying the values for the geometry, material properties, and loading distribution conditions results in:

$$k\begin{bmatrix} 1 & \overset{\text{Element 1}}{-1} & & & \\ -1 & 2 & \overset{\text{Element 2}}{-1} & & \\ & -1 & 2 & \overset{\text{Element 3}}{-1} & \\ & & -1 & 2 & \overset{\text{Element 4}}{-1} \\ & & & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \frac{P_0L}{256}\begin{Bmatrix} 28 \\ 48 \\ 32 \\ 16 \\ 4 \end{Bmatrix}$$

## Introduction

Applying the values for the geometry, material properties, loading distribution, and the boundary conditions results in:

$$k\begin{bmatrix} 1 & 0 & & & \\ 0 & 2 & -1 & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 1 & \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \frac{P_0 L}{256}\begin{Bmatrix} 0 \\ 48 \\ 32 \\ 16 \\ 4 \end{Bmatrix} \longrightarrow u_1 = 0$$

The solution of these equations is:

$$u_1 = 0 \qquad u_2 = 100\frac{P_0 L^2}{1{,}024AE} \qquad u_3 = 152\frac{P_0 L^2}{1{,}024AE}$$

$$u_4 = 172\frac{P_0 L^2}{1{,}024AE} \qquad u_5 = 176\frac{P_0 L^2}{1{,}024AE}$$

## Introduction

Substituting the numerical values for $P_0$, $L$, and $k$ results in :

$$u_1 = 0 \qquad u_2 = 0.4849\text{in.} \qquad u_3 = 0.7371\text{in.}$$

$$u_4 = 0.8341\text{in.} \qquad u_5 = 0.8534\text{in.}$$

The exact solution may be determined from the following expression:

$$u(x) = -\int \frac{1}{EA(x)} \int q(x)dx'dx \qquad \begin{cases} u(0) = 0 \\ AEu'(L) = P \end{cases}$$

$$AEu'(x) = -\int P_0\left(1 - \frac{x}{L}\right)dx' = P_0\left(\frac{x^2}{2L} - x\right) + C_1$$

$$AEu'(L) = 0 \Rightarrow C_1 = \frac{P_0 L}{2}$$

## Introduction

Substituting the numerical values for $P_0$, $L$, and $k$ results in :

$$u_1 = 0 \qquad u_2 = 0.4849\text{in.} \qquad u_3 = 0.7371\text{in.}$$
$$u_4 = 0.8341\text{in.} \qquad u_5 = 0.8534\text{in.}$$

The exact solution may be determined from the following expression:

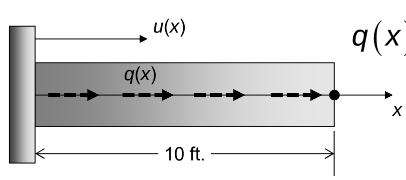
$$u(x) = \frac{P_0 L^2}{AE}\left[\frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{2}\left(\frac{x}{L}\right)\right]$$

$$u_1 = 0 \qquad u_2 = 0.4784\text{in.} \qquad u_3 = 0.7241\text{in.}$$
$$u_4 = 0.8147\text{in.} \qquad u_5 = 0.8276\text{in.}$$

Why the difference?

## Introduction

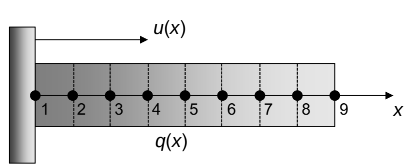
**Example** - Repeat the previous problem using nine equally-spaced nodes (8 elements) to discretize the problem. Solve for the displacement at each node.




$$q(x) = P_0\left(1 - \frac{x}{L}\right)$$

$$E = 29{,}000\,ksi$$
$$A = 1in^2$$
$$P_0 = 10\,kips\,/\,in$$

The discretization of the bar is:




$$f_e = k_e u_e$$

$$\begin{pmatrix} f_i \\ f_{i+1} \end{pmatrix} = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix}\begin{pmatrix} u_i \\ u_{i+1} \end{pmatrix}$$

## Introduction

To handle the distributed load, we will lump the loads into each node. The average loading intensity is computed as:

$$q^* = \frac{q(x_i) + q(x_{i+1})}{2} = \frac{q_i + q_{i+1}}{2}$$

The sum of the nodal loads should approximately satisfy the following relationship:

$$\sum Q_i = \int_o^L q(x)dx = \frac{P_0 L}{2}$$

## Introduction

The individual values for the distributed lumped loads are:

$$Q_1 = \frac{(q_1 + q_2)l_1}{4} = \frac{14P_0L}{256}$$

$$Q_2 = \frac{(q_1 + q_2)l_1}{4} + \frac{(q_2 + q_3)l_2}{4} = \frac{28P_0L}{256}$$

$$Q_3 = 24\frac{P_0L}{256} \qquad Q_4 = 20\frac{P_0L}{256} \qquad Q_5 = 16\frac{P_0L}{256}$$

$$Q_6 = 12\frac{P_0L}{256} \qquad Q_7 = 8\frac{P_0L}{256} \qquad Q_8 = 4\frac{P_0L}{256} \qquad Q_9 = \frac{P_0L}{256}$$

## Introduction

Applying the values for the geometry, material properties, and loading given in this problem results in:

$$
k\begin{bmatrix}
1 & \overset{\text{Element 1}}{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & \overset{\text{Element 2}}{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & \overset{\text{Element 3}}{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & \overset{\text{Element 4}}{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & \overset{\text{Element 5}}{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & \overset{\text{Element 6}}{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & \overset{\text{Element 7}}{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \underset{\text{Element 8}}{1}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}
= \frac{P_0 L}{256}
\begin{Bmatrix} 14 \\ 28 \\ 24 \\ 20 \\ 16 \\ 12 \\ 8 \\ 4 \\ 1 \end{Bmatrix}
$$

## Introduction

Applying the boundary condition results in:

$$u_1 = 0$$

$$
k\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}
= \frac{P_0 L}{256}
\begin{Bmatrix} 0 \\ 28 \\ 24 \\ 20 \\ 16 \\ 12 \\ 8 \\ 4 \\ 1 \end{Bmatrix}
$$

## Introduction

The solution of these equations is:

$$u_1 = 0 \qquad u_3 = 198\frac{P_0L^2}{2{,}048AE} \qquad u_5 = 300\frac{P_0L^2}{2{,}048AE}$$

$$u_7 = 338\frac{P_0L^2}{2{,}048AE} \qquad u_9 = 344\frac{P_0L^2}{2{,}048AE}$$

Substituting the numerical values for $P_0$, $L$, and $k$ results in :

$$u_1 = 0 \qquad u_3 = 0.4801\text{in.} \qquad u_5 = 0.7274\text{in.}$$

$$u_7 = 0.8195\text{in.} \qquad u_9 = 0.8341\text{in.}$$

The exact solution may be determined from the following expression:

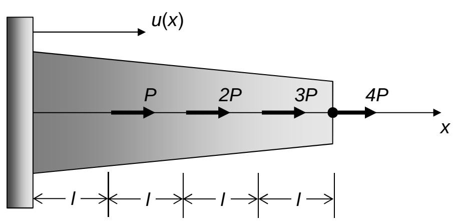
$$u_1 = 0 \qquad u_2 = 0.4784\text{in.} \qquad u_3 = 0.7241\text{in.}$$

$$u_4 = 0.8147\text{in.} \qquad u_5 = 0.8276\text{in.}$$

How does this compare to the 4-element model?

## Introduction

**Problem #1** - Consider a square bar subjected to a series of concentrated loads. Use five equally-spaced nodes to discretize the following problem. Solve for the displacement at each node and compare to the exact solution.




$E = 29{,}000\,ksi$

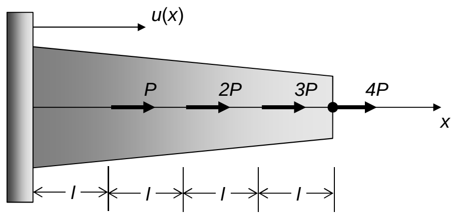
$l = 25\,in$

$P = 5\,kips$

$A(x) = 5\left(2 - \frac{x}{L}\right)in^2$

## Introduction

**Problem #2** - Repeat Problem #1 using twice the number of elements. Compare your results with those obtained in Problem #1 and the exact solution. Explain any differences in the solutions.



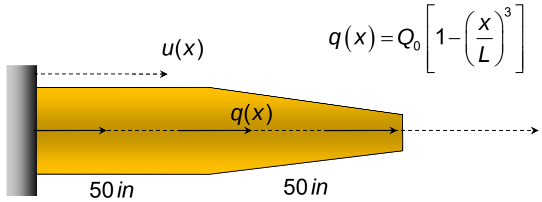

$$E = 29{,}000\,ksi$$
$$l = 25\,in$$
$$P = 5\,kips$$
$$A(x) = 5\left(2 - \frac{x}{L}\right) in^2$$

## Introduction

**Problem #3** - Consider a uniform square bar under a distributed loading. Use five equally-spaced nodes to discretize the following problem. Solve for the displacement at each node.

$$q(x) = Q_0\left[1 - \left(\frac{x}{L}\right)^3\right]$$

$$E = 29{,}000\,ksi$$
$$A = 1\,in^2$$
$$Q_0 = 5\,kips\,/\,in$$



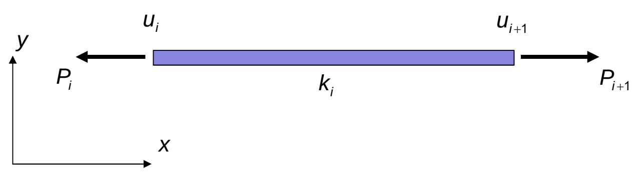

$$A(x) = \begin{cases} 10\,in^2 & \text{if } x \le 50\,in \\ \left(10 - \frac{x}{L}\right) in^2 & \text{if } x > 50\,in \end{cases}$$

**PLANE TRUSS STRUCTURES**

From your experience in structural analysis you are aware of structural elements or members called “two force members”.

These elements are pin connected and transmit only an axial force. There is no shear, bending, or torsional loads transmitted by these members in a structure.

A structure composed of two-force members which behaves elastically may be replaced by a system of connected “springs”. Consider a single two-force member:

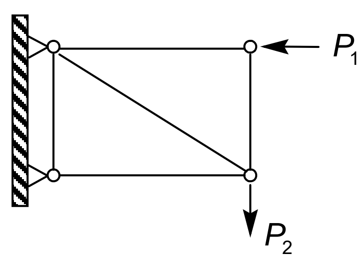



$$P_i = k_i\left(u_i - u_{i+1}\right) \qquad P_{i+1} = k_i\left(u_{i+1} - u_i\right)$$

**PLANE TRUSS STRUCTURES**

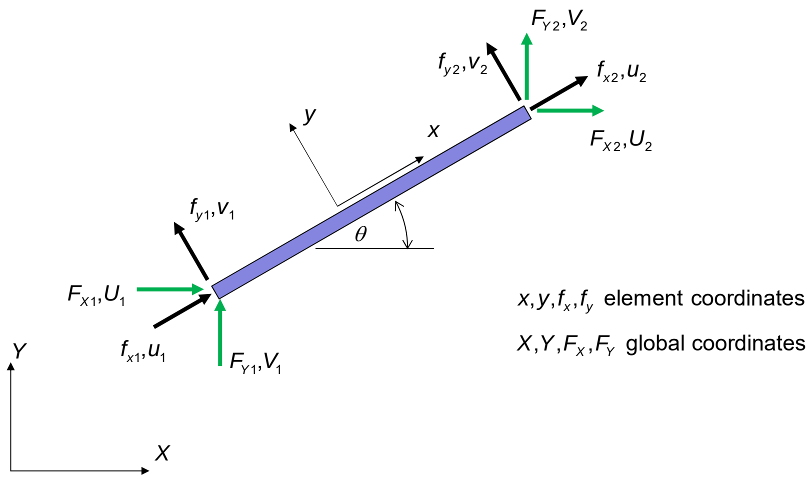
The spring stiffness constant $k_i$ is $(AE/L\,)_i$, where $A$ is an area, $E$ is the modulus of elasticity, and $L$ is the length of the member.

Consider a plane truss with four bars or members or elements:



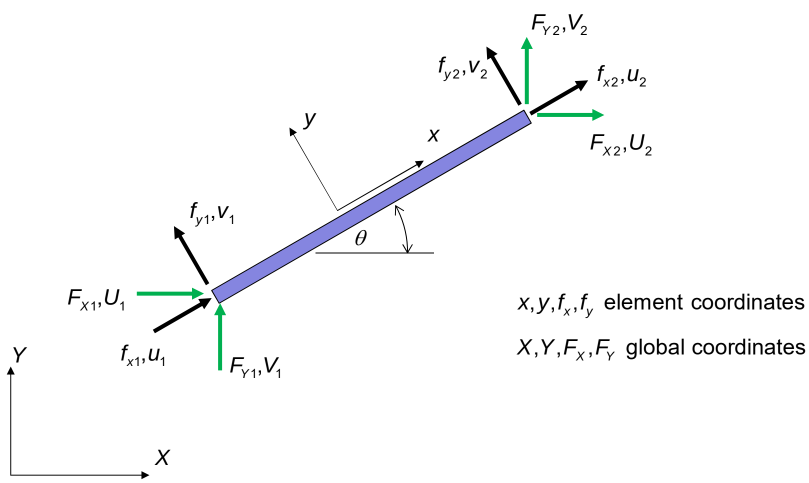

Although each member in the truss will elongate (or contract) and transmit a tensile (or compressive) load, the displacements and the forces are in different directions.

**PLANE TRUSS STRUCTURES**




Although each member in the truss will elongate (or contract) and transmit a tensile (or compressive) load, the displacements and the forces are in different directions.

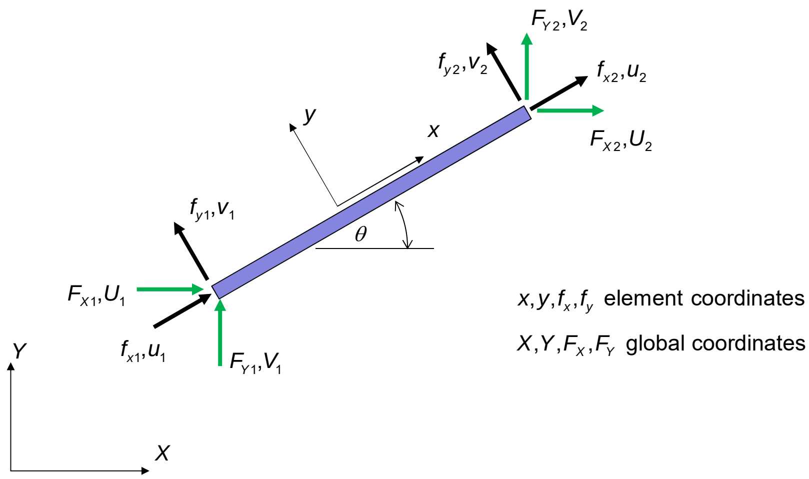
**PLANE TRUSS STRUCTURES**




The global force components may be related to the elemental force components by:

$$F_{X1} = f_{x1}cos\theta - f_{y1}sin\theta \qquad F_{Y1} = f_{x1}sin\theta + f_{y1}cos\theta$$

**PLANE TRUSS STRUCTURES**




The displacements may be related in a similar fashion:

$$U_1 = u_1 cos\theta - v_1 sin\theta \qquad V_1 = u_1 sin\theta + v_1 cos\theta$$

**PLANE TRUSS STRUCTURES**

In matrix form these quantities can be expressed as: $\boldsymbol{F_1} = \boldsymbol{Rf_1}$ $\quad \boldsymbol{U_1} = \boldsymbol{Ru_1}$

$$\boldsymbol{F}_1 = \begin{Bmatrix} F_{X1} \\ F_{Y1} \end{Bmatrix} \qquad \boldsymbol{f}_1 = \begin{Bmatrix} f_{x1} \\ f_{y1} \end{Bmatrix} \qquad \boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\boldsymbol{U}_1 = \begin{Bmatrix} U_1 \\ V_1 \end{Bmatrix} \qquad \boldsymbol{u}_1 = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

The global force and global displacement vectors and $\boldsymbol{R}$ is a transformation matrix for rotation of an axis ( $\boldsymbol{R}^{-1} = \boldsymbol{R}^{T}$ ).

A set of similar quantities can be written for the other end of the element

$$\boldsymbol{f_1} = \boldsymbol{R}^{-1}\boldsymbol{F_1} \qquad \boldsymbol{u_1} = \boldsymbol{R}^{-1}\boldsymbol{U_1}$$

**PLANE TRUSS STRUCTURES**

The stiffness matrix for the axial element in the elemental or local coordinates is:

$$\begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Rewriting the elemental forces-displacement relationship for both *x* and *y* components:

$$\begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Notice the second and fourth equations reflect the fact that only axial loads, in the *x*-direction locally, are possible in the absence of bending, shear, or torsion.

**PLANE TRUSS STRUCTURES**

These equations may be written in partitioned form as:

$$\boldsymbol{f_1} = \boldsymbol{k_{11}u_1} + \boldsymbol{k_{12}u_2} \qquad \begin{Bmatrix} \boldsymbol{f_1} \\ \hline \boldsymbol{f_2} \end{Bmatrix} = \left[\begin{array}{c|c} \boldsymbol{k_{11}} & \boldsymbol{k_{12}} \\ \hline \boldsymbol{k_{21}} & \boldsymbol{k_{22}} \end{array}\right] \begin{Bmatrix} \boldsymbol{u_1} \\ \hline \boldsymbol{u_2} \end{Bmatrix}$$

$$\boldsymbol{f_2} = \boldsymbol{k_{21}u_1} + \boldsymbol{k_{22}u_2}$$

To convert these relationships to global coordinates (*X*, *Y*) we apply the coordinate transformation **R**.

$$\boldsymbol{f_1} = \boldsymbol{R^{-1}F_1} \qquad \boldsymbol{u_1} = \boldsymbol{R^{-1}U_1}$$

$$\boldsymbol{f_1} = \boldsymbol{k_{11}u_1} + \boldsymbol{k_{12}u_2} \Rightarrow \boldsymbol{R^{-1}F_1} = \boldsymbol{k_{11}R^{-1}U_1} + \boldsymbol{k_{12}R^{-1}U_2}$$

$$\boldsymbol{f_2} = \boldsymbol{k_{21}u_1} + \boldsymbol{k_{22}u_2} \Rightarrow \boldsymbol{R^{-1}F_2} = \boldsymbol{k_{21}R^{-1}U_1} + \boldsymbol{k_{22}R^{-1}U_2}$$

Multiply both side by **R**:

$$\boldsymbol{F_1} = \boldsymbol{Rk_{11}R^{-1}U_1} + \boldsymbol{Rk_{12}R^{-1}U_2}$$

$$\boldsymbol{F_2} = \boldsymbol{Rk_{21}R^{-1}U_1} + \boldsymbol{Rk_{22}R^{-1}U_2}$$

**PLANE TRUSS STRUCTURES**

Since $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$

$$\begin{Bmatrix} \boldsymbol{F_1} \\ \hline \boldsymbol{F_2} \end{Bmatrix} = \left[\begin{array}{c|c} \boldsymbol{Rk_{11}R^T} & \boldsymbol{Rk_{12}R^T} \\ \hline \boldsymbol{Rk_{21}R^T} & \boldsymbol{Rk_{22}R^T} \end{array}\right] \begin{Bmatrix} \boldsymbol{U_1} \\ \hline \boldsymbol{U_2} \end{Bmatrix} \qquad \boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In a more convenient form:

$$\begin{Bmatrix} \boldsymbol{F_1} \\ \hline \boldsymbol{F_2} \end{Bmatrix} = \left[\begin{array}{c|c} \boldsymbol{R} & 0 \\ \hline 0 & \boldsymbol{R} \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{k_{11}} & \boldsymbol{k_{12}} \\ \hline \boldsymbol{k_{21}} & \boldsymbol{k_{22}} \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{R^T} & 0 \\ \hline 0 & \boldsymbol{R^T} \end{array}\right] \begin{Bmatrix} \boldsymbol{U_1} \\ \hline \boldsymbol{U_2} \end{Bmatrix}$$

Writing these equations in still a more compact form gives

$$\boldsymbol{F} = \boldsymbol{TkT^TU} = \boldsymbol{KU} \qquad \boldsymbol{T} = \left[\begin{array}{c|c} \boldsymbol{R} & 0 \\ \hline 0 & \boldsymbol{R} \end{array}\right]$$

where $\boldsymbol{K}$ is the global stiffness matrix for a single two-force member or element.

**PLANE TRUSS STRUCTURES**

Substituting the values of $\boldsymbol{R}$ and $\boldsymbol{k}$ and performing the multiplication gives:

$$\boldsymbol{K} = k\left[\begin{array}{cc|cc} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ \hline -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{array}\right] \qquad \begin{array}{l} \lambda = \cos\theta \\ \mu = \sin\theta \end{array}$$

In this case, $\boldsymbol{K}$ is the global stiffness matrix for a single truss element.

In a structure composed of two-force elements, say a truss, we would have to assembly the element global matrices into a global matrix for the entire system.

Before we discuss any problems or work any examples, let’s look at the effect of discretization on the form of the system stiffness matrix.

**PLANE TRUSS STRUCTURES**

**PROBLEM #4** – Show how to develop the global stiffness matrix for 2-D bar elements.

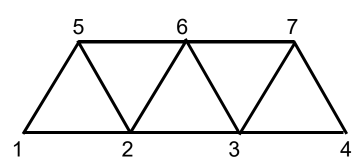
$$\boldsymbol{F} = \boldsymbol{TkT}^T\boldsymbol{U} = \boldsymbol{KU} \qquad \boldsymbol{T} = \left[\begin{array}{c|c} \boldsymbol{R} & 0 \\ \hline 0 & \boldsymbol{R} \end{array}\right]$$

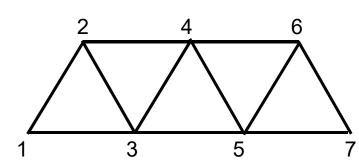
$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\boldsymbol{K} = k\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \qquad \begin{aligned} \lambda &= \cos\theta \\ \mu &= \sin\theta \end{aligned}$$

**PLANE TRUSS STRUCTURES**

Consider the following two ways to number the nodes of the same truss:



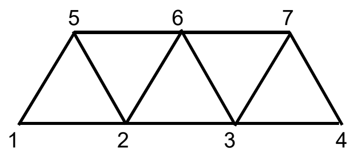



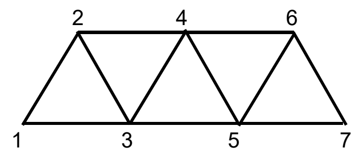

$$\begin{bmatrix} \otimes & \otimes & & & \otimes & & \\ \otimes & \otimes & \otimes & & \otimes & \otimes & \\ & \otimes & \otimes & \otimes & & \otimes & \otimes \\ & & \otimes & \otimes & & & \otimes \\ \otimes & \otimes & & & \otimes & \otimes & \\ & \otimes & \otimes & & \otimes & \otimes & \otimes \\ & & \otimes & \otimes & & \otimes & \otimes \end{bmatrix} \qquad \begin{bmatrix} \otimes & \otimes & \otimes & & & & \\ \otimes & \otimes & \otimes & \otimes & & & \\ \otimes & \otimes & \otimes & \otimes & \otimes & & \\ & \otimes & \otimes & \otimes & \otimes & \otimes & \\ & & \otimes & \otimes & \otimes & \otimes & \otimes \\ & & & \otimes & \otimes & \otimes & \otimes \\ & & & & \otimes & \otimes & \otimes \end{bmatrix}$$

$$\text{Number Scheme \#1} \qquad\qquad \text{Number Scheme \#2}$$

**PLANE TRUSS STRUCTURES**

Consider the following two ways to number the nodes of the same truss:




From these idealizations, it is clear that the second numbering scheme produces a global matrix that has a smaller band width.

Generally, this type of symmetry results in quicker solutions and a reduction in the required memory or storage capacity.

The half-band width of a symmetric set of equations for row *i* and column *j* of the last non-zero entry may be computed as: $(nb)_i = 1 + (j - i)$

where *NB* (half the band width) is the maximum of the $(nb)_i$ over all rows.

**PLANE TRUSS STRUCTURES**

**SOLUTION PROCEDURE**

1. Define a discretization of the truss
   (recall the node numbering scheme we discussed above)

2. Assemble the elemental stiffness and load matrices.
   Each element matrix should be transformed into the global system as previously described.

3. Apply boundary conditions or constraints to the system equations

4. Solve the system equations

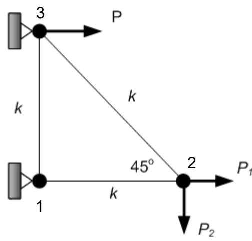
5. Compute the forces in the members. Recall the force displacement relationship $\boldsymbol{f} = \boldsymbol{ku} = \boldsymbol{kT^T U}$

$$f_{x1} = k\left[(U_1 - U_2)cos\theta + (V_1 - V_2)sin\theta\right] \qquad f_{y1} = 0$$

$$f_{x2} = -k\left[(U_1 - U_2)cos\theta + (V_1 - V_2)sin\theta\right] \qquad f_{y2} = 0$$

**PLANE TRUSS STRUCTURES**

**Example** - Develop the element stiffness matrices and system equations for the plane truss below. Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces.




All elements have a constant *k*

**STEP 1.** The node numbering is given in the diagram above (Note that this is the optimum numbering configuration).

**PLANE TRUSS STRUCTURES**

**STEP 2.** Develop the element information

| Member | Node 1 | Node 2 | Elemental Stiffness | $\theta$ |
|---|---|---|---|---|
| 1 | 1 | 2 | $k$ | 0 |
| 2 | 2 | 3 | $k$ | $3\pi/4$ |
| 3 | 1 | 3 | $k$ | $\pi/2$ |

Compute the elemental stiffness matrix for each element. The general form of the matrix is:

$$\boldsymbol{K} = k\left[\begin{array}{cc|cc} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ \hline -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{array}\right] \qquad \begin{array}{l} \lambda = \cos\theta \\ \mu = \sin\theta \end{array}$$

**PLANE TRUSS STRUCTURES**

For element 1:

$$K = k\begin{array}{c} \begin{array}{cccc} U_1 & V_1 & U_2 & V_2 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}$$

$$\boldsymbol{K} = k\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \qquad \begin{array}{l} \lambda = \cos\theta \\ \mu = \sin\theta \end{array}$$

For element 2:

$$K = \frac{k}{2}\begin{array}{c} \begin{array}{cccc} U_2 & V_2 & U_3 & V_3 \end{array} \\ \left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ \hline -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array}$$

For element 3:

$$K = k\begin{array}{c} \begin{array}{cccc} U_1 & V_1 & U_3 & V_3 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ U_1 \\ V_1 \\ U_3 \\ V_3 \end{array}$$

**PLANE TRUSS STRUCTURES**

Assemble the global system matrix by superimposing the elemental global matrices.

Element 1

$$\boldsymbol{K} = \frac{k}{2}\begin{array}{c} \begin{array}{cccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \end{array} \\ \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 3 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -2 & 1 & -1 & -1 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{array}$$

Element 3 Element 2

**PLANE TRUSS STRUCTURES**

The unconstrained (no boundary conditions satisfied) equations are:

$$\frac{k}{2}\left[\begin{array}{cc|cc|cc} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ \hline -2 & 0 & 3 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ \hline 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -2 & 1 & -1 & -1 & 3 \end{array}\right]\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P_1 \\ -P_2 \\ P \\ 0 \end{Bmatrix}$$

**PLANE TRUSS STRUCTURES**

**STEP 3.** The displacement at nodes 1 and 3 are zero in both directions. Applying these conditions to the system equations gives:

$$\frac{k}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P_1 \\ -P_2 \\ 0 \\ 0 \end{Bmatrix}$$

## PLANE TRUSS STRUCTURES

**STEP 4.** Solving this set of equations is fairly easy. The solution is:

$$U_1 = 0 \qquad V_1 = 0 \qquad U_2 = \frac{P_1 - P_2}{k} \qquad V_2 = \frac{P_1 - 3P_2}{k}$$

$$U_3 = 0 \qquad V_3 = 0$$

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

| Member | Node 1 | Node 2 | Elemental Stiffness | $\theta$ |
|---|---|---|---|---|
| 1 | 1 | 2 | $k$ | 0 |
| 2 | 2 | 3 | $k$ | $3\pi/4$ |
| 3 | 1 | 3 | $k$ | $\pi/2$ |

$$f_{x1} = k\left[\left(U_1 - U_2\right)cos\theta + \left(V_1 - V_2\right)sin\theta\right] \qquad f_{y1} = 0$$

$$f_{x2} = -k\left[\left(U_1 - U_2\right)cos\theta + \left(V_1 - V_2\right)sin\theta\right] \qquad f_{y2} = 0$$

## PLANE TRUSS STRUCTURES

**STEP 4.** Solving this set of equations is fairly easy. The solution is:

$$U_1 = 0 \qquad V_1 = 0 \qquad U_2 = \frac{P_1 - P_2}{k} \qquad V_2 = \frac{P_1 - 3P_2}{k}$$

$$U_3 = 0 \qquad V_3 = 0$$

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

*Member (element) 1*

$$f_{x1} = k\left(-\frac{P_1 - P_2}{k}\right) = P_2 - P_1 \qquad f_{y1} = 0$$

$$f_{x2} = k\left(-\frac{P_1 - P_2}{k}\right) = P_1 - P_2 \qquad f_{y2} = 0$$

**PLANE TRUSS STRUCTURES**

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

*Member (element) 2*

$$f_{x2} = k\left[\left(-\frac{P_1 - P_2}{k}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{P_1 - 3P_2}{k}\right)\left(\frac{1}{\sqrt{2}}\right)\right] = -\sqrt{2}P_2$$

$$f_{y2} = 0$$

$$f_{x3} = -k\left[\left(-\frac{P_1 - P_2}{k}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{P_1 - 3P_2}{k}\right)\left(\frac{1}{\sqrt{2}}\right)\right] = \sqrt{2}P_2$$

$$f_{y3} = 0$$

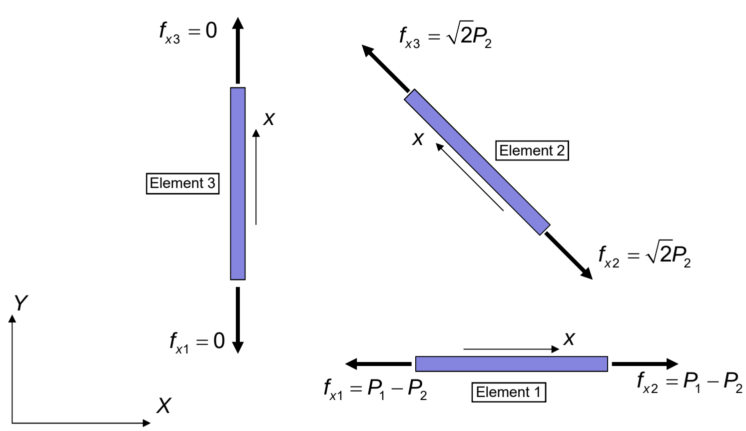
**PLANE TRUSS STRUCTURES**

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

*Member (element) 3*

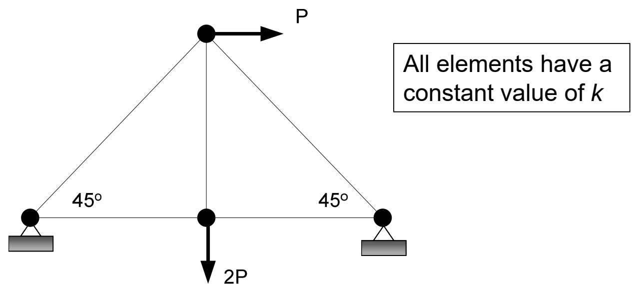
$$f_{x3} = 0 \qquad f_{y3} = 0$$

$$f_{x1} = 0 \qquad f_{y1} = 0$$



**PLANE TRUSS STRUCTURES**

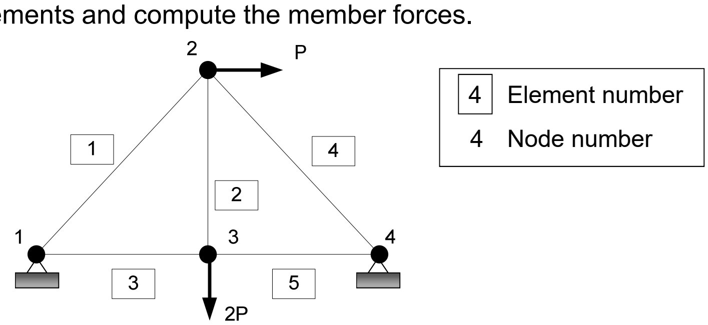
**Example** - Develop the element stiffness matrices and system equations for the plane truss below. Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces.




**STEP 1.** A node numbering configuration is given (note that this is the optimum numbering configuration).

**PLANE TRUSS STRUCTURES**

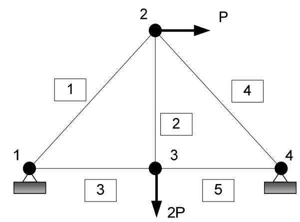
**Example** - Develop the element stiffness matrices and system equations for the plane truss below. Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces.




**STEP 1.** A node numbering configuration is given (note that this is the optimum numbering configuration).

**PLANE TRUSS STRUCTURES**

**STEP 2.** Develop the element information




| Element | Node 1 | Node 2 | Elemental Stiffness | $\theta$ |
|---|---|---|---|---|
| 1 | 1 | 2 | $k$ | $\pi/4$ |
| 2 | 2 | 3 | $k$ | $3\pi/2$ |
| 3 | 1 | 3 | $k$ | 0 |
| 4 | 2 | 4 | $k$ | $7\pi/4$ |
| 5 | 3 | 4 | $k$ | 0 |

**PLANE TRUSS STRUCTURES**

**STEP 2.** Compute the elemental stiffness matrix for each element. The general form of the matrix is:

$$\boldsymbol{K} = k\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \qquad \begin{array}{l} \lambda = \cos\theta \\ \mu = \sin\theta \end{array}$$

| Element | Node 1 | Node 2 | Elemental Stiffness | $\theta$ |
|---|---|---|---|---|
| 1 | 1 | 2 | $k$ | $\pi/4$ |
| 2 | 2 | 3 | $k$ | $3\pi/2$ |
| 3 | 1 | 3 | $k$ | 0 |
| 4 | 2 | 4 | $k$ | $7\pi/4$ |
| 5 | 3 | 4 | $k$ | 0 |

**PLANE TRUSS STRUCTURES**

For elements 1 and 2:

$$K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_1 & V_1 & U_2 & V_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array} \begin{matrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{matrix} \qquad K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_2 & V_2 & U_3 & V_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \end{array} \begin{matrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{matrix}$$

For elements 3 and 4:

$$K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_1 & V_1 & U_3 & V_3 \end{matrix} \\ \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} U_1 \\ V_1 \\ U_3 \\ V_3 \end{matrix} \qquad K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_2 & V_2 & U_4 & V_4 \end{matrix} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{matrix} U_2 \\ V_2 \\ U_4 \\ V_4 \end{matrix}$$

**PLANE TRUSS STRUCTURES**

For element 5:

$$K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_3 & V_3 & U_4 & V_4 \end{matrix} \\ \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} U_3 \\ V_3 \\ U_4 \\ V_4 \end{matrix}$$

The global matrix for element 1 and 2 are:

$$K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 & U_4 & V_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{matrix} \qquad K=\frac{k}{2}\begin{array}{c} \begin{matrix} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 & U_4 & V_4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{matrix}$$

**PLANE TRUSS STRUCTURES**

The global matrix for element 3 and 4 are:

$$
\mathbf{K} = \frac{k}{2}\begin{array}{c} \begin{array}{cccccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 & U_4 & V_4 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{array}
\qquad
\mathbf{K} = \frac{k}{2}\begin{array}{c} \begin{array}{cccccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 & U_4 & V_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{array}
$$

**PLANE TRUSS STRUCTURES**

The global matrix for element 5 is:

$$
\mathbf{K} = \frac{k}{2}\begin{array}{c} \begin{array}{cccccccc} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 & U_4 & V_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{array}
$$

**PLANE TRUSS STRUCTURES**

The unconstrained (no boundary conditions satisfied) equations are:

$$\frac{k}{2}\begin{bmatrix} 3 & 1 & -1 & -1 & -2 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 4 & 0 & -2 & 1 & -1 \\ -2 & 0 & 0 & 0 & 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \\ 0 \\ -2P \\ 0 \\ 0 \end{Bmatrix}$$

**PLANE TRUSS STRUCTURES**

**STEP 3.** Apply the boundary conditions to the system equations:

$$\frac{k}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \\ 0 \\ -2P \\ 0 \\ 0 \end{Bmatrix}$$

**PLANE TRUSS STRUCTURES**

**STEP 4.** Solving this set of equations is fairly easy. The solution is:

$$U_1 = 0 \qquad V_1 = 0 \qquad U_2 = \frac{P}{k} \qquad V_2 = -\frac{2P}{k}$$

$$U_3 = 0 \qquad V_3 = -\frac{4P}{k} \qquad U_4 = 0 \qquad V_4 = 0$$

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

*Member (element) 1*

$$f_{x1} = -P\left(\frac{1}{\sqrt{2}}\right) + 2P\left(\frac{1}{\sqrt{2}}\right) = \frac{P}{\sqrt{2}}$$

$$f_{x2} = P\left(\frac{1}{\sqrt{2}}\right) - 2P\left(\frac{1}{\sqrt{2}}\right) = -\frac{P}{\sqrt{2}}$$

**PLANE TRUSS STRUCTURES**

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

*Member (element) 2*

$$f_{x2} = -P(-2+4) = -2P \qquad f_{x3} = P(-2+4) = 2P$$

*Member (element) 3*

$$f_{x1} = 0 \qquad f_{x3} = 0$$

*Member (element) 4*

$$f_{x2} = P\left(\frac{1}{\sqrt{2}}\right) + 2P\left(\frac{1}{\sqrt{2}}\right) = \frac{3P}{\sqrt{2}}$$

$$f_{x4} = -P\left(\frac{1}{\sqrt{2}}\right) - 2P\left(\frac{1}{\sqrt{2}}\right) = \frac{-3P}{\sqrt{2}}$$

**PLANE TRUSS STRUCTURES**

**STEP 5.** Using the force-displacement relationship the force in each member may be computed.

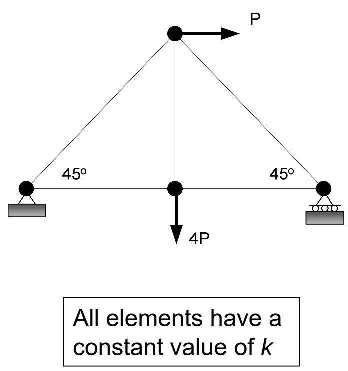
*Member (element) 5*

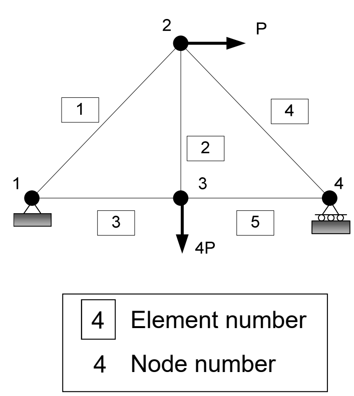
$$f_{x3} = 0 \qquad\qquad f_{x4} = 0$$

| Element | Node 1 | Node 2 | $U_{node\ 1}$ | $U_{node2}$ | $V_{node\ 1}$ | $V_{node\ 2}$ | $f_x$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 0 | *P/k* | 0 | -2*P/k* | 0.707*P* (C) |
| 2 | 2 | 3 | *P/k* | 0 | -2*P/k* | -4*P/k* | 2*P* (T) |
| 3 | 1 | 3 | 0 | 0 | 0 | -4*P/k* | 0 |
| 4 | 2 | 4 | *P/k* | 0 | -2*P/k* | 0 | *2.12P* (C) |
| 5 | 3 | 4 | 0 | 0 | -4*P/k* | 0 | 0 |

**PLANE TRUSS STRUCTURES**

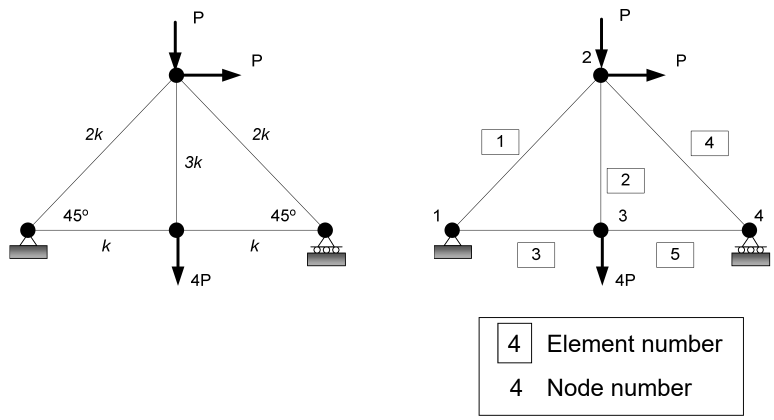
**PROBLEM #5** - Develop the element stiffness matrices and system equations for the plane truss below. Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces.



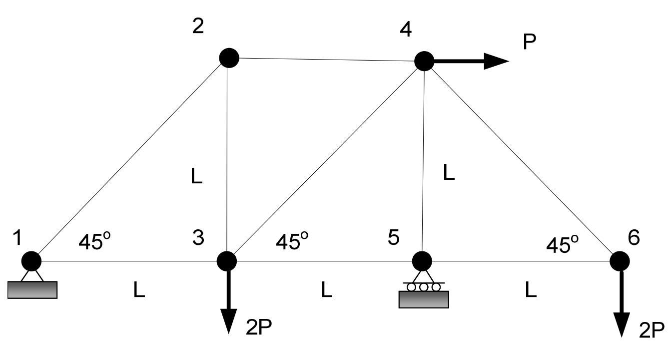
**PLANE TRUSS STRUCTURES**

**PROBLEM #6** - Develop the element stiffness matrices and system equations for the plane truss below. Assume the stiffness of each element is constant. Use the numbering scheme indicated. Solve the equations for the displacements and compute the member forces.




**PLANE TRUSS STRUCTURES**

**PROBLEM #7** - Consider the following two-dimensional plane truss. For the given node numbering scheme, determine the displacements of each node and the member forces. Check your results by using the method of sections and the method of joints from static analysis. For computational purposes assume a $P$ = 10 *kips*, $E$ = 29,000 *ksi*, $L$ = 10 ft., and $A$ = 4 in.$^2$.



# End of Introduction