Discrete Distributions

Probability Distribution-

Types of Probability Distributions:

a.) Discrete –

b.) Continuous -

Let’s take a look at what a simple probability distribution looks like:

<table>
<thead>
<tr>
<th>X</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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</thead>
</table>

Figure 1. Example of a Discrete Distribution.

5 Types of Discrete Distributions:

Binomial Distribution – Probability of exactly \( x \) successes in \( n \) trials.

Negative Binomial – Probability that it will take exactly \( n \) trials to produce exactly \( x \) successes. (Special case of the negative binomial).
Hypergeometric Distribution – Probability of exactly $x$ successes in a sample of size $n$ drawn without replacement.

Poisson Distribution – Probability of exactly $x$ successes in a “unit” or continuous interval.

Probability Mass Function- For a discrete random variable $X$, with possible values $x_1, x_2, \ldots, x_n$, a probability mass function is a function such that:

1. $f(x_i) \geq 0$
2. $\sum_{i=1}^{n} f(x_i) = 1$
3. $f(x_i) = P(X = x_i)$

Cumulative Distribution Function:

For a discrete random variable $X$, $F(x)$ satisfies the following properties:

1. $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$
2. $0 \leq F(x) \leq 1$
3. If $x \leq y$, then $F(x) \leq F(y)$

Example:
Suppose that a day’s production of 850 manufactured parts contain 50 that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable $x$ equal the number of nonconforming parts in the sample. What is the cumulative distribution function of $X$?
Mean and Variance of Discrete Random Variables:

Mean or expected value:
\[ \mu = E(x) = \sum_x x f(x) \]

Variance of X:
\[ \sigma^2 = V(x) = E(X - \mu)^2 \]

Standard Deviation:
\[ \sigma = \sqrt{\sigma^2} \]

Binomial Distribution:

Bernoulli trial- a trial with only two possible outcomes.

\[ P(X = x) = \binom{n}{x} p^x q^{n-x} \]

Where \( P(X = x) \) = probability of exactly \( x \) success in \( n \) trials
- \( n \) = number of trials
- \( x \) = number of successes
- \( p \) = probability of success for any given trial
- \( q \) = probability of failure for any given trial

\[ \binom{n}{x} = \frac{n!}{x!(n-x)!} \quad (Binomial \ Coefficient) \]

Binomial Coefficient – possible number of arrangements of \( n \) things taken \( x \) at a time.

Expectation- \( E(x) = np \)

Variance: \( \sigma^2 = npq \)
Example:
In how many ways can a committee of 4 people be selected from 10 people?

Example:
The drainage system of a city has been designed for a rainfall intensity that will be exceeded on an average once in 50 years. What is the probability that the city will be flooded at most 2 out of 10 years?

Example: The probability that a certain kind of component will survive a given shock test is $\frac{3}{4}$. Find the probability that exactly two of the next four components tested survive.
Negative Binomial:

\[ P(n) = \binom{n-1}{x-1} p^x q^{n-x} \]

Where \( P(X = x) = \) probability of exactly x success in n trials
- \( n = \) number of trials
- \( x = \) number of successes
- \( p = \) probability of success for any given trial
- \( q = \) probability of failure for any given trial

Expectation: \( E(x) = \frac{x}{p} \)

Variance: \( \sigma^2 = \frac{xq}{p^2} \)

Example:
A high-performance aircraft contains three identical computers. Only one is used to operate the aircraft; the other two are spares that can be activated in case the primary system fails. During one hour of operation, the probability of a failure in the primary computer is 0.0005. Assuming that each hour represents an independent trial,

a.) What is the mean time to failure of all three computers?
b.) What is the probability that all three computers fail within a 5-hour flight?
Example:
Cotton linters used in the production of rocket propellant are subjected to a nitration process that enables the cotton fibers to go into solution. The process is 90% effective in that the material produced can be shaped as desired in a later processing stage with probability 0.9. What is the probability that exactly 20 lots will be produced in order to obtain the third defective lot?

Geometric Distribution:

\[ P(n) = pq^{n-1} \]

Expectation: \( E(n) = \frac{1}{p} \)

Variance: \( \sigma^2 = \frac{q}{p^2} \)

Example:
In a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?
**Hypergeometric:**
Given a set of \( N \) items, \( K \) of which have a trait of interest ("successes"), we wish to select a sample of \( n \) items without replacement from the set \( N \). The variable \( x \) represents the number of successes in the \( n \) items.

\[
P(X = x) = \binom{K}{x} \binom{N - K}{n-x} \binom{N}{n}
\]

**Expectation:**  
\[E(x) = n \left( \frac{K}{N} \right) \Rightarrow p = \frac{K}{N} \Rightarrow E(x) = np\]

**Variance:**  
\[
\sigma^2 = n \left( \frac{K}{N} \right) \left( \frac{N-K}{N} \right) \left( \frac{N-n}{N-1} \right) = np(1-p) \left( \frac{N-n}{N-1} \right)
\]

**Example:**  
A particular part that is used as an injection device is sold in lots of 10. The producer deems the lot acceptable if no more than one defective part is in the lot. Some lots are sampled, and the sampling plan involves random sampling and testing 3 of the parts out of 10. If none of the three parts are defective, the lot is accepted. Comment on the utility of this plan.
Example:
Lots of 40 components each are called unacceptable if they contain as many as three defectives or more. The procedure for sampling the lot is to select 5 components at random and to reject the lot if a defective is found. What is the probability that exactly 1 defective is found in the sample if there are three defectives in the entire lot?

**Poisson Distribution:**

\[ P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \]

Where \( x \) is the number of successes in a continuous unit
\( \lambda \) is the average number of successes per unit

*Make sure the “unit” for both \( x \) and \( \lambda \) are the same.*

Expectation- \( E(x) = \lambda \)

Variance- \( \sigma^2 = \lambda \)  *mean and variance are the same for Poisson!*
Example:
During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Example:
From records of the past 50 years, it is observed that tornadoes occur in a particular area an average of two times a year. What is the probability of no tornadoes in the next year? What is the probability of exactly 2 tornadoes next year? What is the probability of no tornadoes in the next 50 years?
Examples: Discrete Distributions

1. A pediatrician wishes to recruit five couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. If 20% of randomly selected couples agree to participate, what is the probability that 15 couples must be asked before five are found who agree to participate?

2. It is known that 20% of all copies of a particular textbook fail a certain binding strength test. If 5 texts are randomly selected and tested:
   a.) What is the probability that exactly one fails?
   b.) What is the probability that at most two fail?
   c.) What is the probability that between two and four (inclusive) fail?

3. Geophysicists determine the age of a zircon by counting number of uranium fission tracks on a polished surface. A particular zircon is of such age that the average number of tracks per square centimeter is five. What is the probability that a 2-cm² sample of this zircon will reveal at most three tracks, thus leading to an underestimation of the material?

4. Five animals from a species thought to be near extinction in a certain region have been caught, tagged, and released to mix into the population.
   a) If there are 25 of these animals in the region, what is the probability that two animals will have to be caught to obtain one of the tagged ones?
   b) What is the probability that 10 will have to be caught to obtain two tagged ones?
   c) If a random sample of 10 of these animals is caught, what is the probability that two animals in the sample of 10 will be tagged ones?
   d) What is the probability that no more than two will be tagged ones (out of the sample of 10?)

5. A nuclear plant releases a detectable amount of radioactive gases twice a month on the average.
   a) Find the probability that there will be at most four such emissions during a month.
   b) What is the expected # of emissions during a three-month period?
   c) If, in fact, 12 or more emissions are detected during a three-month period, do you think that there is a reason to be suspicious of the reported average figure of twice a month? Explain, on the basis of the probability involved.