APPENDIX B

Derivation of Poisson Count Distribution*

*This appendix is reprinted with the permission of the Transportation Research Board and is reproduced from the reference: Daniel G. Gerlough and Matthew J. Huber, *Traffic Flow Theory—A Monograph*, Transportation Research Board Special Report 165, 1975, Appendix B, pages 202–203.

Consider a line that can represent in a general case either distance or time; for the present purposes consider it to represent time (Figure B.1). Specifically, consider the occurrence of random arrivals where the average rate of arrival (i.e., probability density) is λ. Let $P_i(t)$ = the probability of i arrivals up to the time t, and $P_n(\Delta t) = \lambda\,\Delta t$ = the probability of one arrival in the incremental period Δt. Because it is assumed that Δt is of such short duration, the probability of more than one arrival in Δt is negligible; therefore, $(1-\lambda\,\Delta t)$ = the probability of no arrival in Δt. Then,

$$P_i(t+\Delta t) = \text{the probability that } i \text{ arrivals have taken place to the time } (t+\Delta t)$$

$$= [\text{Prob}(i-1 \text{ arrivals in } t)\cdot\text{Prob}(1 \text{ arrival in } \Delta t)] + [\text{Prob}(i \text{ arrivals in } t)$$

$$\cdot\,\text{Prob}(0 \text{ arrivals in } \Delta t)]$$

$$P_i(t+\Delta t) = P_{i-1}(t)\cdot P_1(\Delta t) + P_i(t)\cdot P_0(\Delta t)$$

$$= P_{i-1}(t)\lambda\,\Delta t + P_i(t)\,(1-\lambda\,\Delta t)$$

$$= [P_{i-1}(t) - P_i(t)](\lambda\,\Delta t) + P_i(t)$$

and

$$\frac{P_i(t+\Delta t) - P_i(t)}{\Delta t} = \lambda[P_{i-1}(t) - P_i(t)]$$

Letting $\Delta t \to 0$,

$$\frac{dP_i(t)}{dt} = \lambda[P_{i-1}(t) - P_i(t)] \tag{B.1}$$

Now,

$P_{-1}(t)=0$ (i.e., impossible to have <0)

$P_0(0) = 1$ (i.e., no arrivals up to time $t=0$)

$P_i(0) = 0$ for $i \geq 1$ (zero probability of i arrivals at time $t=0$)

Setting $i = 0$ in equation (B.1),

$$\frac{dP_0(t)}{dt} = \lambda[0-P_0(t)]$$

$$\frac{dP_0(t)}{P_0(t)} = -\lambda\,dt$$

$$\ln P_0(t) = -\lambda t + c$$

$$P_0(t) = e^{-\lambda t+c}$$

Since $P_0(0) = 1$ and $1 = e^0 = e^c$, $c = 0$, and

$$P_0(t) = e^{-\lambda t}$$

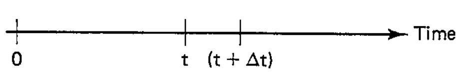

Figure B.1 Schematic Representation of Uniform Probability Density

Setting $i = 1$ in equation (B.1) and inserting the above value for $P_0(t)$,

$$\frac{dP_1(t)}{dt} = \lambda[e^{-\lambda t} - P_1(t)]$$

$$\frac{dP_1(t)}{dt} + \lambda P_1(t) = \lambda e^{-\lambda t}$$

Using method of operators for solving this differential equation*

$$(D+\lambda)P_1(t) = \lambda e^{-\lambda t}$$

$$P_1(t) = \frac{1}{D+\lambda}\lambda e^{-\lambda t}$$

$$= (\lambda t)e^{-\lambda t} + C_2 e^{-\lambda t}$$

But

$$P_1(0) = 0 \qquad \therefore C_2 = 0$$

$$\therefore P_1(t) = (\lambda t)e^{-\lambda t}$$

For $i = 2$,

$$\frac{dP_2(t)}{dt} = \lambda[P_1(t) - P_2(t)]$$

$$\frac{dP_2(t)}{dt} + \lambda P_2(t) = \lambda P_1(t) = \lambda(\lambda t)e^{-\lambda t}$$

$$P_2(t) = \frac{1}{D+\lambda}\lambda(\lambda t)e^{-\lambda t}$$

$$= \frac{\lambda^2 t^2}{2}e^{-\lambda t} + C_3 e^{-\lambda t}$$

But

$$P_2(0) = 0 \qquad \therefore C_3 = 0$$

$$P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$$

Similarly,

$$P_3(t) = \frac{(\lambda t)^3 e^{-\lambda t}}{3!}$$

$$P_4(t) = \frac{(\lambda t)^4 e^{-\lambda t}}{4!}$$

*Any standard method may be used for solution of this differential equation. The method of operators is particularly simple; see any standard text. The form $y = [1/(D+A)]\,u(x)$ results in a solution

$$y = e^{-Ax}\int e^{Ax}u(x)\,dx + ce^{-Ax}$$

$$P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

If $\lambda t = m$, the result is the most familiar form of the Poisson distribution:

$$P(x) = \frac{m^x e^{-m}}{x!}$$

This relationship states the probability that exactly x arrivals will occur during an interval (of length t) when the mean number of arrivals is m (per interval of t).

B.1 Population Mean of Poisson Distribution

In the foregoing m is defined as the mean *arrival rate*. To determine the mean value of the distribution, begin with the definition of the population mean μ for a discrete distribution:

$$\mu = \sum_{x=0}^{\infty} xP(x), \qquad \text{for } P(x) = \frac{f(x)}{\sum_{x=0}^{\infty} f(x)} \tag{B.2}$$

where $f(x)$ is the frequency of occurrence of x. For the Poisson distribution, substitute

$$P(x) = \frac{m^x e^{-m}}{m!}$$

Thus

$$\mu = \sum_{x=0}^{\infty} \frac{xm^x e^{-m}}{m!} \tag{B.3}$$

$$= 0 + me^{-m} + \frac{2m^2 e^{-m}}{2!} + \frac{3m^3 e^{-m}}{3!} \cdots$$

$$= me^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} \cdots\right]$$

$$= me^{-m}e^{m}$$

$$= m \tag{B.4}$$

B.2 Population Variance of Poisson Distribution

By definition, the *population* variance, σ^2, may be expressed:

$$\sigma^2 = \frac{\Sigma f(x)(x-\mu)^2}{\Sigma f(x)} \tag{B.5}$$

$$= \sum_{x=0}^{\infty} (x-\mu)^2 P(x) \tag{B.6}$$

Because the population mean is m, this variance may be stated:

$$\begin{aligned}\sigma^2 &= \Sigma\,(x-m)^2 P(x)\\ &= \Sigma\,(x^2-2xm+m^2)P(x)\\ &= \Sigma\,x^2P(x) - 2m\,\Sigma\,xP(x) + m^2\,\Sigma\,P(x)\end{aligned}$$

The last term reduces to m^2 because $\Sigma P(x)=1$. The middle term reduces to $-2m^2$ because $\Sigma xP(x)$ has been shown equal to m in the derivation of the population mean. The first term may be reduced by the following steps:

$$\begin{aligned}\Sigma x^2P(x) &= \Sigma\,[x(x-1)+x]P(x)\\ &= \Sigma\,x(x-1)P(x) + \Sigma xP(x)\\ &= A+B\\ B &= \Sigma xP(x) = m\\ A &= \sum_{x=0}^{\infty} x(x-1)\frac{m^x e^{-m}}{x!}\\ &= \left[0+0+\frac{2m^2e^{-m}}{2!}+\frac{6m^3e^{-m}}{3!}+\frac{12m^4e^{-m}}{4!}+\cdots\right]\\ &= m^2e^{-m}\left[1+m+\frac{m^2}{2!}+\cdots\right]\\ &= m^2e^{-m}e^{m} = m^2\\ \Sigma x^2P(x) &= m^2+m\\ \sigma^2 &= [m^2+m]-[2m^2]+[m^2]\\ \sigma^2 &= m \qquad \text{(B.7)}\end{aligned}$$

Thus, for the Poisson distribution the population variance equals the population mean.