

CIVL 7/8012
In-class problem solutions
Multivariate distributions

1. PMF of precipitation X and runoff Y is discretized as follows:

	X=1	X=2	X=3	Totals
Y=10	0.05	0.15	0	0.2
Y=20	0.1	0.25	0.25	0.6
Y=30	0	0.1	0.1	0.2
Totals	0.15	0.5	0.35	1

- a. $P(X \geq 2, Y > 20)$
 $= P(X=2, Y=30) + P(X=3, Y=30)$
 $= 0.1 + 0.1$
 $= 0.2$
- b. $P(X=2, Y \geq 20)$
 $= P(X=2, Y=20) + P(X=2, Y=30)$
 $= 0.25 + 0.1$
 $= 0.35$
- c. $P(X=1) = 0.05 + 0.1 + 0 = 0.15$
 $P(Y=10) = 0.05 + 0.15 + 0 = 0.20$
 $P(X=1) * P(Y=10) = 0.03$
 $P(X=1, Y=10) = 0.05$
 Since,
 $P(X=1) * P(Y=10) \neq P(X=1, Y=10)$ They are not independent
- d. Marginal PMF of runoff;
 $P(Y=10) = 0.05 + 0.15 + 0 = 0.20$
 $P(Y=20) = 0.10 + 0.25 + 0.25 = 0.6$
 $P(Y=30) = 0 + 0.1 + 0.1 = 0.2$
- e. $P(X=2 | Y) = P(X=2 | Y=10) + P(X=2 | Y=20) + P(X=2 | Y=30)$
 $= 0.15 + 0.25 + 0.10$
 $= 0.5$
- f. Correlation coefficient, $\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{V(X)*V(Y)}}$
 $Cov(X, Y) = E(XY) - \mu_X \mu_Y$
 $= (10 * 1 * 0.05 + 10 * 2 * 0.15 + 0 +$
 $20 * 1 * 0.1 + 20 * 2 * 0.25 + 20 * 3 * 0.25 +$
 $0 + 30 * 2 * 0.1 + 30 * 3 * 0.1) - 20 * 2$
 $= 45.5 - 40$
 $= 5.5$

$$\begin{aligned}
E(X) &= 1 * 0.15 + 2 * 0.5 + 3 * 0.35 = 2.2 \\
E(Y) &= 10 * 0.2 + 20 * 0.6 + 30 * 0.2 = 20 \\
V(X) &= 1^2 * 0.15 + 2^2 * 0.5 + 3^2 * 0.35 - E(X)^2 = 3.1 \\
V(Y) &= 10^2 * 0.2 + 20^2 * 0.6 + 30^2 * 0.2 - E(Y)^2 = 420 \\
\rho_{XY} &= \frac{5.5}{\sqrt{3.1 * 420}} = 0.1524
\end{aligned}$$

2.

a. Integrating over y we get

$$\int_0^1 \frac{6(x+y^2)dy}{5} = \frac{6}{5} \left[xy + \frac{y^3}{3} \right]_0^1 = \frac{2(3x+1)}{5}$$

$$b. f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{(6/5)(x+y^2)}{(2/5)(3x+1)} = \frac{3(x+y^2)}{3x+1}$$

$$\text{Then, } P(Y > 0.5|X = 0.5) = 3 \int_0^1 \frac{(0.5+y^2)dy}{1.5+1} = \frac{3}{2.5} \left[0.5y + \frac{y^3}{3} \right]_0^1 = 0.65$$

c.

$$\int_0^1 \int_0^1 xy f_{XY}(x,y) dx dy = \frac{6}{5} \int_0^1 \int_0^1 (x^2y + xy^3) dx dy = \frac{2}{5} \int_0^1 y dy + \frac{3}{5} \int_0^1 y^3 dy = \frac{1}{5} + \frac{3}{20} = 0.35$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0.35 * \frac{3}{5} * \frac{3}{5} = -0.01$$

$$V(X) = E(X^2) - (E(X))^2 = \left[\frac{13}{30} - \frac{3^2}{5} \right] = 0.0733$$

$$V(Y) = E(Y^2) - (E(Y))^2 = \left[\frac{11}{25} - \frac{3^2}{5} \right] = 0.08$$

$$\text{Correlation coefficient, } \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{V(X) * V(Y)}} = \frac{-0.01}{\sqrt{0.0733 * 0.08}} = -0.131$$