

*Analysis of Members with Axial Loads and Moments*  
(Length effects Disregarded, “Short Column”)

**A. Reading Assignment**

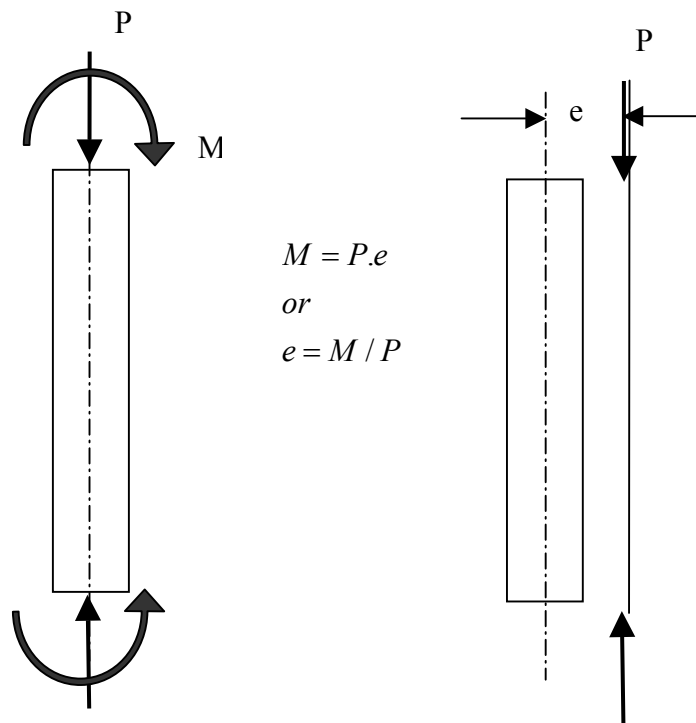
Chapter 9 of text

Chapter 10 of ACI

**B. Presentation of the *INTERACTION DIAGRAM* or *FAILURE ENVELOPE***

We have seen that a given section can have a maximum capacity in either axial load or flexure. It is logical to suppose that for some combination of axial load and moment, a maximum capacity can also be found.

Consider a member with some applied axial “P” and some applied moment “M.”



Short Column Assumption: “ $e$ ” is constant along the member.

Consider a section at ultimate moment condition and apply an axial load.

Condition “a” - Pure Moment

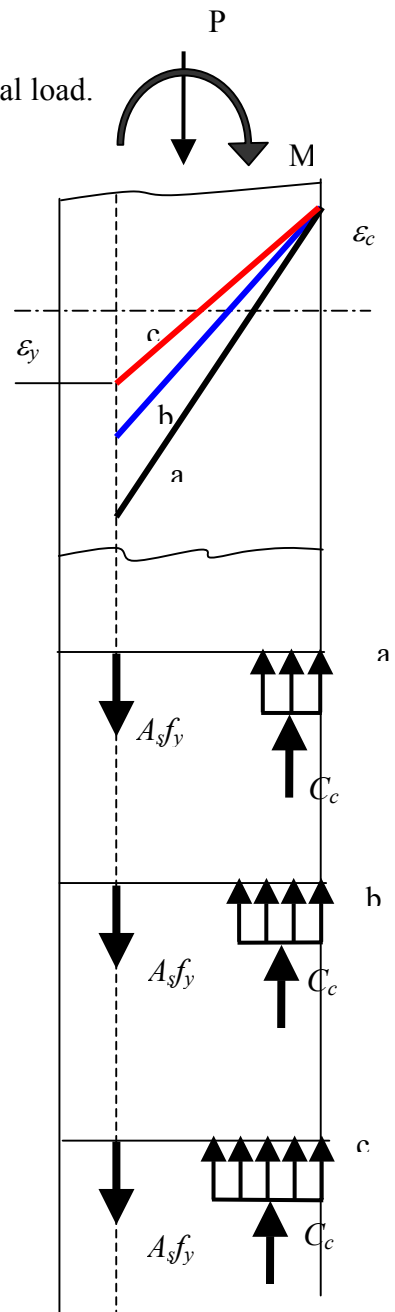
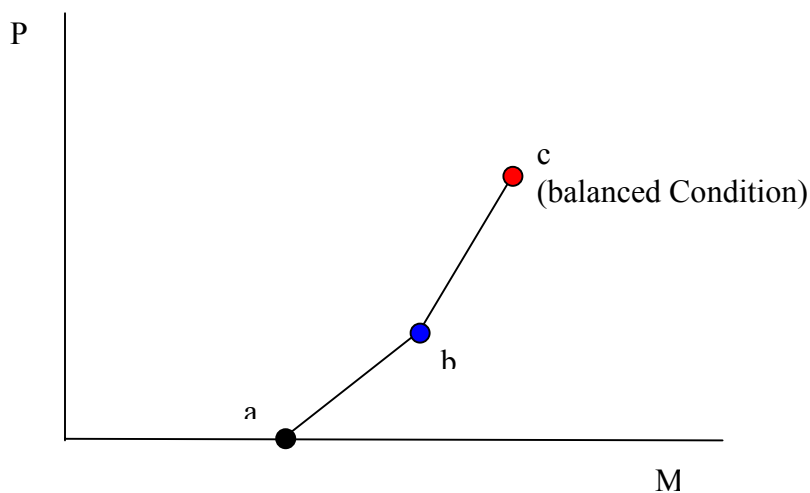
Condition “b” - Moment, Axial load

Condition “c” - Moment, Axial load  
(Balanced condition)

If we take moment about section centroid in all cases:

1. Moment of  $A_s f_y$  remains constant.
2.  $C_c$  increases as we go from strain condition “a” to strain condition “c”
3. Moment of  $C_c$  increases
4. Summation internal forces increases with  $C_c$ .

This can be termed “positive interaction,” since increase in one capacity results in increase in other capacity.

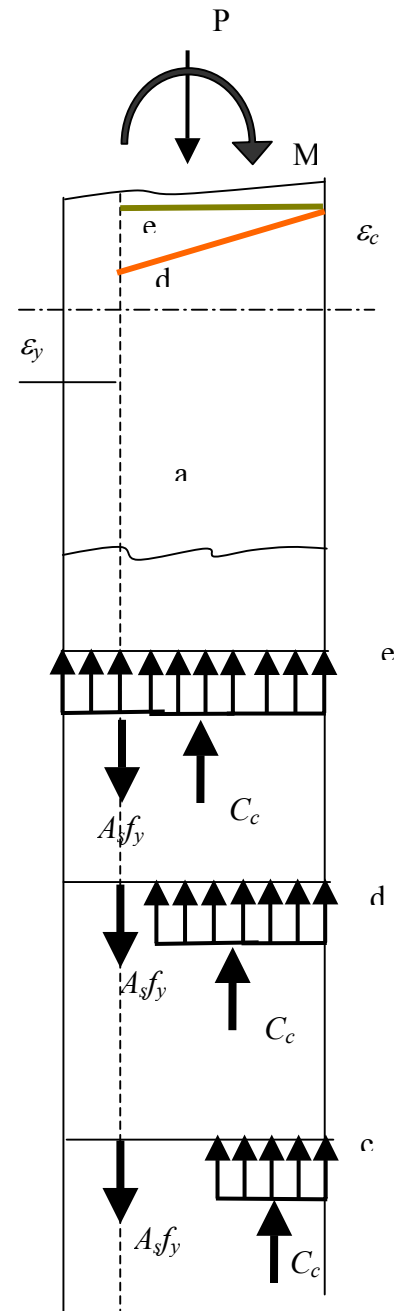
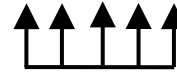


Consider a section at ultimate moment condition and apply an axial load.

Condition “e” - Pure Axial load

Condition “d” - Moment, Axial load

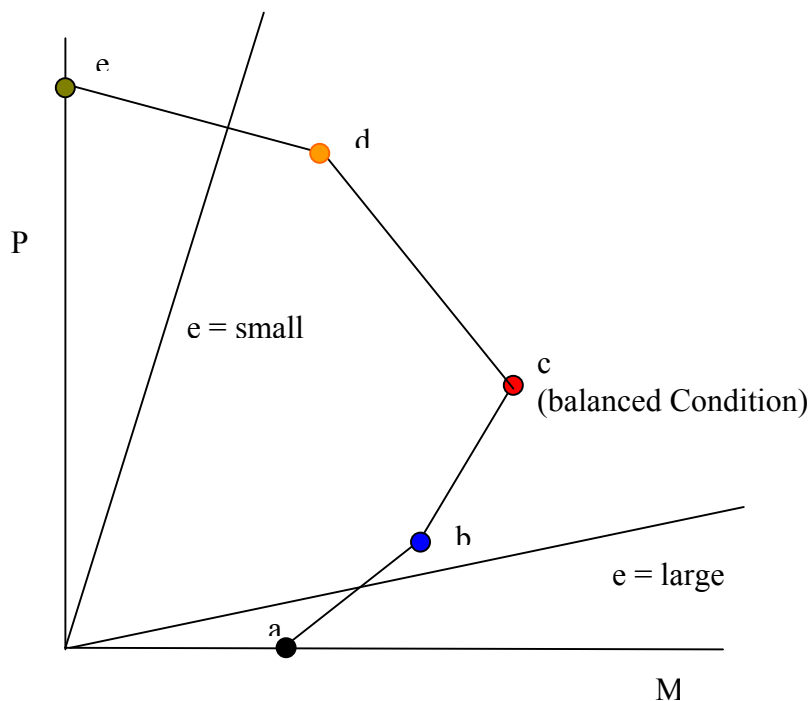
Condition “c” - Moment, Axial load  
(Balanced condition)

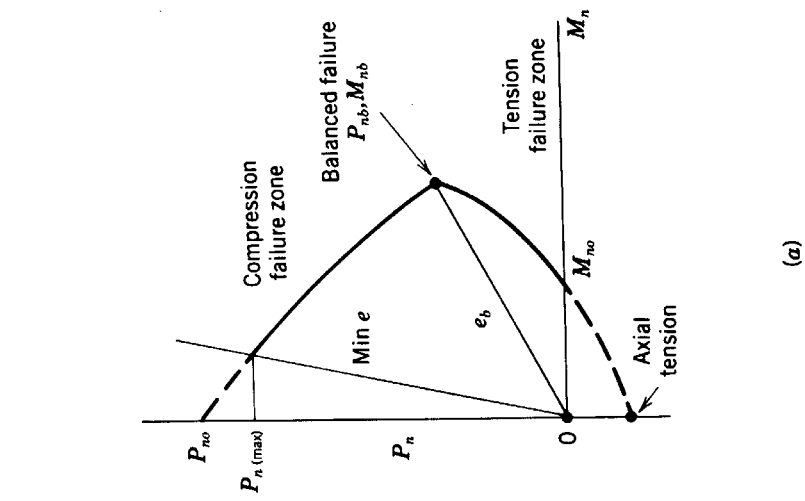


If we take moment about section centroid in all cases:

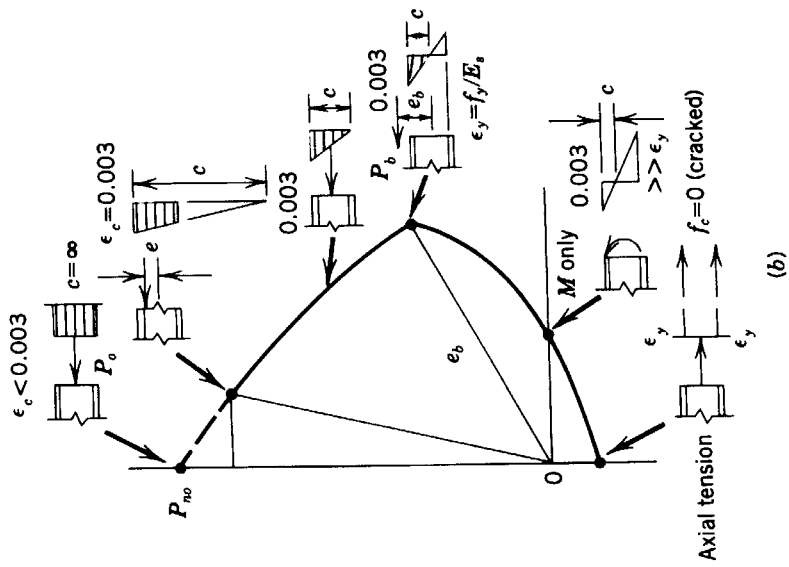
1. Moment of  $A_s f_y$  remains constant.
2.  $C_c$  decreases as we go from strain condition “e” to strain condition “c”
3.  $\Sigma M$  increases.

This can be termed “negative interaction,” since increase in on capacity results in decrease in other capacity.

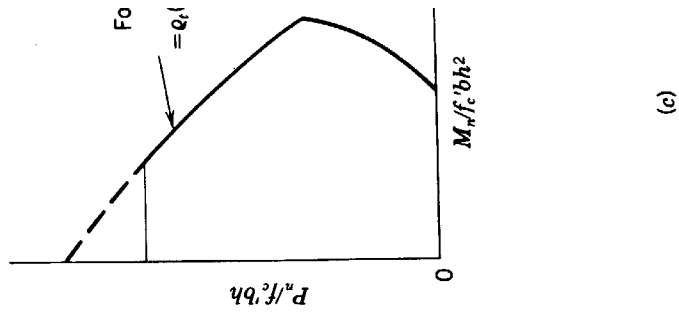




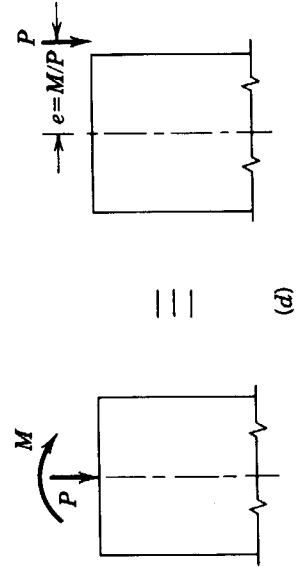
(a)



(b)



(c)



(d)

Col.

## Strain Limits Method for Analysis and Design (ACI 318-2002).

In “Strain Limits Method,” sometime referred to as the “Unified Method,” the nominal flexural strength of a concrete member is reached when the net compressive strain in the extreme compression fiber reaches the ACI code-assumed limit of 0.003 in/in (ACI 10.2.3). It also hypothesized that when the net tensile strain in the extreme tension steel,  $\varepsilon_t = 0.005$  in/in, the behavior is fully ductile. The concrete beam sections characterized as “Tension-Controlled,” with ample warning of failure as denoted by excessive deflection and cracking.

If the net tensile strain in the extreme tension fibers,  $\varepsilon_t$ , is small, such as in compression members, being equal or less than a “Compression-Controlled” strain limit, a brittle mode of failure is expected with a sudden and explosive type of failure. Flexural members are usually tension-controlled. However, some sections such as those subjected to small axial loads, but large bending moments, the net tensile strain,  $\varepsilon_t$ , in the extreme tensile fibers, will have an intermediate or transitional value between the two strain limit states, namely, between the compression-controlled strain limit of

$$\varepsilon_t = \frac{f_y}{E_s} = \frac{60ksi}{29,000ksi} = 0.002 \quad (1.1)$$

and the tension-controlled strain limit  $\varepsilon_t = 0.005$  in/in. Figure 5.1 (ACI Figure. R9.3.2 page 100) shows these three zones as well as the variation in the strength reduction factors applicable to the total range of behavior.

### Variation of $\phi$ as a Function of Strain

Variation of the  $f$  value for the range of strain between  $\varepsilon_t = 0.002$  in/in and  $\varepsilon_t = 0.005$  in/in can be linearly interpolated:

$$0.65 \leq (\phi = 0.48 + 83\varepsilon_t) \leq 0.90 \quad \text{Tied Column}$$

$$0.70 \leq (\phi = 0.57 + 67\varepsilon_t) \leq 0.90 \quad \text{Spiral Column}$$

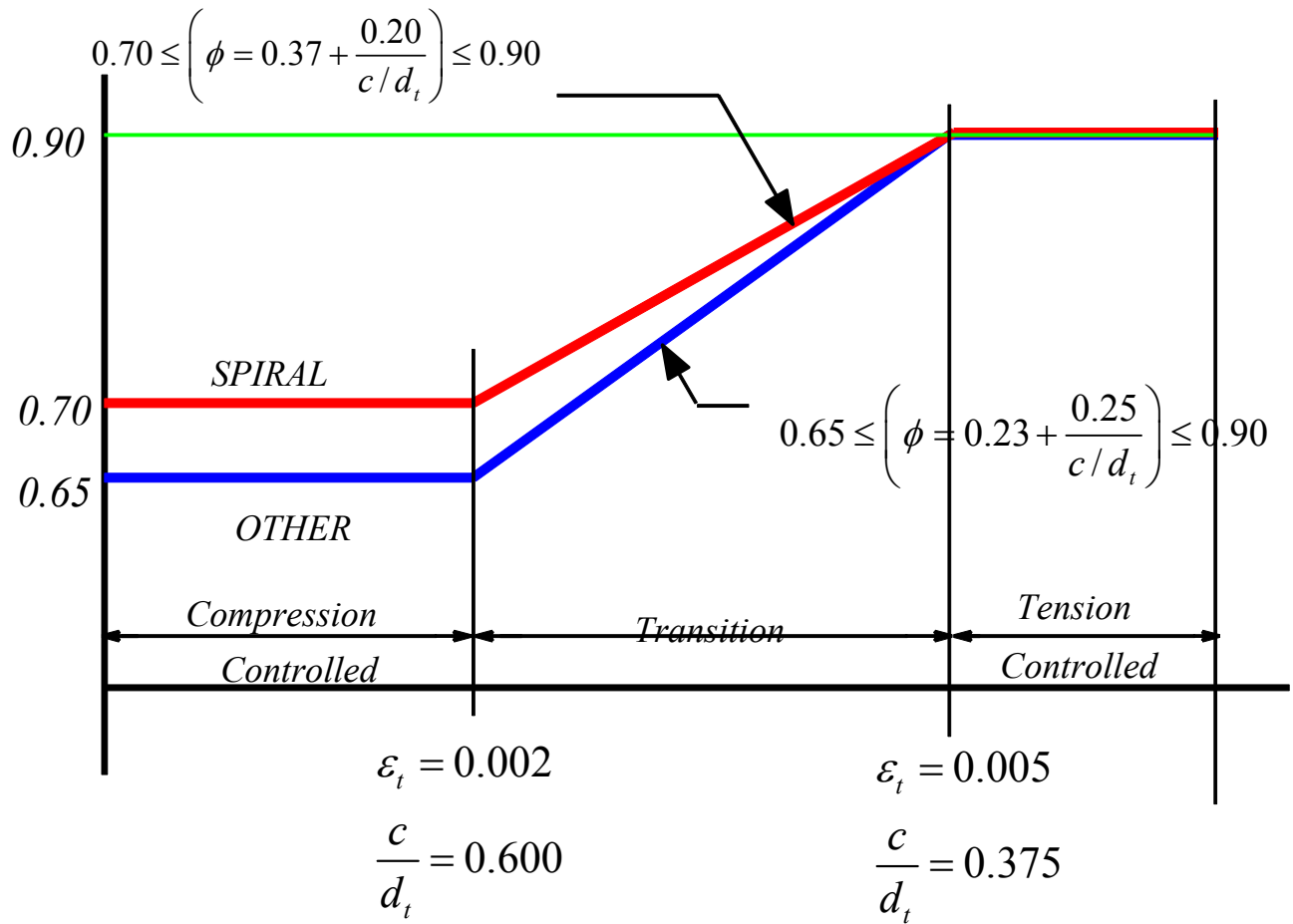
### Variation of $\phi$ as a Function of Neutral Axis Depth Ratio $c/d$

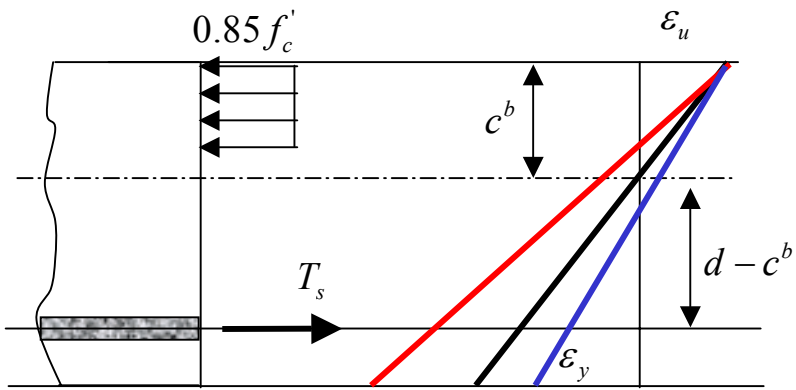
$$0.65 \leq \left( \phi = 0.23 + \frac{0.25}{c/d_t} \right) \leq 0.90$$

Tied Column

$$0.70 \leq \left( \phi = 0.37 + \frac{0.20}{c/d_t} \right) \leq 0.90$$

Spiral Column

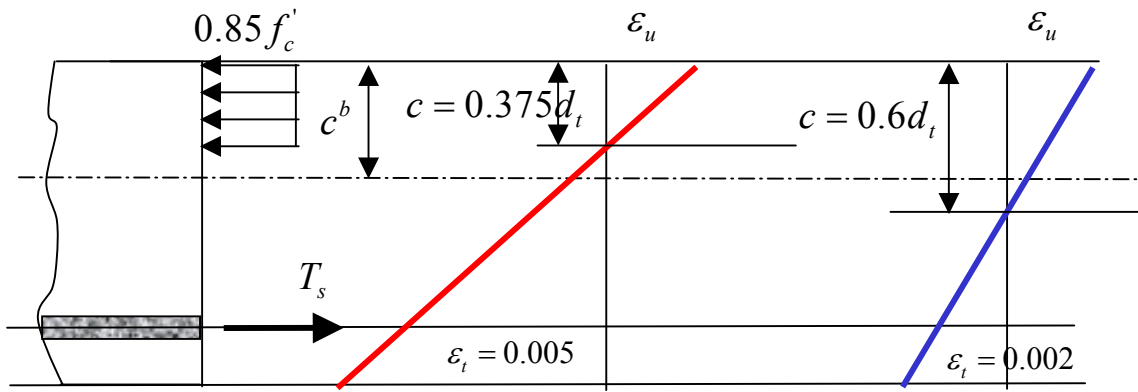
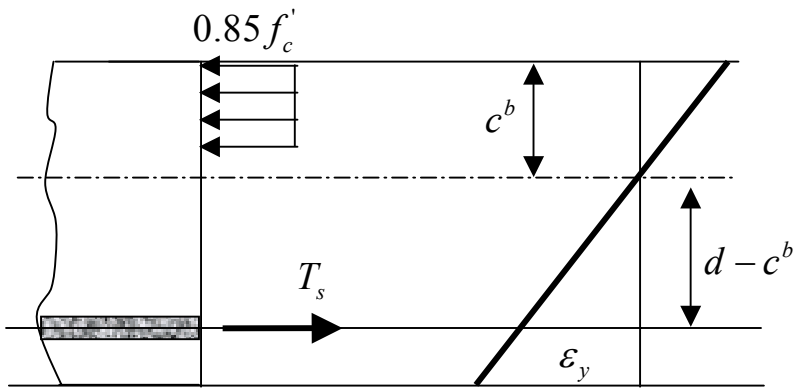




$$\frac{c^b}{d_t} = \frac{87,000}{87,000 + f_y}$$

$$f_y = 60,000 \text{ psi}$$

$$\frac{c^b}{d_t} = \frac{87}{87 + 60} = 0.60$$



Tension  
Failure

Balanced  
Condition

## Stresses and Strains

For tension steel

$$\epsilon_s = \epsilon_u \frac{d-c}{c}$$

$$f_s = \epsilon_s E_s = \epsilon_u \frac{d-c}{c} E_s \leq f_y$$

for compression steel:

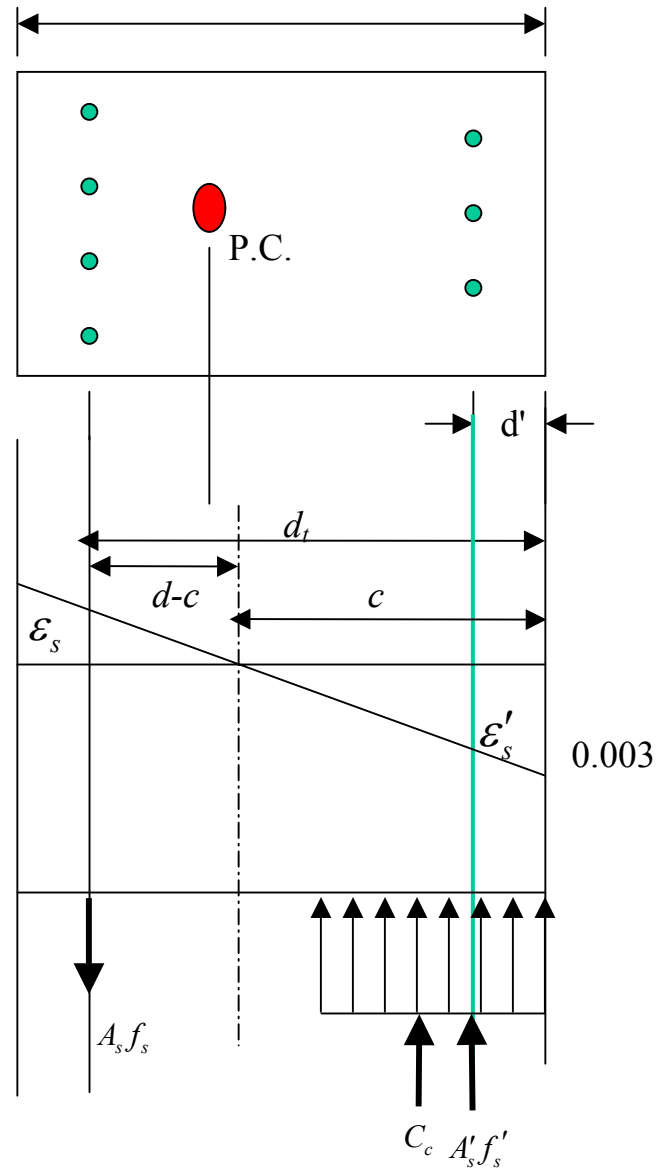
$$\epsilon'_s = \epsilon_u \frac{c-d'}{c}$$

$$f'_s = \epsilon'_s E_s = \epsilon_u \frac{c-d'}{c} E_s \leq f_y$$

where

$$a = \beta_1 c < h \quad \text{see ACI 10.2.7.3 and}$$

$$C_c = 0.85 f'_c a b$$

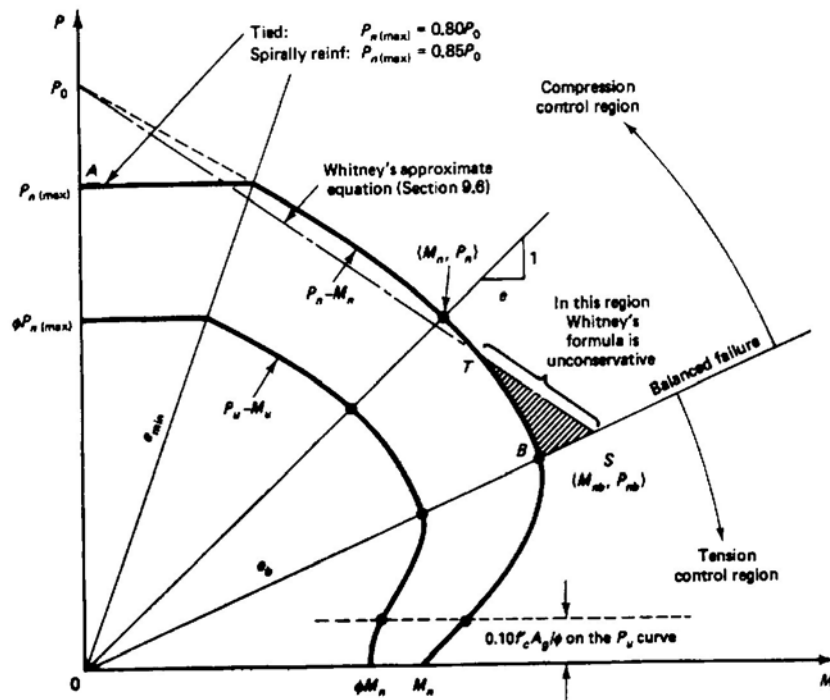


### NOTE:

In contrast to beams we cannot restrict column design such that yielding before failure rather than crushing failure always be the result of over loading. Type of failure of columns depend on the value of eccentricity “e.”



### C. Code Allowable Interaction Diagram (ACI 9.3.2)



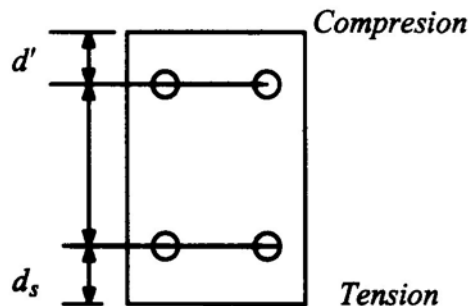
### Description of Code Provisions for Combined Flexure and Axial Load

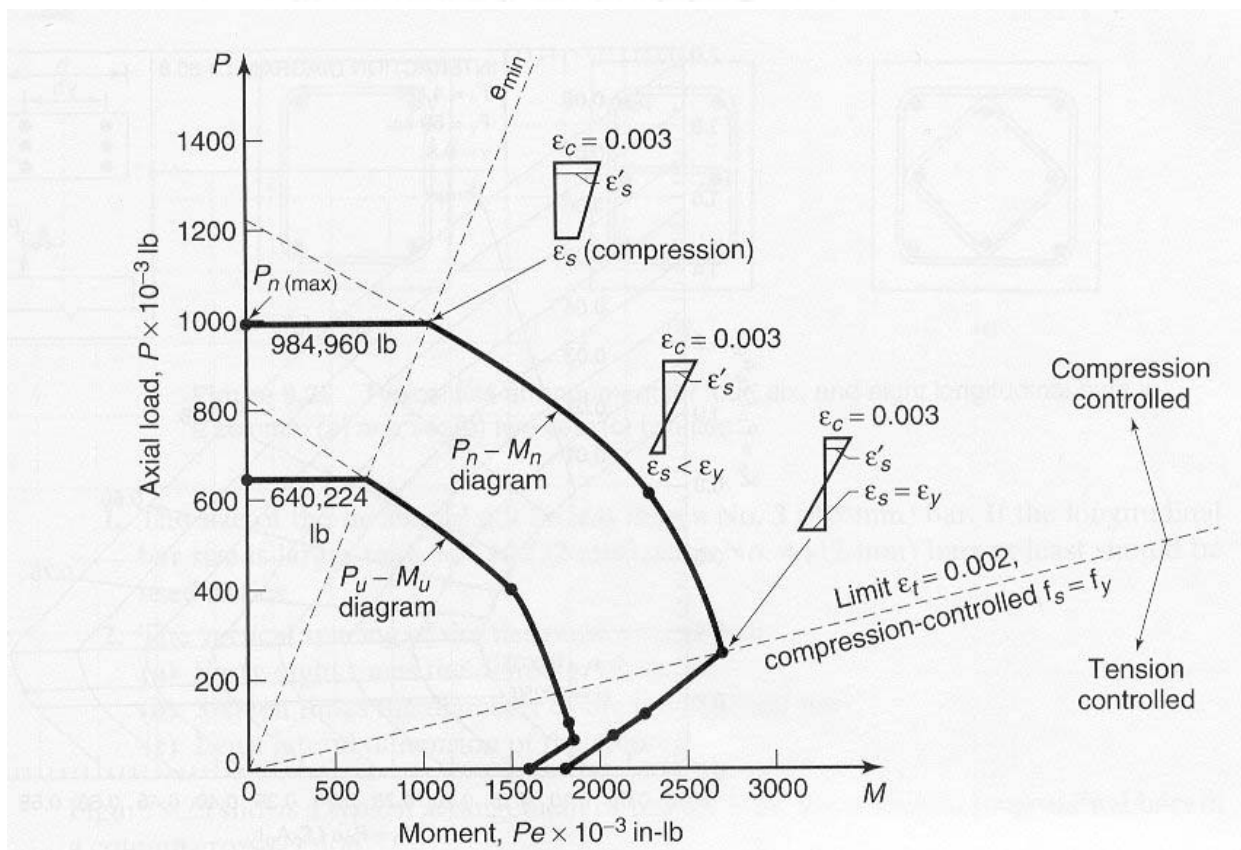
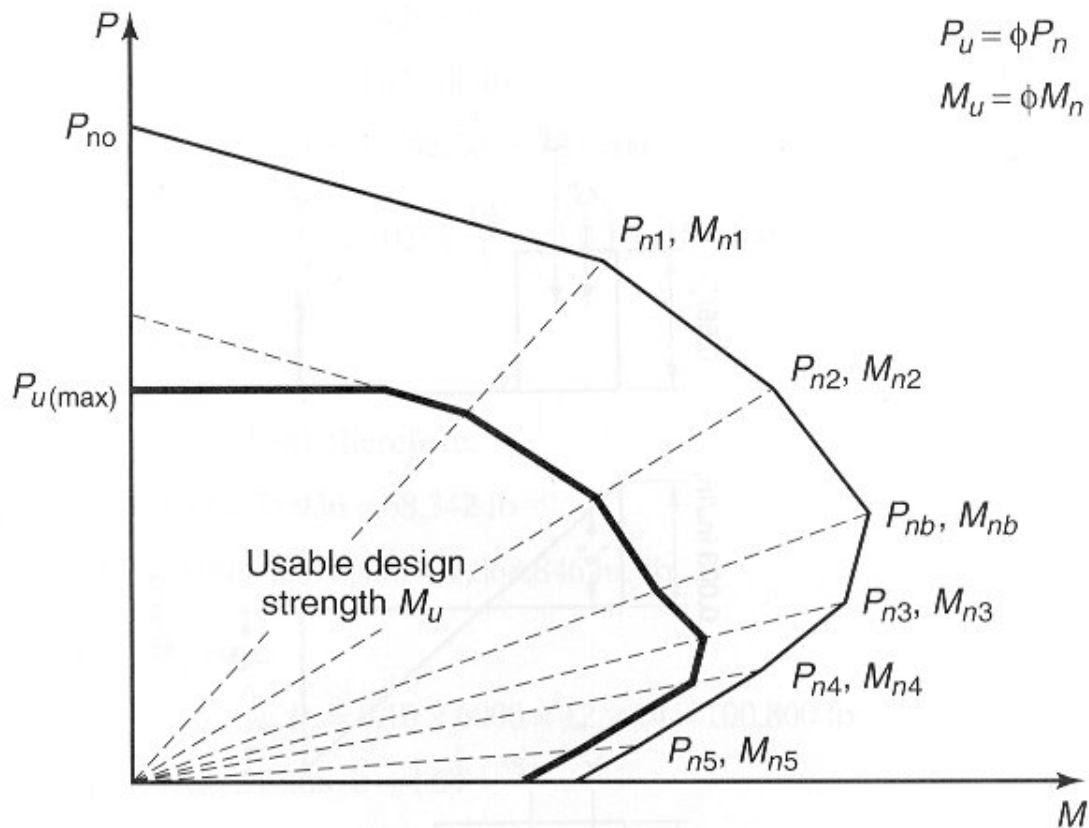
May increase  $\phi$  under special conditions:

If  $f_y \leq 60$  ksi and if  $(h - d' - d_s)/h \geq 0.7$  with symmetrical reinforcement, then  $\phi$  may be increased linearly to 0.9 as  $\phi P_N$  decreases from  $0.1 f'_c A_g$  to zero.

for other reinforced members:

$\phi$  increases linearly to 0.9 as  $\phi P_N$  decreases from  $0.1 f'_c A_g$  to  $\phi P_b$  whichever is smaller





### E. Define Plastic Centroid:

Point at which resultant of resisting forces acts when strain is uniform over the section  $M = 0$  when  $e = 0$ , as measured from this point for typical symmetrical reinforced sections, plastic centroid will be at section geometric centroid. For other sections a small calculation is needed (see the example below). In other words, the plastic centroid is the centroid of resistance of the section if all the concrete is compressed to the maximum stress ( $0.85f'_c$ ) and all the steel is compressed to the yield stress ( $f_y$ ) with uniform strain over the section.

### Example

For the section given below determine the location of the plastic centroid.

#### Given:

$$f'_c = 4,000 \text{ ksi}$$

$$f_y = 60,000 \text{ psi}$$

*Tied Column*

#### Solution:

$$C_c = 0.85f'_c A_c$$

$$C_c = 0.85 \times (4\text{ksi}) \times (26 \times 15 - 14)$$

$$C_c = 1280 \text{ kips}$$

$$P_{No} = 10 \times 60 + 4 \times 60 + 1280 = 2,120 \text{ kips}$$

$$2120 \times f = 600 \times 10 - 240 \times 10 = 3,600$$

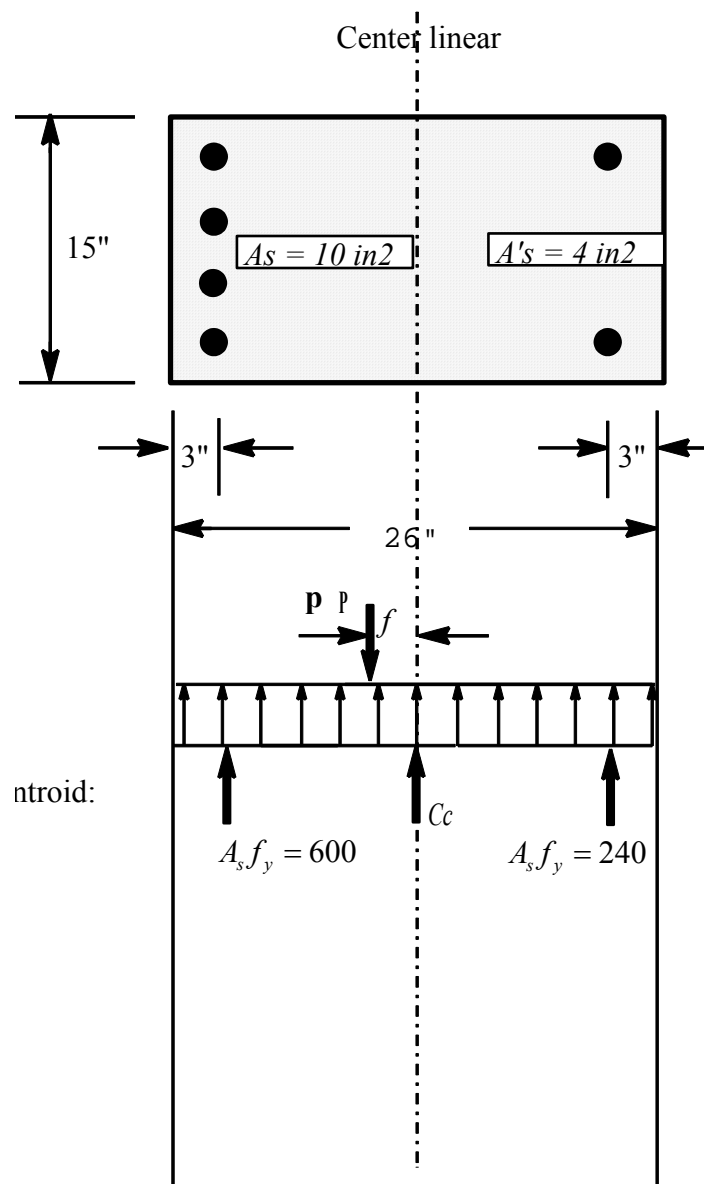
$$f = 1.7 \text{ in}$$

Could also take moment about PC.

$$\sum M = 0$$

$$1280 \times f + 240(f + 10) - 600 \times (10 - f) = 0$$

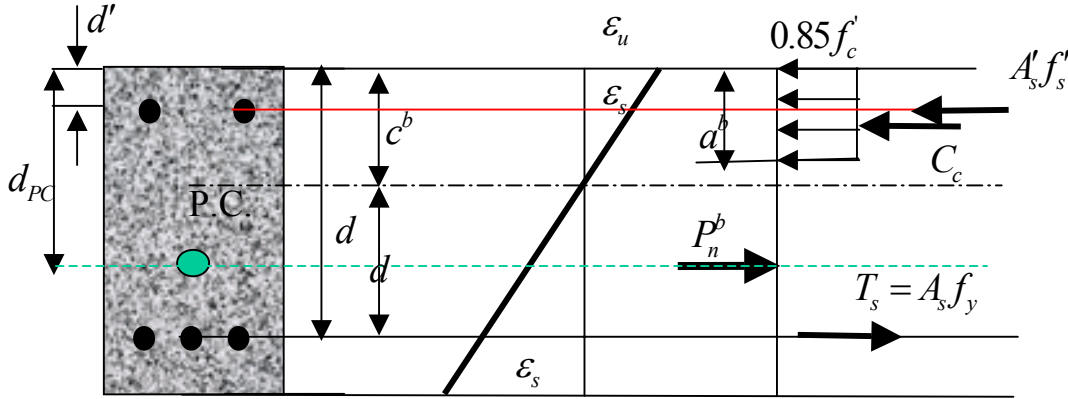
$$f = 1.7 \text{ in}$$



## F. Construction of Interaction Diagram

Recall that we have previously analyzed the situations in which nominal moment or nominal axial load existed. The calculation associated with these conditions will not change.

Recall the balanced strain condition. The important thing to note here is that any section can be put under appropriate moment and axial load to cause balanced strain:



From geometry:

$$c^b = 0.600d$$

where

$$a^b = \beta_1 c^b$$

and

$$f'_s = \epsilon'_s E_s = 0.003 \left( \frac{c^b - d'}{c^b} \right) E_s \leq f_y$$

From statics:

$$P_n^b = 0.85 f'_c a^b b + A'_s f'_s - A_s f_y$$

and

$$M_n^b = 0.85 f'_c a^b b \left( d_{pc} - \frac{a^b}{2} \right) + A'_s f'_s (d_{pc} - d') + A_s f_y (d - d_{pc})$$

Note that, whenever axial load is present, moment must be taken about the plastic centroid if consistent results are to be obtained. It must be recalled that the moment value calculated above is actually an internal, resisting moment, which is in equilibrium with an external applied moment of the same magnitude. We could, if we wished, calculate the internal resisting moment about any point if we at the same time remembered to include the influence of the external axial load, assumed applied at the plastic centroid. If we chose to take moments about compression steel in the above case, for example, this calculation of  $M_n^b$  would become:

$$M_n^b = A_s f_y (d - d') - 0.85 f'_c a^b b \left( \frac{a^b}{2} - d' \right) + P_n^b (d_{pc} - d')$$

It is generally easier to take moment about the plastic centroid. Thus, avoiding need for consideration of the external axial load.

### Example. Calculation of Points on Interaction Diagram

Consider the following section for which we have calculated the plastic centroid location

**Given:**

$$f_c = 4,000 \text{ ksi}$$

$$f_y = 60,000 \text{ psi}$$

*Tied Column*

**Balanced Condition:**

$$A_s f_y = 400 \text{ kips}$$

$$A_s' f_y = 160 \text{ kips}$$

$$c = 0.6 \times 23 = 13.8 \text{ in}$$

$$a = 0.85 \times 13.8 = 11.73 \text{ in}$$

$$\epsilon_s' = \epsilon_u \frac{c - d'}{c}$$

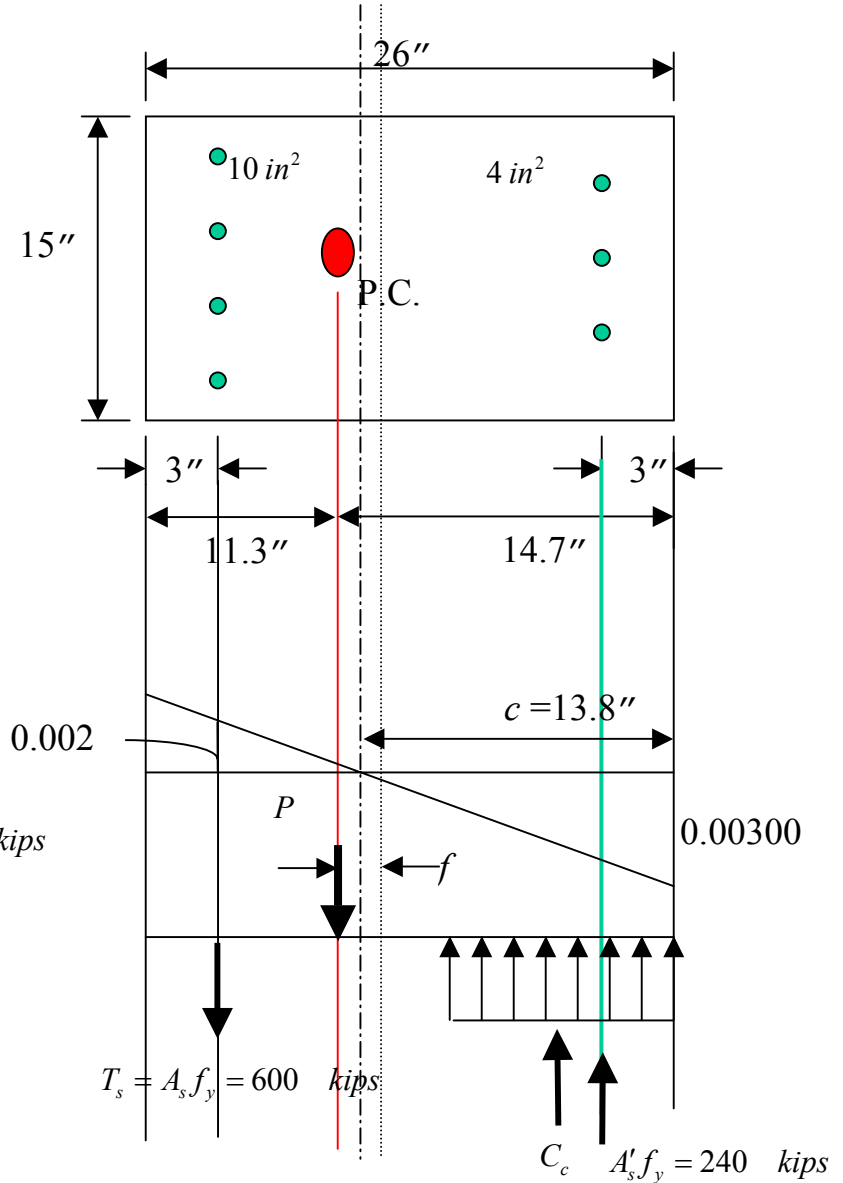
$$\epsilon_s' = 0.003 \times \frac{13.8 - 3.0}{13.8} = 0.00235 > 0.00207$$

$$C_c = 0.85 f_c' A_c$$

$$C_c = 0.85 \times 4 \text{ ksi} \times [0.85 \times 13.8] \times 15 = 598.2 \text{ kips}$$

$$C_s = 60 \times 4 = 240 \text{ kips}$$

$$T_s = 60 \times 10 = 600 \text{ kips}$$



$$P_b = C_s + C_c - T_s = 598.2 + 240 - 600 = 238.2 \text{ kips}$$

Take moment about P.C.:

$$M_n^b = 598.2 \left( 14.7 - \frac{11.73}{2} \right) + 240 (14.7 - 3) + 600 (11.3 - 3) = 13,070 \text{ in-kips}$$

when axial load is present, moment may be taken about P.C.

## Ultimate Moment Condition (No Axial Loads)

Assume compression steel yields:

$$0.85 \times 4 \times 15 \times a + 240 = 600$$

$$a = \frac{360}{51} = 7.06 \text{ inches}$$

$$c = \frac{7.06}{0.85} = 8.3 \text{ inches}$$

$$\varepsilon'_s = \varepsilon_u \frac{c - d'}{c}$$

$$\varepsilon'_s = 0.003 \times \frac{8.3 - 3.0}{8.3} = 0.00192 \geq 0.00207 \quad \text{Assumption that compression steel yields is}$$

wrong. Therefore, the compression steel will not yield.

Since the compression steel does not yield,  $C_s = A'_s \times f'_s = 4f'_s$

$$0.85 \times 4 \times 15 \times a + 4 \times f'_s = 600$$

$$0.85 \times 4 \times 15 \times (0.85c) + 4 \times \varepsilon_u \frac{c - d'}{c} E_s = 600$$

$$43.35c + 4 \times 0.003 \frac{c - 3}{c} 29000 = 600$$

$$43.35c + 348 \frac{c - 3}{c} = 600$$

$$43.35c^2 + 348c - 1044 = 600c$$

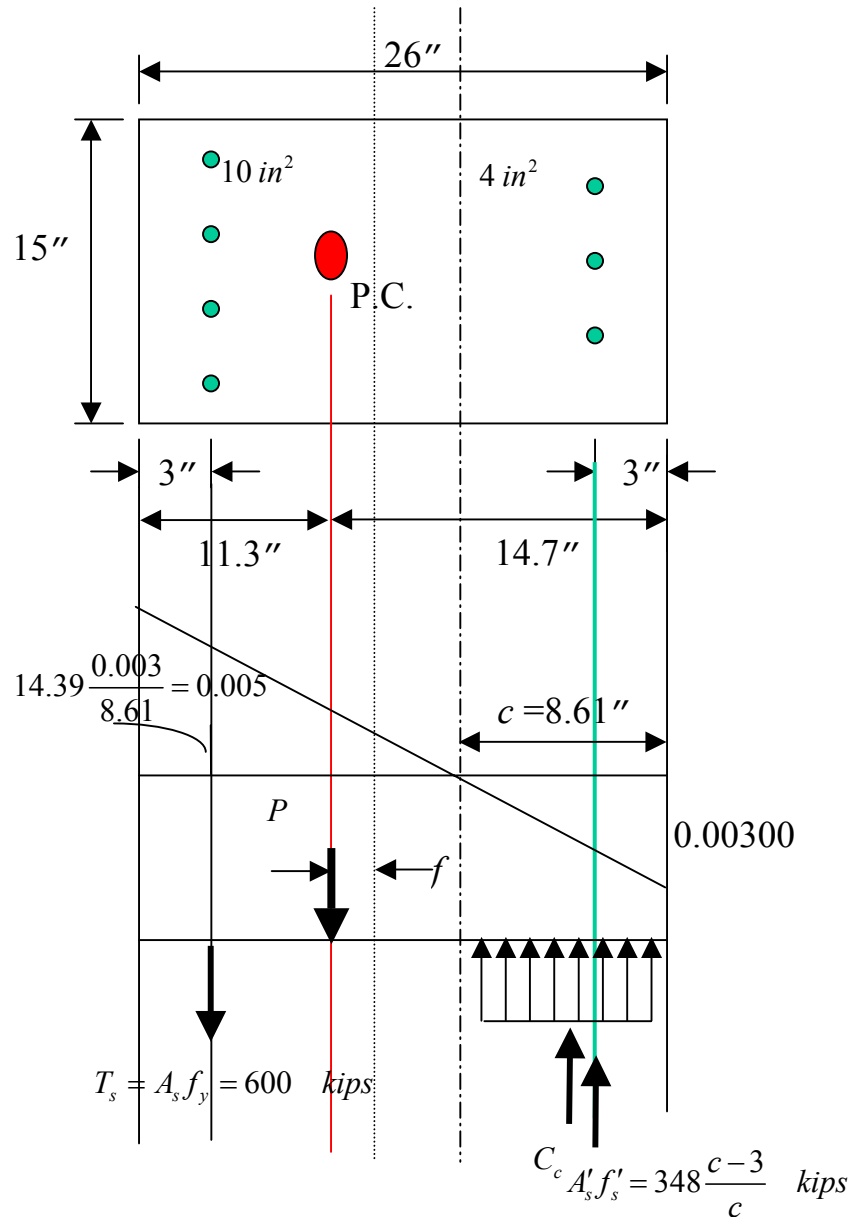
$$43.35c^2 - 252c - 1044 = 0$$

$$c = \frac{252 \mp \sqrt{252^2 + 4 \times 43.35 \times 1044}}{2 \times 43.35} = 8.61 \text{ inches}$$

$$P_n = 0$$

$$C_c = 43.35c = 43.35(8.61) = 373.2 \text{ kips}$$

To be consistent, always take moment about PC.



$$M_n^b = 226.7(14.7 - 3) + 373.2(14.7 - \frac{0.85 \times 8.61}{2}) + 600(11.3 - 3) = 11,753 \quad in - kips$$

**Case**  $\varepsilon_s = 0.003$        $\varepsilon_u = 0.003$

$$c = 11.5 \text{ inches}$$

$$a = 0.85c = 0.85 \times 11.5 = 9.78 \text{ inches}$$

Check to find out whether the compression steel yield or not.

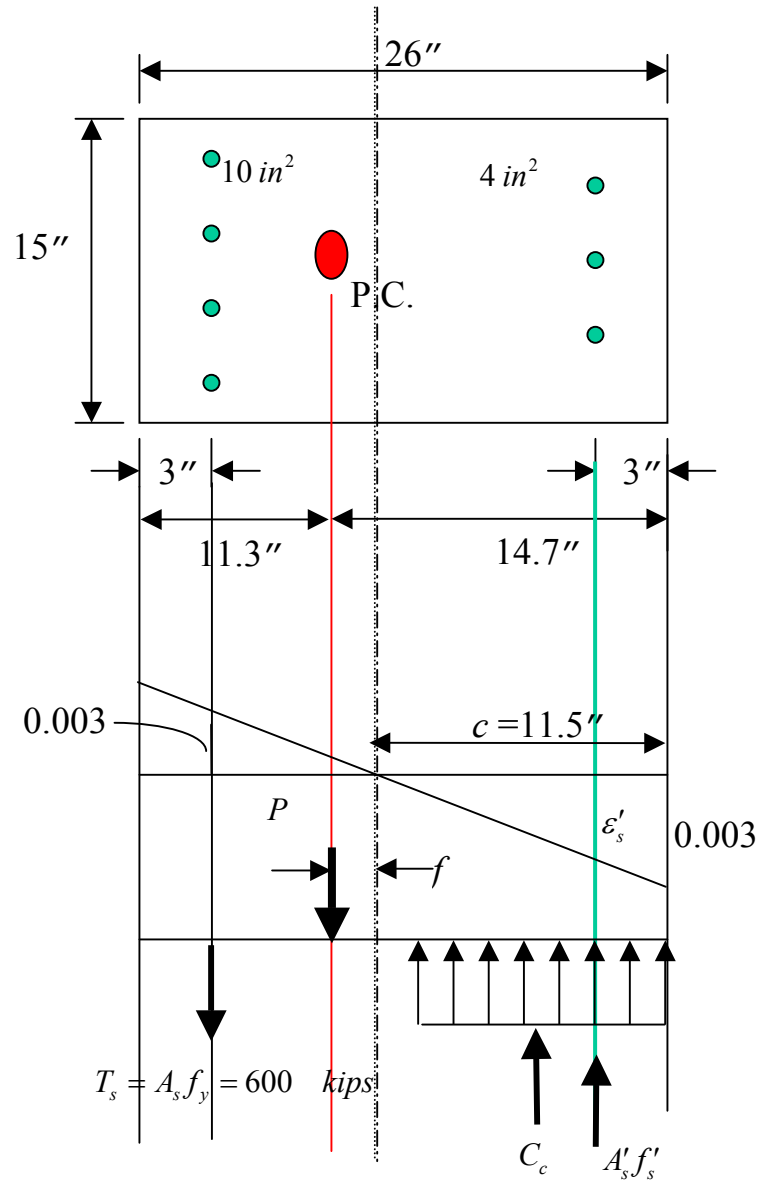
$$\varepsilon'_s = \varepsilon_u \frac{c - d'}{c}$$

$$\varepsilon'_s = 0.003 \times \frac{11.5 - 3.0}{11.5} = 0.00220 \geq 0.00207$$

Therefore, compression steel yields

$$C_c = 0.85(4)(15)(9.78) = 499 \text{ kips}$$

$$P_n = 499 + 240 - 600 = 139 \text{ kips}$$



$$M_n = 499(14.7 - \frac{9.78}{2}) + 240(14.7 - 3) + 600(11.3 - 3) = 12,683 \text{ in-kips}$$



Case  $\varepsilon_s = 0$        $\varepsilon_u = 0.003$

$$c = 23 \text{ inches}$$

$$a = 0.85c = 0.85 \times 23 = 19.55 \text{ inches}$$

Check to find out whether the compression steel yield or not.

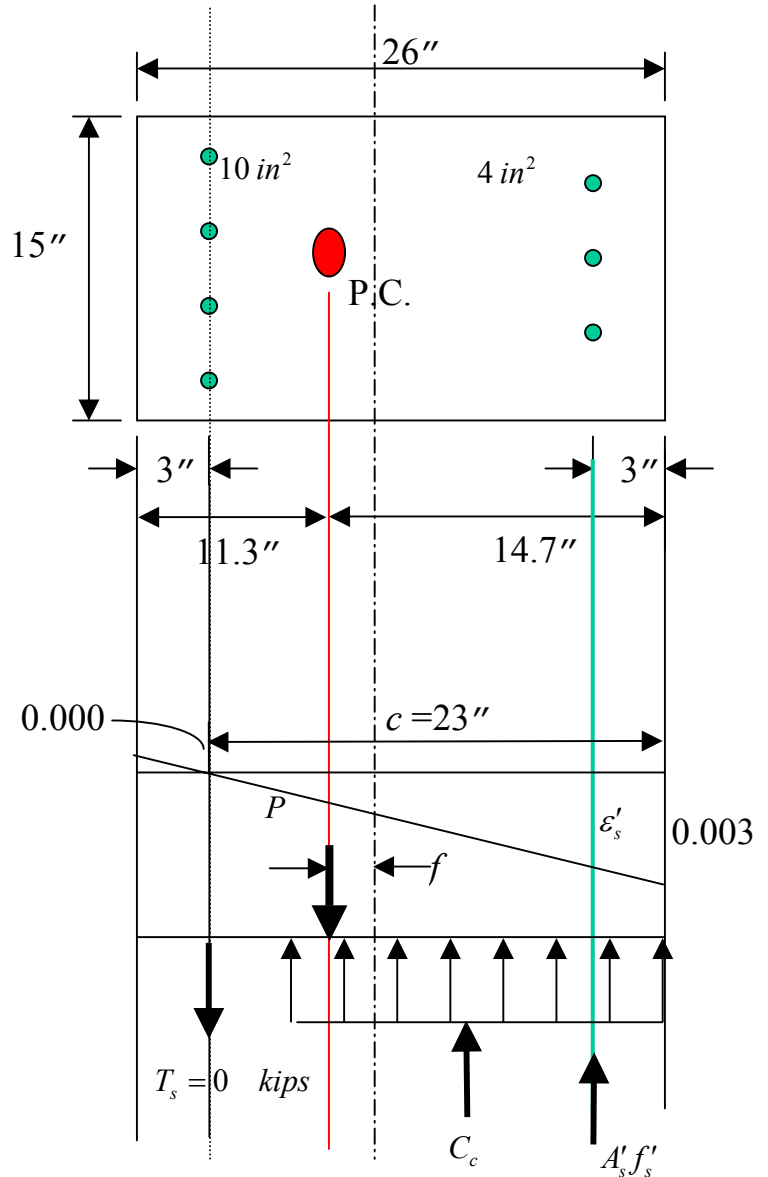
$$\varepsilon'_s = \varepsilon_u \frac{c - d'}{c}$$

$$\varepsilon'_s = 0.003 \times \frac{19.55 - 3.0}{19.55} = 0.00254 \geq 0.00207$$

Therefore, compression steel yields

$$C_c = 0.85(4)(15)(19.55) = 997 \text{ kips}$$

$$P_n = 997 + 240 = 1237 \text{ kips}$$



$$M_n = 997\left(14.7 - \frac{19.55}{2}\right) + 240(14.7 - 3) = 7,720 \text{ in-kips}$$

$$a = 0.85c = 0.85 \times 17.25 = 14.66 \text{ inches}$$

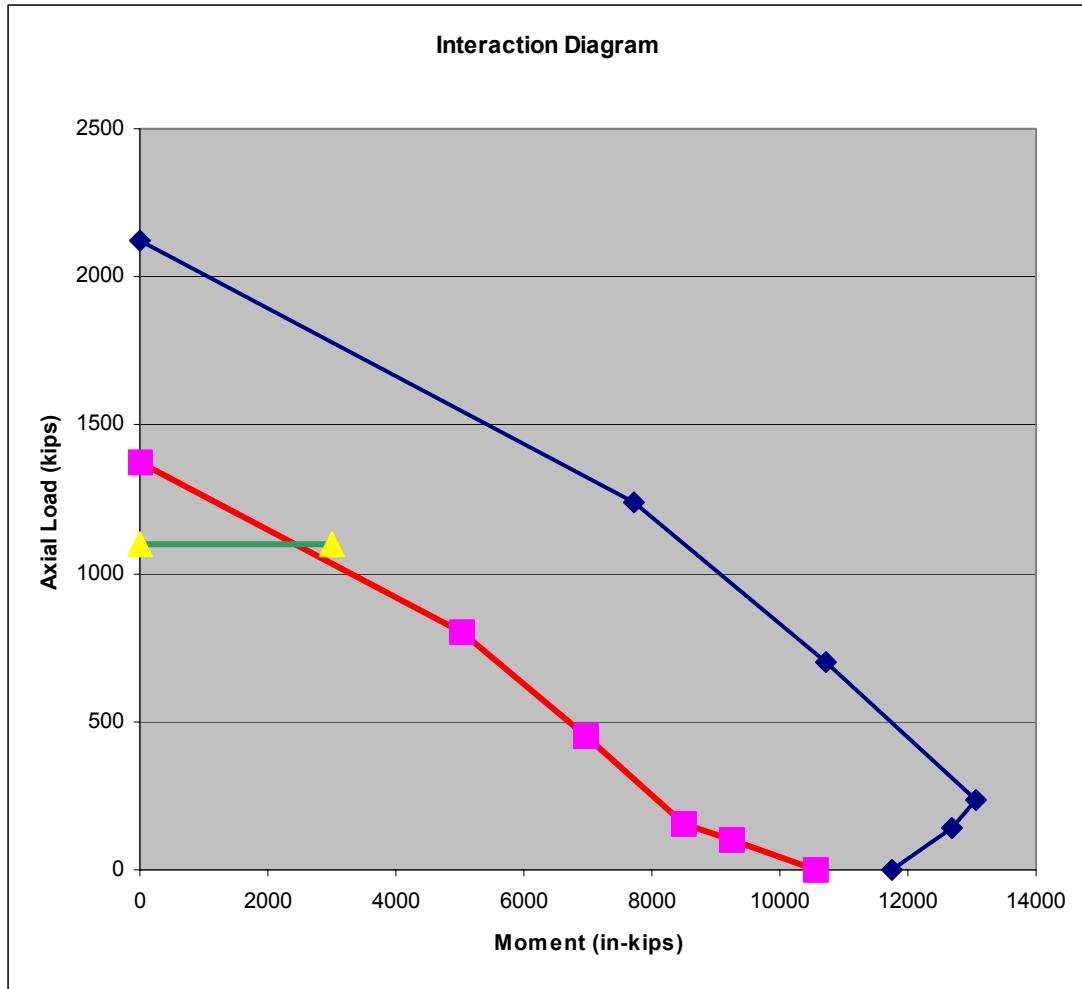
$$\varepsilon'_s = \varepsilon_u \frac{c - d'}{c}$$

Therefore, compression steel yields

$$C_c = 0.85(4)(15)(14.66) = 748 \text{ kips}$$

$$M_n = 748(14.7 - \frac{14.66}{2}) + 240(14.7 - 3) + 290(11.3 - 3) = 10,728 \text{ in-kips}$$





$\varepsilon_u$	$\varepsilon_s$	$P$	$M_n$	$\phi$	$\phi P_n$	$\phi M_n$
0.003	-0.003	2120	0	0.65	1378	0
0.003	0	1237	7730	0.65	804	5024
0.003	0.001	698	10728	0.65	454	6973
0.003	0.00207	238	13070	0.65	155	8496
0.003	0.003	139	12683	0.73	101	9259
0.003	0.005	0	11753	0.90	0	10578

$$\varepsilon_t = 0.003 \rightarrow \phi = 0.48 + 83\varepsilon_t = 0.48 + 83 \times 0.003 = 0.73$$

$$\varepsilon_t = 0.001 \rightarrow \phi = 0.48 + 83\varepsilon_t = 0.48 + 83 \times 0.001 = 0.563 \rightarrow \phi = 0.65$$

$$\text{limit for pure axial load} \quad 0.80\phi = 0.8 \times 1378 = 1100 \text{ kips}$$