5. Flexural Analysis and Design of Beams

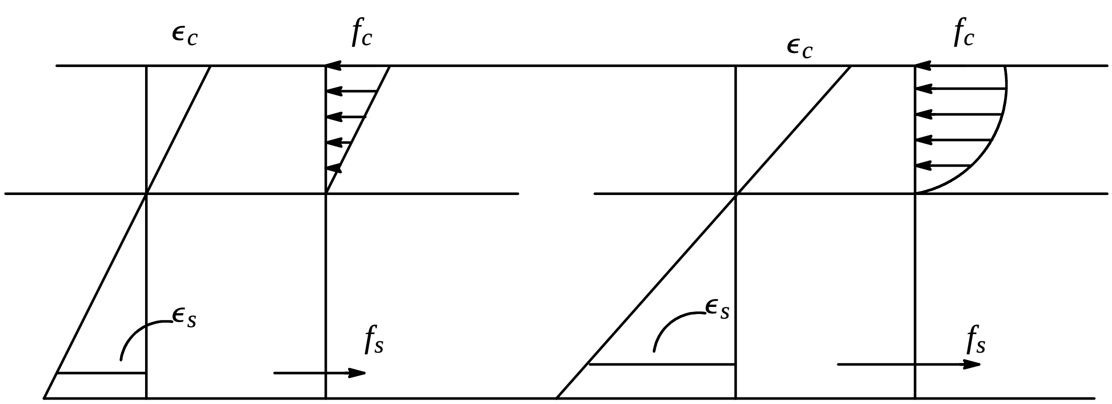
5.1. Reading Assignment

Chapter 3 of text

5.2. Introduction

It is of interest in structural practice to calculate those stresses and deformations which occur in a structure in service under design load. For reinforced concrete beams this can be done by the methods just presented, which assume elastic behavior of both materials. It is equally, if not more, important that the structural engineer be able to predict with satisfactory accuracy the ultimate strength of a structural member. By making this strength larger by an appropriate amount than the largest loads which can be expected during the lifetime of the structure, an adequate margin of safety is assured. Until recent times, methods based on elastic analysis like those just presented have been used for this purpose. It is clear, however, that at or near the ultimate load, **stresses are no longer proportional to strains**.

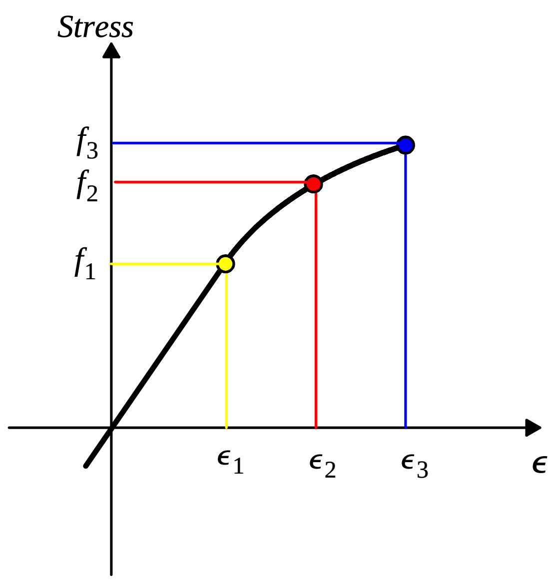
At high loads, close to ultimate, the distribution of stresses and strains is that of figure 2 rather that the elastic distribution of stresses and strains given in figure 1 below. More realistic methods of analysis, based on actual inelastic rather than an assumed elastic behavior of the materials and results many experimental research, have been developed to predict the ultimate strength.

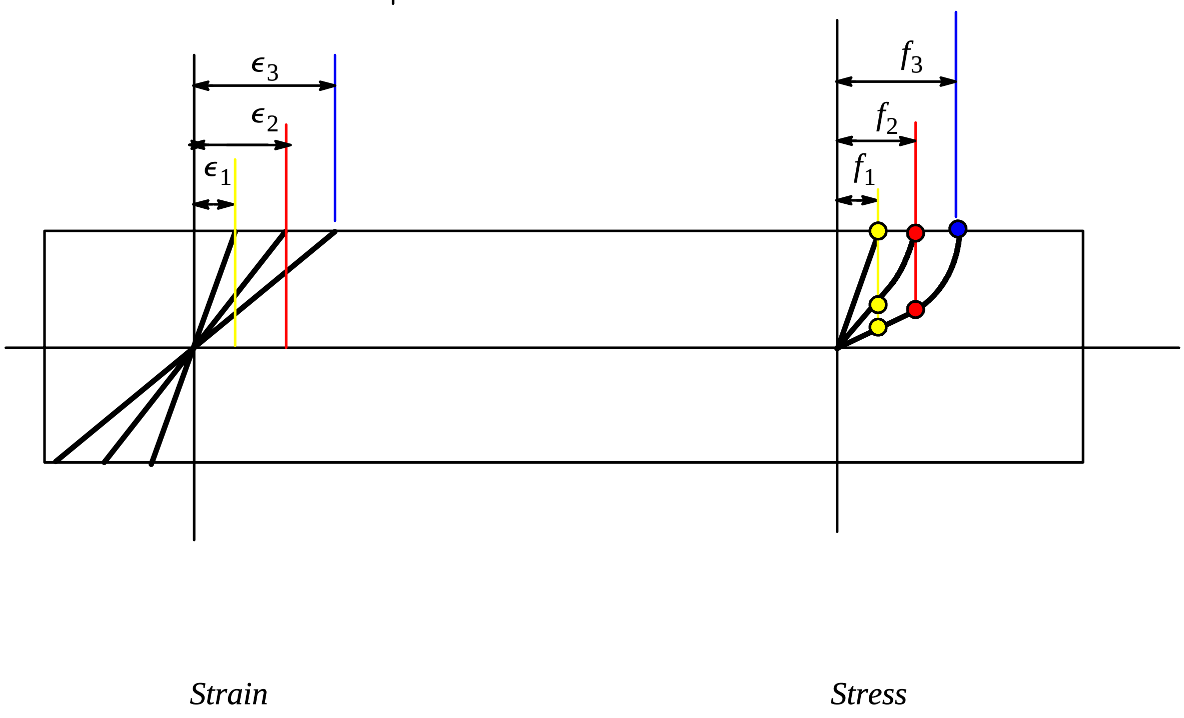


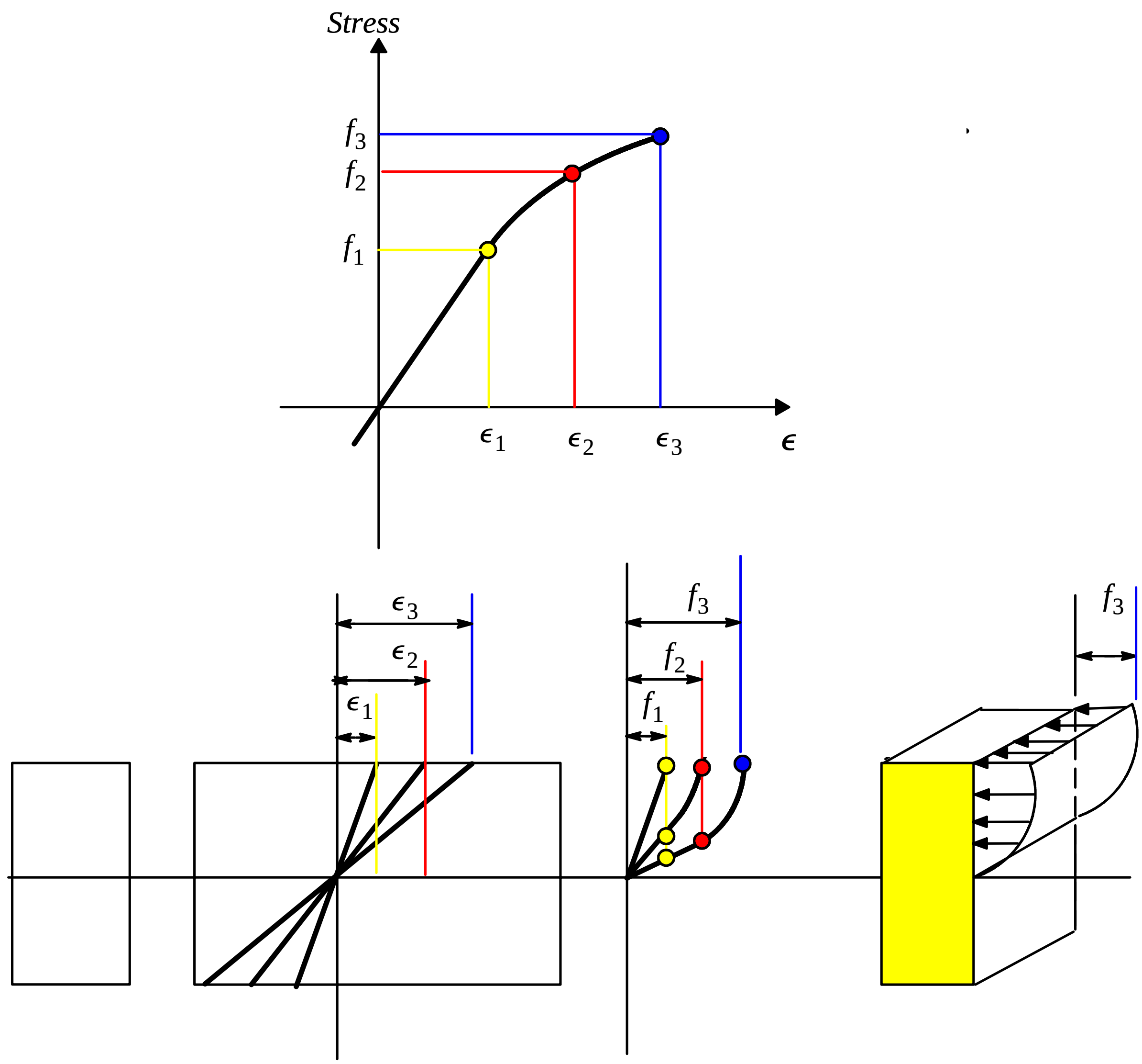
1

2

As progressively increasing bending moments are applied to the beam, the strains will increase as exemplified by ε_1, ε_2, and ε_3 as shown below. Corresponding to these strains and their linear variation from the neutral axis, the stress distribution will look as shown.







Strain *Stress*

Figure 5.1. Cracks, Strains, and Stresses in test beam (From Nawy's Book).

5.1

5.3. Flexure Strength

As it was mentioned earlier it is important that the structural engineer be able to predict with satisfactory accuracy the ultimate strength of a structural member. It is important to know that at or near the ultimate load, **stresses are no longer proportional to strains**.

Actual inspection of many concrete stress-strain curves which have been published, show that the geometrical shape of the stress distribution is quite varied and depends on a number of factors such as cylinder strength, the rate, and duration of loading.

Below is a typical stress distribution at the ultimate load.

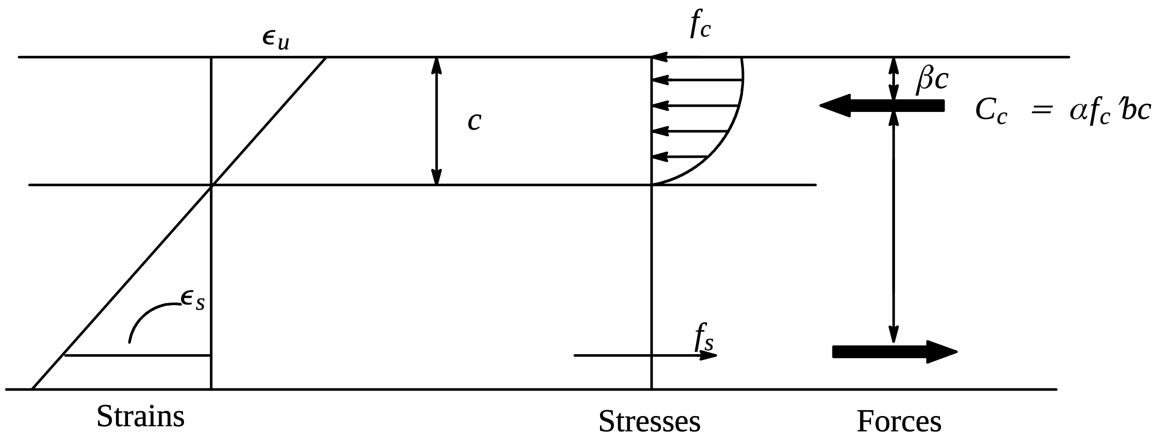


Figure 5.2. Strain, Stress, and Force Diagrams

5.4. Two Different Types of Failure

There are two possible ways that a reinforced beam can fail:

- Beam will fail by tension of steel
  Moderate amount of reinforcement is used. Steel yields suddenly and stretches a large amount, tension cracks become visible and widen and propagate upward (Ductile Failure)
- Compression failure of concrete
  Large amount of reinforcement is used. Concrete fails by crushing when strains become so large (0.003 to 0.004). Failure is sudden, an almost explosive nature and occur with no warning (Brittle Failure).

In a rectangular beam the area that is in compression is *bc*, and the total compression force on this area can be expressed as $C = f_{av}bc$, where f_{av} is the average compression stress on the area *bc*. Evidently, the average compression stress that can be developed before failure occurs becomes larger the higher the cylinder strength f_c' of the particular concrete. Let

$$\alpha = \frac{f_{av}}{f_c'} \tag{5.7}$$

then

$$C_c = f_{av}bc = \alpha f_c' bc \tag{5.8}$$

compression force is applied at βc distance from top fiber, and *c* is the distance of the N.A. from top fiber.

Based on research we have:

$$\alpha = 0.72 - \frac{f_c' - 4,000}{1000} \times 0.04 \quad \textit{and} \quad 0.56 < \alpha < 0.72$$

$$\beta = 0.425 - \frac{f_c' - 4,000}{1000} \times 0.025 \quad \textit{and} \quad 0.324 < \beta < 0.425$$

FORCES

From equilibrium we have $C_c = T$ or

$$\alpha f_c' bc = A_s f_s \tag{5.9}$$

$$M = TZ = A_s f_s (d - \beta c) \tag{5.10}$$

or

$$M = C_c Z = \alpha f_c{}' bc\,(d - \beta c) \tag{5.11}$$

5.5. Tension Failure

$$f_s = f_y \quad \textit{steel yielding} \tag{5.12}$$

From Eq. (5.9) we have

$$c = \frac{A_s f_y}{\alpha b f_c{}'} \times \frac{d}{d} = \frac{A_s}{bd}\frac{f_y d}{\alpha f_c{}'} = \rho \frac{f_y}{f_c{}'}\frac{d}{\alpha} \tag{5.13}$$

Substitute c from Eq. (5.13) in Eq. (5.10)

$$M_n = A_s f_y \left(d - \rho \frac{\beta}{\alpha} \frac{f_y}{f_c{}'} d \right) \tag{5.14}$$

with the specific, experimentally obtained values for α and β we always have

$$\frac{\beta}{\alpha} = 0.59 \ \text{ for } \ f_c{}' = 4{,}000\,\textit{psi or any other strength} \tag{5.15}$$

Therefore, Eq. (5.14) simplifies as

$$M_n = A_s f_y \left(d - 0.59\rho \frac{f_y}{f_c{}'} d \right) \tag{5.16}$$

or

$$\boxed{M_n = \rho b d^2 f_y \left(1 - 0.59\rho \frac{f_y}{f_c{}'} \right)} \tag{5.17}$$

where M_n = nominal moment capacity.

5.6. Compression Failure

In this case, the criterion is that the compression strain in the concrete becomes $\varepsilon_u = 0.003$, as previously discussed. The steel stress f_s, not having reached the yield point, is proportional to the steel strain, ε_s; i.e. according to Hooke's law:

$$\epsilon_u = 0.003 \qquad (ACI\ 10.2.3), \qquad and \qquad f_s < f_y \tag{5.18}$$

$$f_s = E_s\,\epsilon_s \qquad \text{Hooks law, since } f_s < f_y \tag{5.19}$$

from similar triangles we have

$$\frac{\epsilon_u}{c} = \frac{\epsilon_s}{d - c} \quad \rightarrow \quad \epsilon_s = \epsilon_u \frac{d - c}{c} \tag{5.20}$$

substitute Eq. (5.20) in Eq. (5.19)

$$f_s = E_s\,\epsilon_s = E_s\,\epsilon_u \frac{d - c}{c} < f_y \tag{5.21}$$

From Eq. (5.9) we have

$$\alpha f_c' bc = A_s f_s = A_s\,E_s\,\epsilon_u \frac{d - c}{c} \tag{5.22}$$

Using Eq. (5.22) solve for c, and then find M_n, the nominal moment capacity.

5.7. Balance Steel Ratio

We like to have tension failure, because it gives us warning, versus compression failure which is sudden. Therefore, we want to keep the amount of steel reinforcement in such manner that the failure will be of tension type.

Balanced steel ratio, ρ_b represents the amount of reinforcement necessary to make a beam fail by crushing of concrete at the same load that causes the steel to yield. This means that neutral axis must be located at the load which the steel starts yielding and concrete starts reaching its compressive strain of $\varepsilon_u = 0.003$. (ACI 10.2.3)

$$c^b = \frac{\epsilon_u}{\epsilon_y + \epsilon_u} d \tag{5.23}$$

$$T = C \quad \rightarrow \quad A_s^b\,f_y = \alpha\,f_c{}'\,bc^b \tag{5.24}$$

$$A_s^b\,f_y = \rho_b\,bd\,f_y = \alpha\,f_c{}'\,b\,\frac{\epsilon_u}{\epsilon_u + \epsilon_y} d \tag{5.25}$$

$$\boxed{\overline{\rho_b} = \alpha\,\frac{f_c{}'}{f_y}\,\frac{\epsilon_u}{\epsilon_u + \epsilon_y}} \tag{5.26}$$

5.8. Strain Limits Method for Analysis and Design (ACI 318–2002).

In "Strain Limits Method," sometime referred to as the "Unified Method," the nominal flexural strength of a concrete member is reached when the net compressive strain in the extreme compression fiber reaches the ACI code-assumed limit of 0.003 in/in (ACI 10.2.3). It also hypothesized that when the net tensile strain in the extreme tension steel, ε_t = 0.005 in/in, the behavior is fully ductile. The concrete beam sections characterized as "Tension-Controlled," with ample warning of failure as denoted by excessive deflection and cracking.

If the net tensile strain in the extreme tension fibers, ε_t, is small, such as in compression members, being equal or less than a "Compression-Controlled" strain limit, a brittle mode of failure is expected with a sudden and explosive type of failure. Flexural members are usually tension-controlled. However, some sections such as those subjected to small axial loads, but large bending moments, the net tensile strain, ε_t, in the extreme tensile fibers, will have an intermediate or transitional value between the two strain limit states, namely, between the compression-controlled strain limit of

$$\epsilon_t = \frac{f_y}{E_s} = \frac{60\ ksi}{29,000\ ksi} = 0.002 \tag{5.27}$$

and the tension-controlled strain limit ε_t = 0.005 in/in. Figure 5.3 (ACI Figure. R9.3.2 page 100) shows these three zones as well as the variation in the strength reduction factors applicable to the total range of behavior.

5.8.1. Variation of Φ as a Function of Strain

Variation of the ϕ value for the range of strain between $\varepsilon_t = 0.002$ in/in and $\varepsilon_t = 0.005$ in/in can be linearly interpolated:

$$0.65 \le (\phi = 0.48 + 0.84\epsilon_t) \le 0.90 \qquad \textit{Tied Column}$$

$$0.70 \le (\phi = 0.57 + 0.67\epsilon_t) \le 0.90 \qquad \textit{Spiral Columnm} \tag{5.28}$$

5.8.2. Variation of Φ as a Function of Neutral Axis Depth Ratio c/d

$$0.65 \le \left(\phi = 0.23 + \frac{0.25}{c/d_t}\right) \le 0.90 \qquad \textit{Tied Column}$$

$$0.70 \le \left(\phi = 0.37 + \frac{0.20}{c/d_t}\right) \le 0.90 \qquad \textit{Spiral Columnm} \tag{5.29}$$

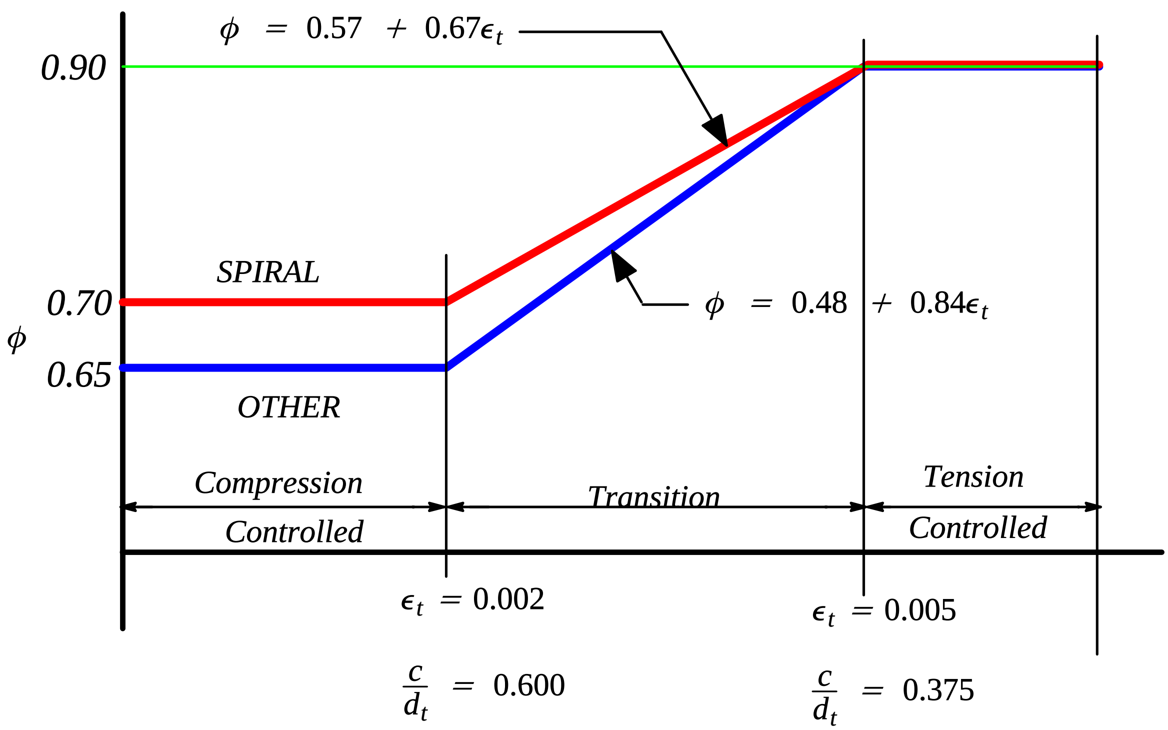
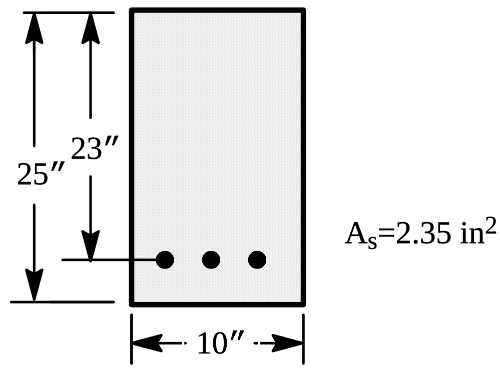


Figure 5.3. Example. Calculate Nominal Moment Capacity of a Beam for Fy = 60 ksi

5.9. Example. Calculate Nominal Moment Capacity of a Beam

Determine the nominal moment M_n at which the beam given below will fail.



Given

$$f_c{}' = 4,000 \ psi$$

$$f_y = 60,000 \ psi$$

Solution

$$\rho = \frac{A_s}{bd} = \frac{2.35}{10 \times 23} = 0.0102$$

$$c = \rho \ \frac{f_y}{f_c{}'}\frac{d}{\alpha}$$

$$c = 0.0102 \times \frac{60}{4} \times \frac{23}{0.72} = 4.89 \ in$$

$$\frac{c}{d} = \frac{4.89}{23} = 0.213 < \frac{c}{d_t} = 0.375 \qquad \textit{Tension failure}$$

$$M_n = \rho \ f_y \ bd^2(1 - 0.59\frac{\rho \ f_y}{f_c{}'})$$

$$M_n = (0.0102) \times (60 \ ksi) \times (10 \ in) \times (23 \ in)^2 \times \left(1 - 0.59 \times (0.0102) \times \frac{60}{4}\right)$$

$$= 2,950,000 \ lb - in = 246 \ k - ft$$

5.10. Prediction of Nominal Strength in Flexure by Equivalent Rectangular Stress Block

- Represents an extension of the empirical method.
- Simpler than empirical method - No secondary calculation necessary to locate centroid (always at stress block center).
- Allows for considerations and analyses of non-rectangular sections.
- Must be developed such that it gives the same answer as empirical method - requires <u>same total compression force and same centroid location.</u>
- Development of the method:

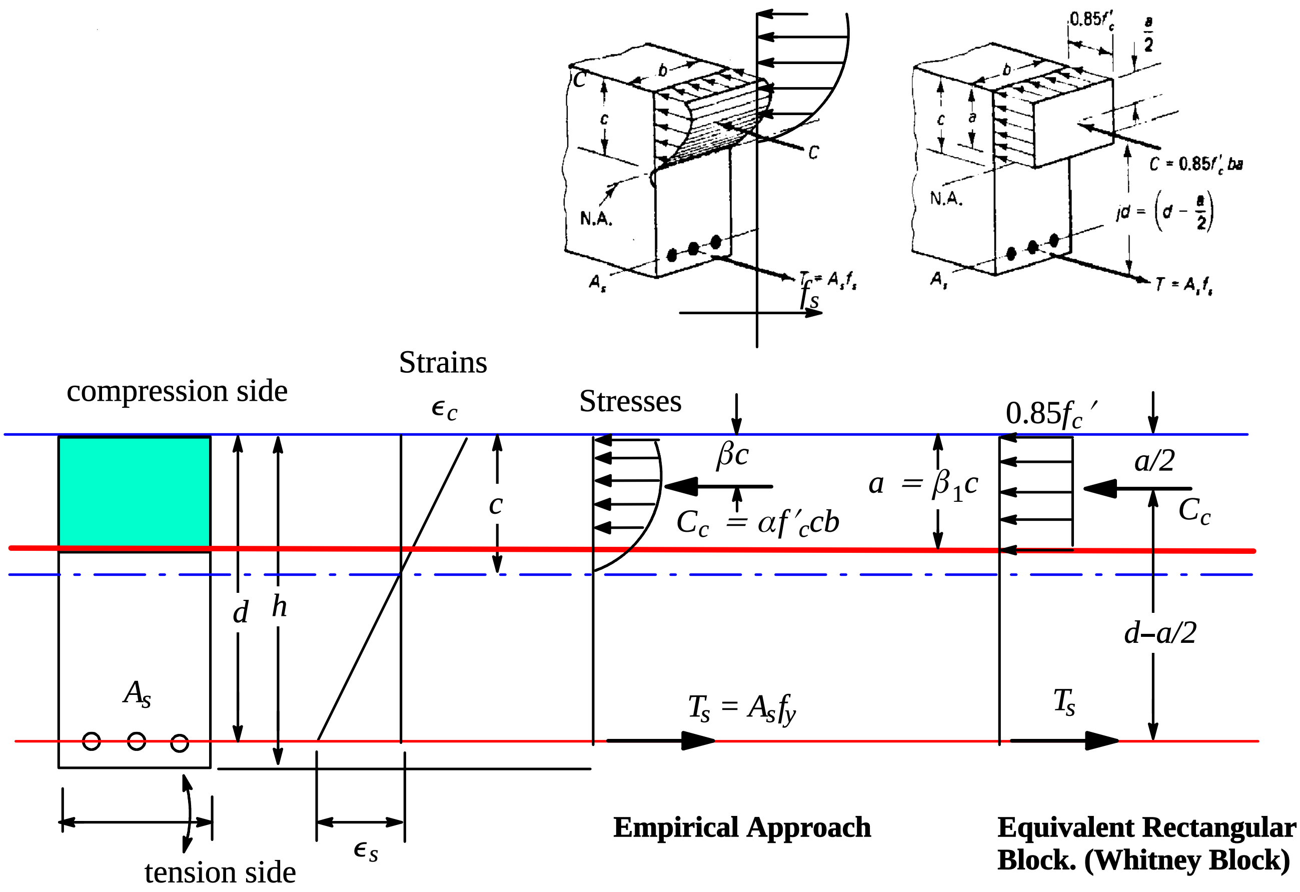


Figure 5.4. Equivalent Rectangular Block (From Nawy's Book).

Require the forces to have the same location:

$$a = \beta_1 c \qquad (ACI\ \ 10.2.7)$$

$$C_c = \alpha f_c{}' cb = \gamma f_c{}' ab \quad \text{from which } \gamma = \alpha \frac{c}{a}$$

$$\gamma = \frac{\alpha}{\beta_1}$$

$$\beta_1 = 2\beta \quad \text{and} \quad \gamma = 0.85 \qquad ACI\ 10.2.7$$

ACI 10.2.7.3

$$\beta_1 = 0.85 - 0.05\frac{f_c{}' - 4000}{1000} \quad \text{and } 0.65 \le \beta_1 \le 0.85 \qquad ACI\ 10.2.7.3$$

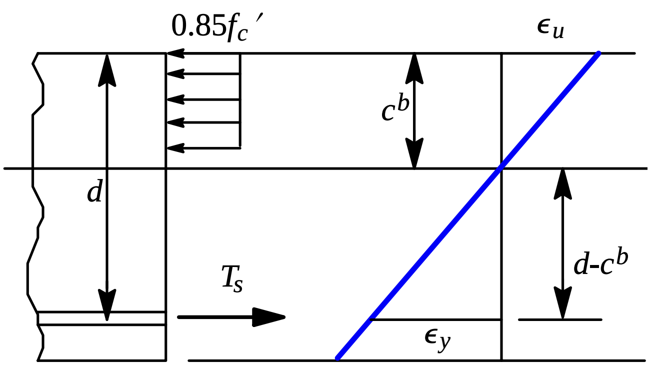
$$C_c = 0.85 f_c{}' ab \quad \text{remember } \gamma = 0.85 \text{ and } a = \beta_1 c$$

For balanced steel ratio we have

$$T = C$$

$$\overline{\rho_b} f_y bd = 0.85 f_c{}' a^b b = 0.85 f_c{}' \beta_1 bc^b$$

$$\overline{\rho_b} = 0.85 \beta_1 \frac{f_c{}'}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_y}$$



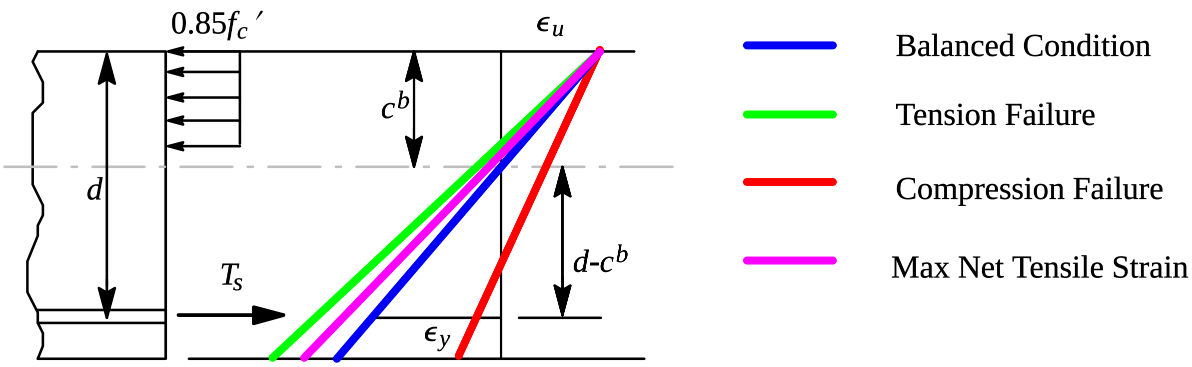
substituting $\varepsilon_u = 0.003$ and $E_s = 29{,}000$ ksi

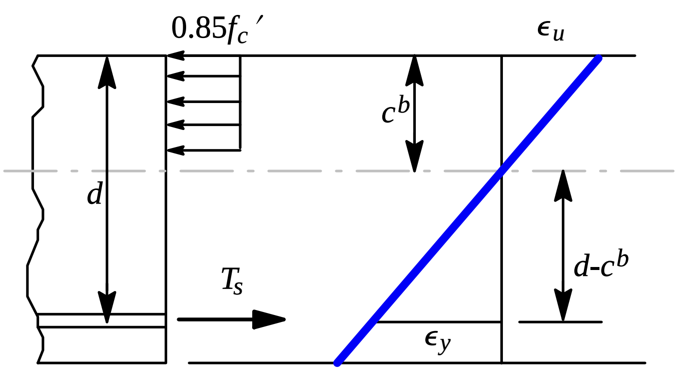
$$\overline{\rho_b} = 0.85\beta_1 \frac{f_c{}'}{f_y}\left(\frac{87{,}000}{87{,}000 + f_y}\right) \qquad (5.30)$$

See Eq. 8-1 of ACI 8.4.3

ACI 10.3.5. Maximum Net Tensile Strain

For nonprestressed flexural members and prestressed members with axial load less than $0.10 f'_c A_a$ the net tensile strain ϵ_t at nominal strength shall not be less than 0.04.

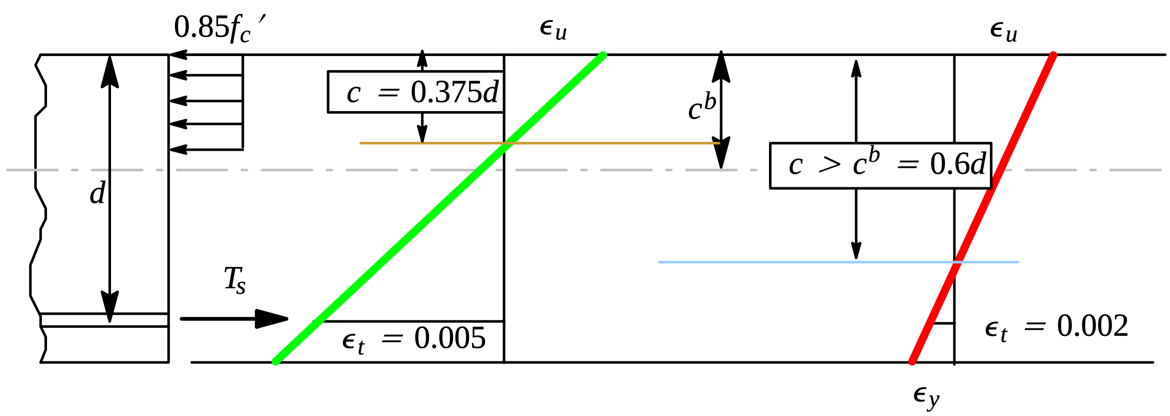


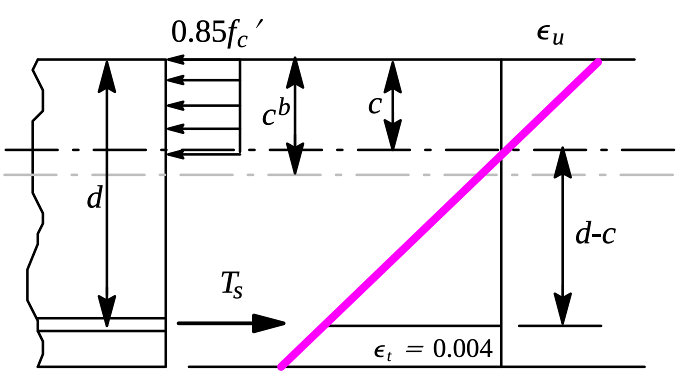


$$\frac{c^b}{d_t} = \frac{87,000}{87,000 + f_y}$$

$$f_y = 60,000 \quad psi$$

$$\frac{c^b}{d_t} = \frac{87}{87 + 60} = 0.60$$



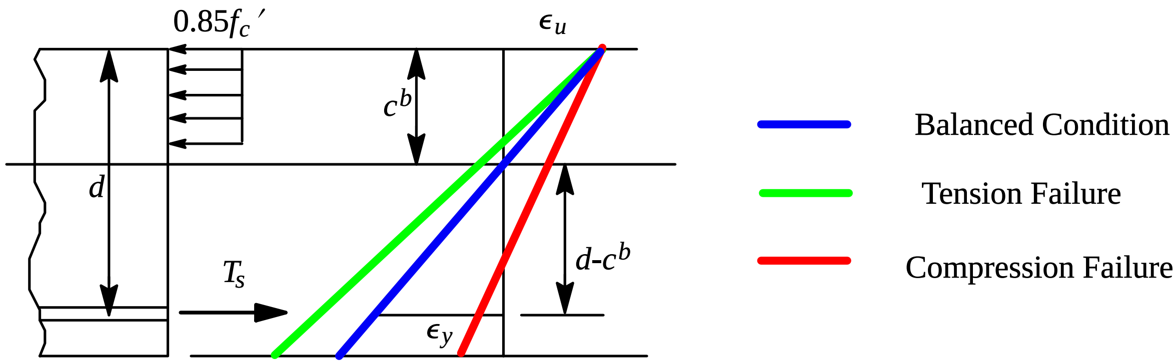


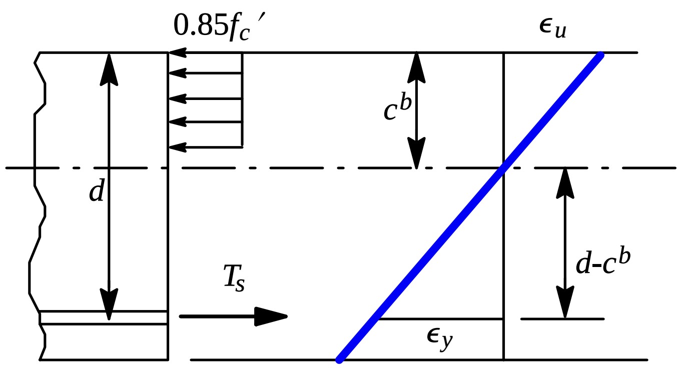
Max net Tensile Strain

$$f_y = 60,000 \quad psi$$

$$\frac{c}{d_t} = \frac{0.003}{0.003 + 0.004} = 0.429$$

$f_y = 40ksi$

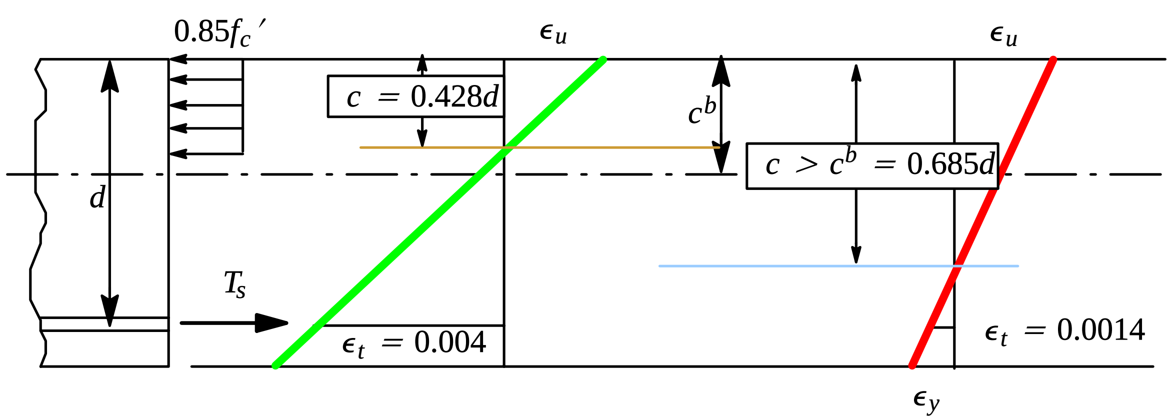




$$\frac{c^b}{d_t} = \frac{87,000}{87,000 + f_y}$$

$$f_y = 40 \quad ksi$$

$$\frac{c^b}{d_t} = \frac{87}{87 + 40} = 0.685$$



$$c = 0.625 \times 0.685 = 0.428$$

5.10.1. Example

Consider the same example problem given in Section 5.9.

$$\rho \quad = \quad \frac{A_s}{bd} \quad = \quad \frac{2.35}{10 \times 23} \quad = \quad 0.0102$$

$$0.85 f_c{}' \, ab \quad = \quad A_s \, f_y \quad \rightarrow \quad a \quad = \quad \frac{(2.35 \ \ in^2) \times (60,000 \ \ psi)}{0.85 \times (4,000 \ \ psi) \times (10 \ \ in)} \quad = \quad 4.15 \ \ in$$

$$c \ = \ a/\beta_1 \ = \ 4.15/0.85 \ = 4.88$$

$$\frac{c}{d} \ = \ \frac{4.88}{23} = 0.212 \ < 0.375$$

Tension failure

Therefore the nominal moment capabit will be:

$$M_n \ = A_s f_y \, (d \ - \frac{a}{2}) \ = (2.35 \ \ in^2)) \times (60,000 \ \ psi) \times (23 \ - 2.07) \ = 246,000 \ \ lb\text{–}ft = 246 \ \ kip\text{–}ft$$

$$= \ 2,950,000 \ \ lb \ - in \quad = \quad 246 \ \ k \ - ft$$

$$\phi \ = 0.9$$

$$M_u \ = \phi M_n \ = 0.9 \times 246 \quad = \quad 221.4 \ \ kip \ - ft$$

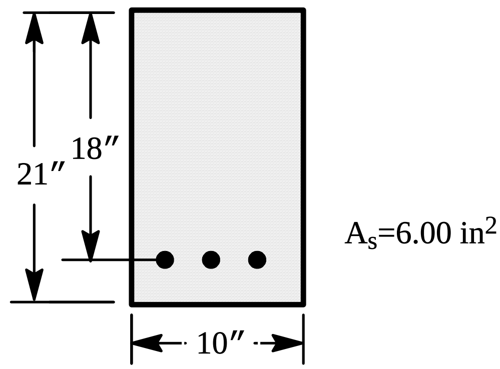
5.10.2. Example. Calculate Nominal Moment Capacity of a Beam

Determine if the beam shown below will fail in tension or compression.

Given

$$f_c{}' = 4,000 \ psi$$

$$f_y = 60,000 \ psi$$



Solution

$$a = \frac{A_s \ f_y}{0.85 f_c{}' \ b}$$

$$a = \frac{6 \times 60 \ ksi}{0.85 \times 4 \ ksi \times 10} = 10.59 \ in$$

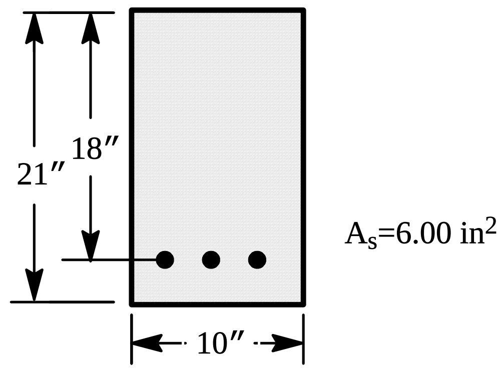
$$c = \frac{a}{\beta_1} = \frac{10.59}{0.85} = 12.46$$

$$\frac{c}{d} = \frac{12.46}{18} = 0.69 > 0.6$$ *Compression failure*

Hence, A_s does not yield and the strain is smaller than 0.02 in/in. Brittle failure results. This beam does not satisfy ACI Code requirement.

5.10.3. Example. Calculate Nominal Moment Capacity of a Beam

Determine if the beam shown below will fail in tension or compression.



Given

$$f_c' = 4,000 \; psi$$

$$f_y = 40,000 \; psi$$

Solution

$$a = \frac{A_s \; f_y}{0.85 f_c' \; b}$$

$$a = \frac{6 \times 40 \; ksi}{0.85 \times 4 \; ksi \times 10} = 7.06 \; in$$

$$c = \frac{a}{\beta_1} = \frac{7.06}{0.85} = 8.31 \; in$$

$$\frac{c}{d} = \frac{8.31}{18} = 0.46 > 0.428 < 0.685 \quad \textit{Transition Zone}$$

Hence, the beam is in the transition zone, tension steel yeilds. A reduced value of ϕ should be used.