4. Flexural Members - Elastic

4.1. Reading Assignment:

Section 3.3 of Text.

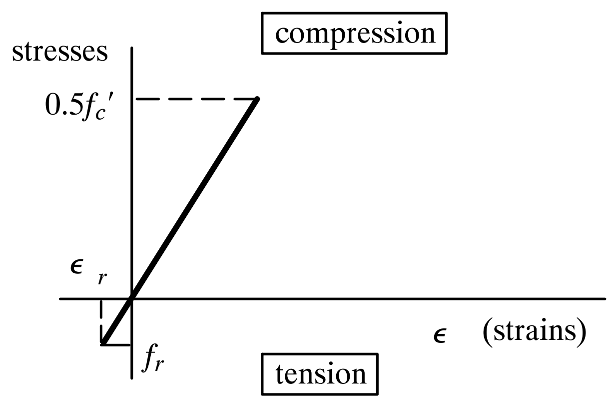
4.2. Unreinforced Concrete Beam:

1. Assumptions:

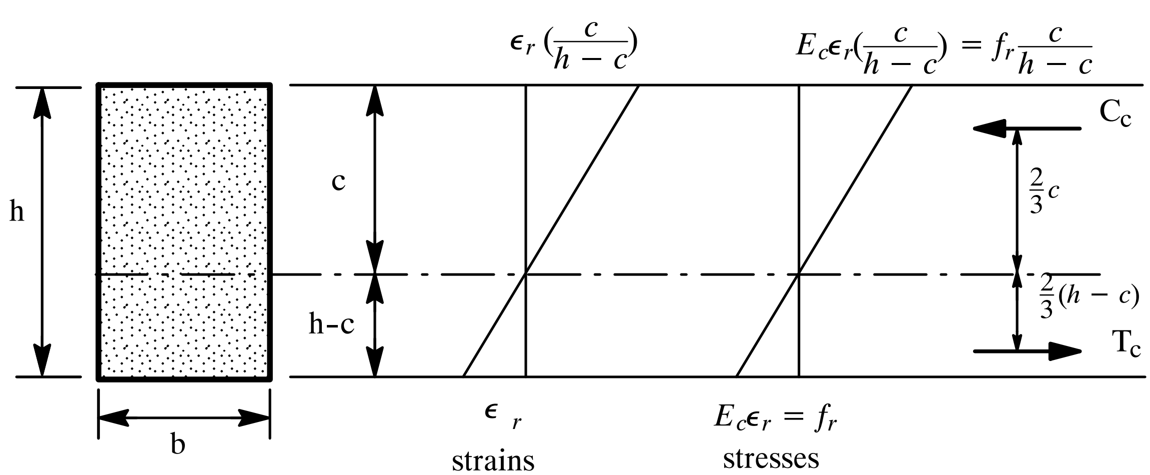
a Strains vary linearly with distance from N.A.

b Stress-strain relationship linear

c. Concrete is capable of some tension.



2. Plain concrete beam. - before cracking



$$C_c = \frac{1}{2}f_r\left(\frac{c}{h - c}\right)cb$$

$$T_c = \frac{1}{2} f_r (h - c) b$$

Equilibrium $C_c = T_c$

$$\frac{1}{2} f_r \left(\frac{c}{h - c} \right) cb = \frac{1}{2} f_r (h - c) b$$

simplify

$$c^2 = (h - c)^2 \quad \rightarrow \quad c = \mp (h - c) \quad \rightarrow \quad c = \frac{h}{2}$$

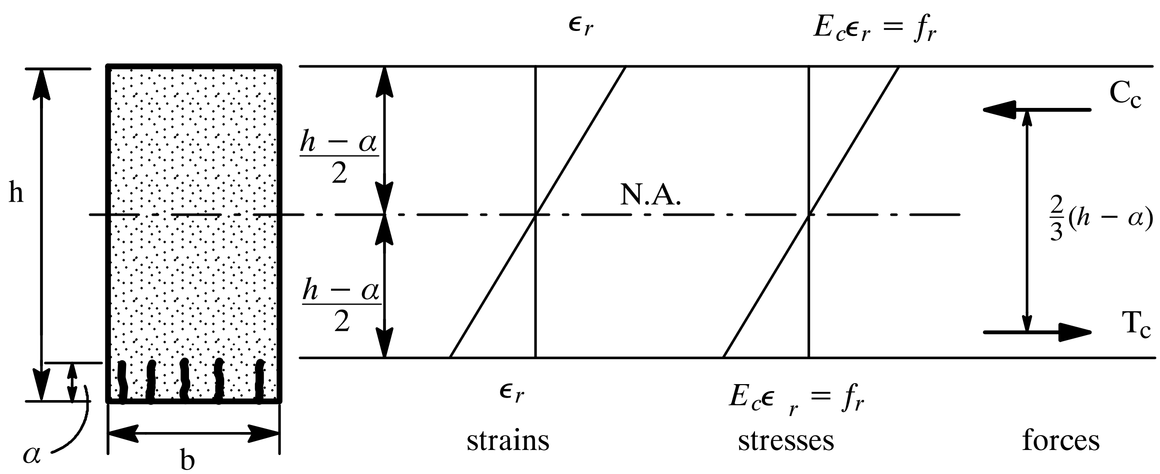
Therefore, moment when cracking is about to occur:

$$M_{cr} = (Moment\ Arm) \times Force$$

$$M_{cr} = \frac{2}{3} h T_c = \frac{2}{3} h C_c = \frac{1}{6} f_r h^2 b$$

3. Plain concrete beam - After cracking

let cracking extend α distance into beam.



$$C_c = T_c = \frac{1}{2} f_r \left(\frac{h - \alpha}{2} \right) b$$

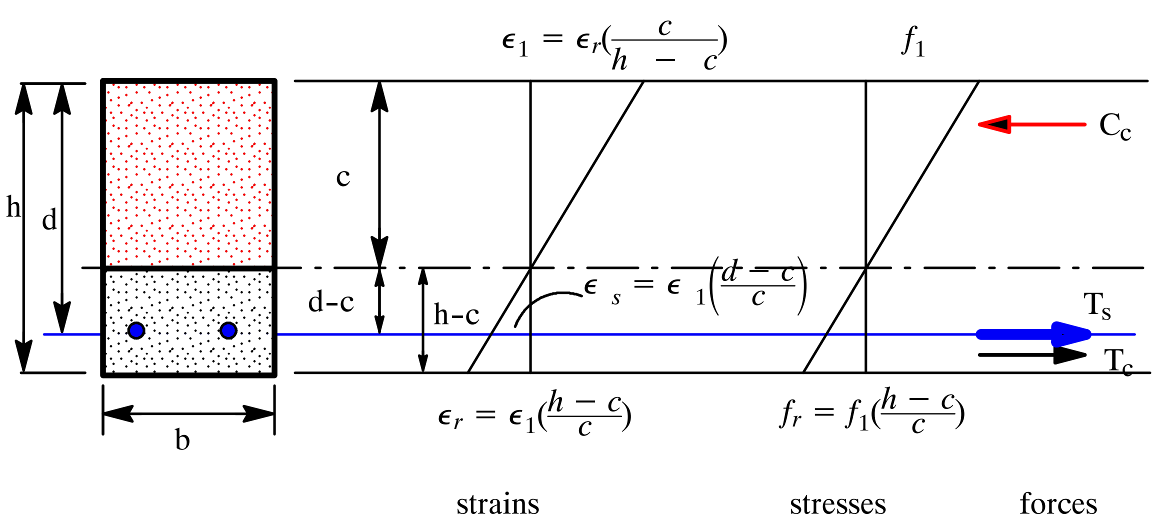
$$M = \frac{2}{3} (h - \alpha) C_c = \frac{1}{6} f_r b (h - \alpha)^2$$

When $\alpha = 0$, we get the same result as the one without cracking.

4.3. Reinforced Concrete Beam - Uncracked

1. Assumptions:
   a. Strains vary linearly with distance from N.A.
   b. Linear stress-strain relationship;
   c. Strain compatibility between steel and concrete $\varepsilon = \varepsilon_s = \varepsilon_c$

2. Reinforced concrete beam. Before cracking



Equilibrium:

Summation of forces in tension must be equal to the summation of forces in compression. Express all forces in terms of a single stress f_1:

$$C_c = \frac{1}{2} f_1 cb$$

$$T_c = \frac{1}{2} f_1 \left(\frac{h - c}{c}\right)(h - c)b = \frac{1}{2} f_1 \frac{b}{c}(h - c)^2$$

The tension force in steel, T_S, can be determined as (compensate for area of concrete taken by steel):

$$T_s = f_s A_s - f_c A_s$$

$$T_s = f_s A_s - \epsilon_1 \frac{(d - c)}{c} E_c A_s$$

$$= E_s \epsilon_s A_s - \epsilon_1 \frac{(d - c)}{c} E_c A_s$$

$$= E_s \epsilon_1 (\frac{d - c}{c}) A_s - \epsilon_1 \frac{(d - c)}{c} E_c A_s$$

substitute for ε_1 and simplify

$$T_s = E_s \frac{f_1}{E_c} (\frac{d - c}{c}) A_s - \frac{f_1}{E_c} \frac{(d - c)}{c} E_c A_s$$

$$T_s = f_1 (\frac{d - c}{c}) \left[\frac{E_s A_s}{E_c} - \frac{E_c A_s}{E_c} \right] \quad \leftarrow \quad \textit{factor out}$$

$$T_s = f_1 (\frac{d - c}{c}) [(n - 1) A_s] \quad \leftarrow \quad \textit{substitute} \text{ for } n = \frac{E_s}{E_c}$$

Equilibrium:

$$\sum T = \sum C \qquad or \qquad C_c = T_s + T_c$$

$$\frac{1}{2} f_1 \, cb = f_1 (\frac{d - c}{c}) A_s (n - 1) + \frac{1}{2} f_1 \frac{(h - c)^2}{c} b$$

Multiply both sides by $2c/f_1$:

$$c^2 b = 2 (d - c) A_s (n - 1) + (h - c)^2 b$$

We would like to obtain the ratio *c/d* in terms of known section properties. Expand the previous equation:

$$c^2 b = 2 \, d A_s (n - 1) - 2c (n - 1) A_s + h^2 b - 2hcb + c^2 b$$

Simplify and divide by bd^2 we get

$$0 \;=\; 2\,\frac{d}{d}\frac{A_s}{bd}(n - 1) \;-\; 2\frac{c}{d}(n - 1)\frac{A_s}{bd} \;+\; \frac{h^2 b}{d^2 b} \;-\; \frac{2h}{d}\frac{c}{d}\frac{b}{b} \tag{4.1}$$

define ρ as the reinforcement ratio:

$$\rho \;=\; \frac{A_s}{bd}$$

then Eq. (4.1) given above can be written as:

$$0 \;=\; 2\,\rho_s(n - 1) \;-\; 2\frac{c}{d}(n - 1)\rho \;+\; (\frac{h}{d})^2 \;-\; 2\frac{h}{d}\frac{c}{d}$$

simplify and solve for c/d:

Uncracked

$$\boxed{\frac{c}{d} \;=\; \frac{2\rho(n - 1) \;+\; (h/d)^2}{2\rho(n - 1) \;+\; 2(h/d)}} \tag{4.2}$$

Note:

Knowing Eq. (4.2), we can solve for c; solve for $f_1 = f_r\, c / (h-c)$

Knowing c, we can solve for C_c, T_s, and T_c;

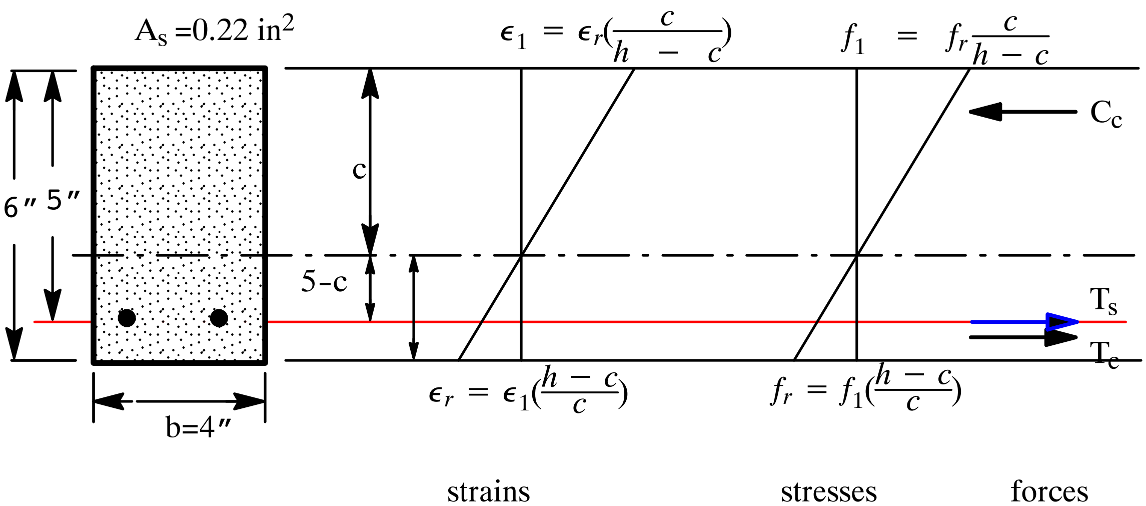
Knowing forces, C_c, T_s, and T_c; we can find moment capacity of the section.

$$M_{capacity} \;=\; C_c(\frac{2}{3}c) \;+\; T_s(d - c) \;+\; T_c(h - c)\frac{2}{3} \qquad \text{For any concrete tension} < f_r$$

If $M_{applied} > M_{at\ Fr}$ Tension stress in concrete will be greater than f_r and section will become "cracked Section."

4.4. Example 1. Calculate Cracking Moment (M_{cr})

Calculate the moment of the section shown below when maximum tensile stress in concrete is equal to f_r (Cracking Moment)



Given Material Properties

f'_c = 3200 psi

f_r = 500 psi = 0.5 ksi

$$E_c = 57,000\sqrt{3200} = 3,220,000 \ psi = 3,220 \ ksi$$

Solution

$$\rho = \frac{A_s}{bd} = \frac{0.22(in^2)}{4(in) \times 5(in)} = 0.011$$

$$\frac{E_s}{E_c} = n = \frac{29,000 \ (ksi)}{3,220 \ (ksi)} = 9.01 \approx 9$$

$$\frac{h}{d} = \frac{6}{5} = 1.2$$

$$\frac{c}{d} = \frac{2\rho(n - 1) + (\frac{h}{d})^2}{2\rho(n - 1) + 2(\frac{h}{d})} = \frac{2 \times 0.011 \times 8 + (1.2)^2}{2 \times 0.011 \times 8 + 2 \times (1.2)} = 0.627$$

$$c = 0.627d = 0.627 \times 5(in) = 3.14 \ inches$$

$$f_1 = f_r\left(\frac{c}{h - c}\right) = 0.5 \times \frac{3.14}{6 - 3.14} = 0.549 \ \ ksi$$

Having f_1, we can easily calculate all forces:

$$C_c = \frac{1}{2}f_1 cb = \frac{1}{2} \times 0.549(ksi) \times 3.14(in) \times 4(in) = 3.45 \ \ kip$$

$$T_s = f_1\left(\frac{d - c}{c}\right)(n - 1)A_s$$

$$= 0.549(ksi) \times \frac{5(in) - 3.14(in)}{3.14(in)} \times (9 - 1) \times 0.22(in^2) = 0.57 \ \ kips$$

$$T_c = \frac{1}{2}f_1\left(\frac{h - c}{c}\right)(h - c)b$$

$$= \frac{1}{2} \times 0.549(ksi) \times \frac{[6(in) - 3.14(in)]^2}{3.14(in)} \times 4(in) = 2.86 \qquad kips$$

Check equilibrium, does it satisfy $C_c = T_s + T_c$?

$C_c = 0.57 + 2.86 = 3.43$ kips $= C_c$ kips ; the difference is due to rounding error associated with calculating "c"

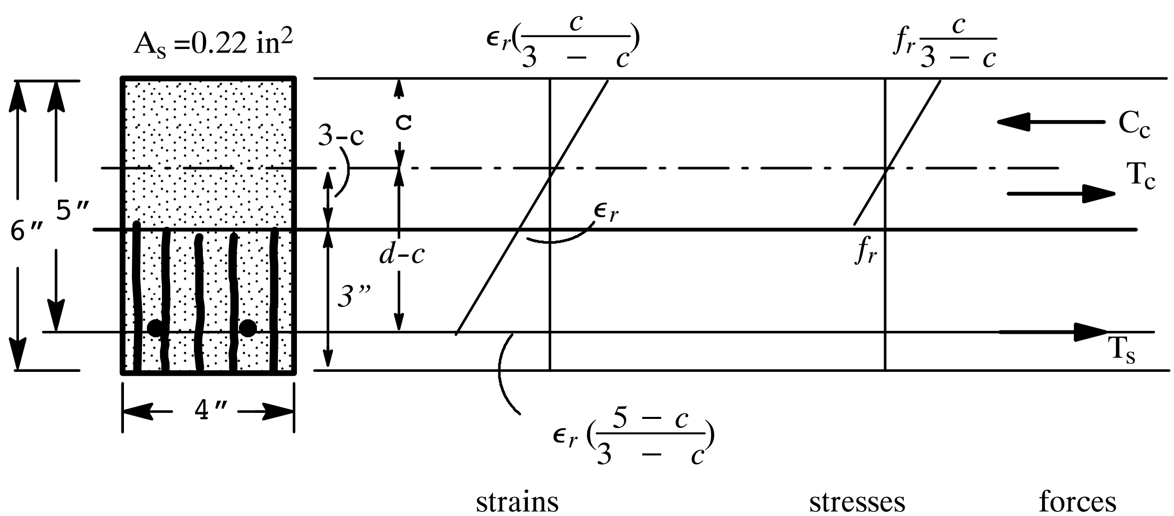
Calculate moment about N.A. (or any point on the cross section)

Force Kips	Moment Arm inches	Moment in-kips
$C_c = 3.45 \ \ kips$	$\frac{2}{3}c = \frac{2}{3} \times 3.14 = 2.09$	= 7.21
$T_s = 0.57$	$d - c = 5 - 3.14 = 1.86$	= 1.06
$T_c = 2.86$	$\frac{2}{3}(h - c) = \frac{2}{3}(6 - 3.14) = 1.91$	= 5.46

Total M = 13.73 in-kips

4.5. Example 2. Calculate Moment Capacity of a Cracked Beam

Consider the section from the previous example after cracking has progressed 3 inches into beam.



Given:

From previous example problem we have:

f'_c = 3200 psi

f_r = 500 psi = 0.5 ksi

$$E_c = 57,000\sqrt{3200} = 3,220,000 \ psi = 3,220 \ ksi$$

Solution

$$\epsilon_r = \frac{500 \ (psi)}{3,220,000 \ (psi)} = 0.000155$$

$$\epsilon_s = \epsilon_r\frac{5-c}{3-c} = 0.000155\frac{5-c}{3-c}$$

Calculate forces (kips)

$$C_c = \frac{1}{2}f_r\frac{c}{3-c} \times c \times 4(in) = \frac{1}{2} \times 0.5(ksi) \times \frac{c^2}{3-c} \times 4 = \frac{c^2}{3-c}$$

$$T_c = \frac{1}{2}f_r(3-c) \times 4(in) = 3-c$$

$$T_s = 0.000155 \times \left(\frac{5-c}{3-c}\right) \times 29,000(ksi) \times 0.22(in^2) = 0.99\left(\frac{5-c}{3-c}\right)$$

Let $\sum T = \sum C$

$$\frac{c^2}{3 - c} = 3 - c + 0.99\frac{5 - c}{3 - c}$$

Solve for "c" we get:
c = 2.0 inches

Calculate forces;

$$C_c = \frac{c^2}{3 - c} = \frac{2^2}{3 - 2} = 4 \ kips$$

$$T_c = 3 - c = 3 - 2 = 1 \ kips$$

$$T_s = 0.99\left(\frac{5 - c}{3 - c}\right) = 2.97 \ kips$$

Check equilibrium, does it satisfy $C_s = T_s + T_c$?

C_s = 2.97 + 1 = 3.97 kips = C_c = 4.0 kips ; the difference is due to rounding error associated with calculating "c"

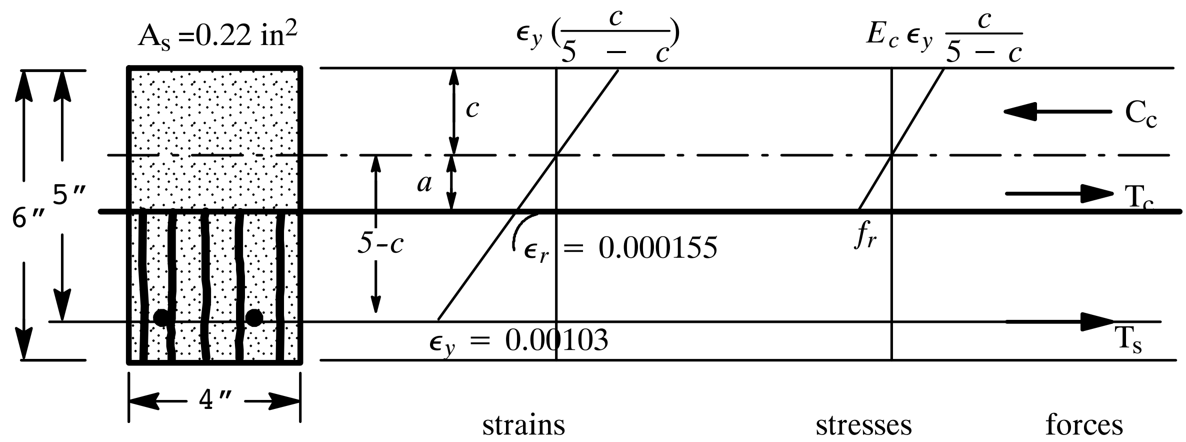
Calculate moment about N.A. (or any point on the cross section)

Force Kips	Moment Arm inches	Moment in-kips
$C_c = 4 \ kips$	$\frac{2}{3}c = \frac{2}{3} \times 2 = 1.33$	= 5.32
$T_s = 2.97$	$d - c = 5 - 2 = 3$	= 8.91
$T_c = 1$	$\frac{2}{3}(3 - c) = \frac{2}{3}(3 - 2) = 0.67$	= 0.67

Total M = 14.9 in-kips

4.6. Example 3. Calculate Moment Capacity of a Beam when Tension Steel Yields (M_y)

Calculate yield moment (when tension steel is yielding). Assume linear stress-strain relationship for concrete.



Given:

From previous example problem we have:

f'_c = 3200 psi $\quad$ f_y = *30,000 psi*

f_r = *500 psi*

$$E_c = 57,000\sqrt{3200} = 3,220,000 \ psi = 3,220 \ ksi$$

Solution

Calculate important parameters

$$\epsilon_r = \frac{500(psi)}{3,220,000(psi)} = 0.000155$$

$$\epsilon_y = \frac{f_y}{E_s} = \frac{30 \ ksi}{29,000 \ ksi} = 0.00103$$

$$a = \frac{0.000155}{0.00103}(5 - c) = 0.151(5 - c)$$

Calculate forces (kips)

$$C_c = 0.00103\left(\frac{c}{5 - c}\right)(3220 \ ksi) \times \frac{1}{2}4(c) = 6.66\frac{c^2}{5 - c}$$

$$T_c = \frac{1}{2}(0.5 \ ksi)[0.151(5 - c)(4 \ in)] = 0.151(5 - c)$$

$$T_s = A_s f_y = (0.22 \ in^2) \times (30 \ ksi) = 6.6 \ kips$$

Let $\sum T = \sum C$

$$6.66\frac{c^2}{5 - c} = 0.151(5 - c) + 6.6$$

$$c^2 + 1.246c - 5.651 = 0$$

Solve for "c"

$$c = \frac{-1.246 \pm \sqrt{1.246^2 + 5.651 \times 4}}{2} \rightarrow \boxed{c = 1.83 \ in}$$

Calculate forces;

$$C_c = 6.66\frac{1.83^2}{5 - 1.83} = 7.08 \quad kips$$

$$T_c = 0.151(5 - 1.83) = 0.48 \quad kips$$

$$T_s = 6.6 \ kips$$

Check equilibrium, does it satisfy $C_s = T_s + T_c$?

$T_s + T_c = 6.6 + 0.48 = 7.08$ kips $= C_c = 7.08$ kips o.k.

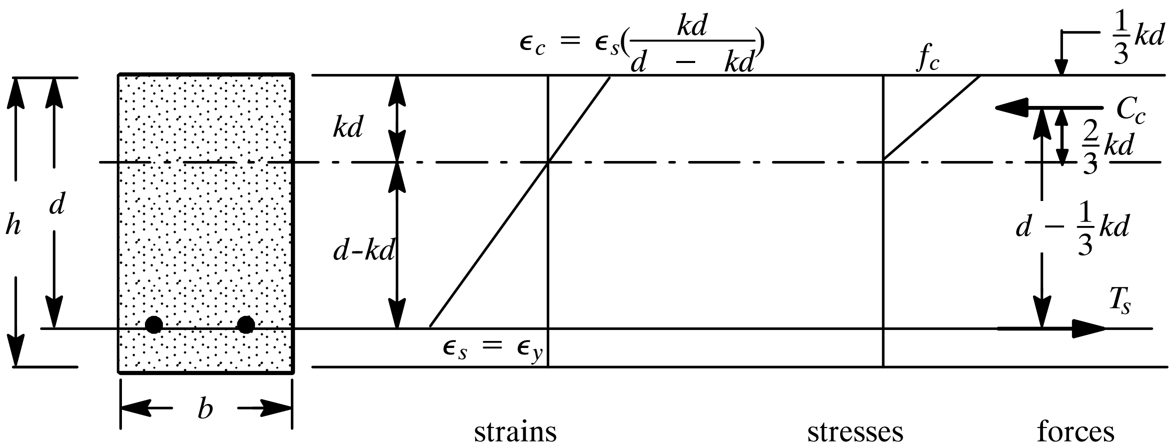
Calculate moment about N.A. (or any point on the cross section)

Force Kips	Moment Arm inches	Moment in-kips
$C_c = 7.08 \ kips$	$\frac{2}{3}c = \frac{2}{3} \times 1.83 = 1.22$	= 8.64
$T_s = 6.6$	$d - c = 5 - 1.83 = 3.17$	= 20.92
$T_c = 0.48$	$\frac{2}{3}a = \frac{2}{3} \times 0.151(5\text{-}1.83) = 0.32$	= 0.15

Total M = 29.71 in-kips

4.7. Linear Stress-Strain Relationship for Concrete in Compression

When the tension stress exceeds the modulus of rupture, cracks form. If the concrete compression stresses less than approximately $0.5f'_c$ and the steel stress has not reached the yield point, both steel and concrete behave elastically. This situation generally happens under service loads. Since the contribution of tension in concrete is negligible in most case, it is assumed that tension cracks have progressed all the way to the neutral axis and that sections plane before bending are plane in the bent member. Therefore, we will assume that concrete tension capacity is zero.



Equilibrium: C = T

$$\frac{1}{2}f_c(kd)b = A_s f_s$$

From geometry we have:

$$\epsilon_c = \epsilon_s \frac{kd}{d - kd} = \epsilon_s \frac{k}{1 - k}$$

for linear stress-strain relationship we have

$$f_c = \epsilon_c E_c$$

therefore we have

$$C = T$$

$$\frac{1}{2}f_c(kd)b = \frac{1}{2}\epsilon_c E_c(kd)b = \frac{1}{2}(\epsilon_s \frac{k}{1 - k}) E_c kdb = A_s f_s$$

divide both sides by "bd" and note $\rho = \dfrac{A_s}{bd}$

$$\frac{1}{2}\epsilon_s \frac{k}{1 - k} E_c k = \rho f_s$$

divide both sides by $\epsilon_s E_c$ and note $\dfrac{f_s}{\epsilon_s E_c} = \dfrac{E_s}{E_c} = n$

then we will have:

$$\frac{1}{2}k \times \frac{k}{1 - k} = \rho n$$

simplify

$$k^2 = 2(1 - k)\rho n$$

$$k^2 + 2\rho n k - 2\rho n = 0$$

solve for k

$$k = \frac{-2\rho n \pm \sqrt{4(\rho n)^2 + 4(2\rho n)}}{2}$$

simplify

$$\boxed{k = \sqrt{(\rho n)^2 + 2\rho n} - \rho n} \tag{4.3}$$

Remember this equation is good only when concrete behaves linearly.

The yield moment capacity of the section can be determined by taking moment about steel location:

$$M_y = A_s f_s (d - \frac{kd}{3})$$

$$M_y = A_s f_s d(1 - \frac{k}{3})$$