Chapter 9.
Shear and Diagonal Tension

9.1. Reading Assignment

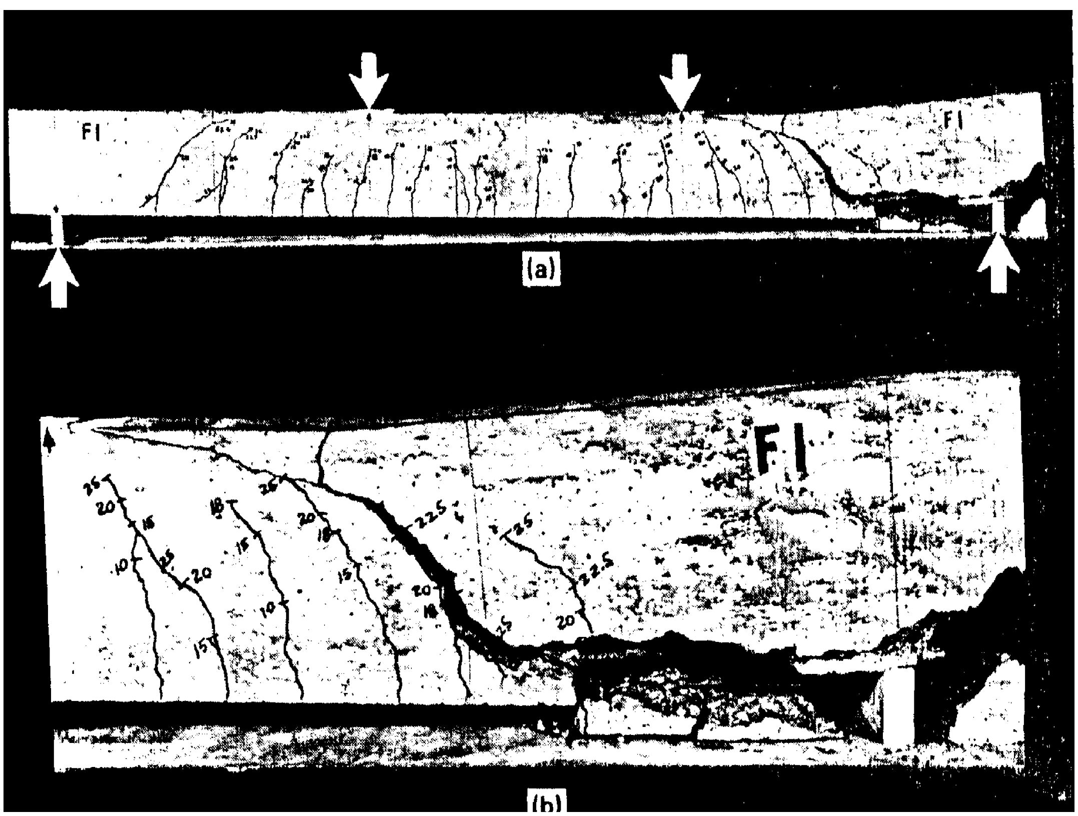
Text Chapter 4; Sections 4.1 - 4.5

Code Chapter 11; Sections 11.1.1, 11.3, 11.5.1, 11.5.3, 11.5.4, 11.5.5.1, and 11.5.6

9.2. Introduction of Shear Phenomenon

Beams must have an adequate safety margin against other types of failure, some of which may be more dangerous than flexural failure. Shear failure of reinforced concrete, more properly called "diagonal tension failure" is one example.

If a beam without properly designed shear reinforcement is overloaded to failure, shear collapse is likely to occur suddenly with no advance warning (brittle failure). Therefore, concrete must be provided by "special shear reinforcement" to insure flexural failure would occur before shear failure. In other words, we want to make sure that beam will fail in a ductile manner and in flexure not in shear.



Shear failure of reinforced concrete beam: (a) overall view, (b) detail near right support

9.3. Review of Shear

Consider a homogenous beam in two sections as shown below.

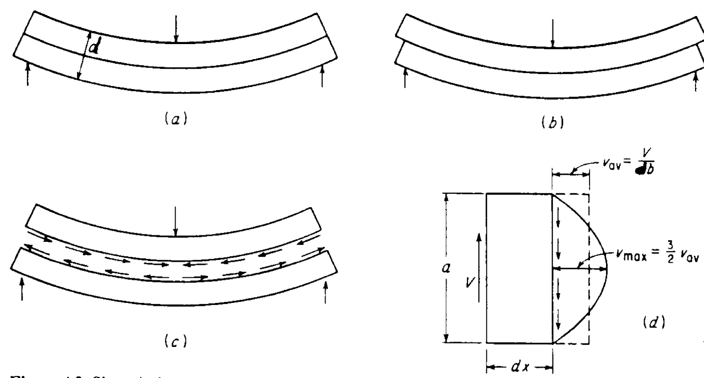
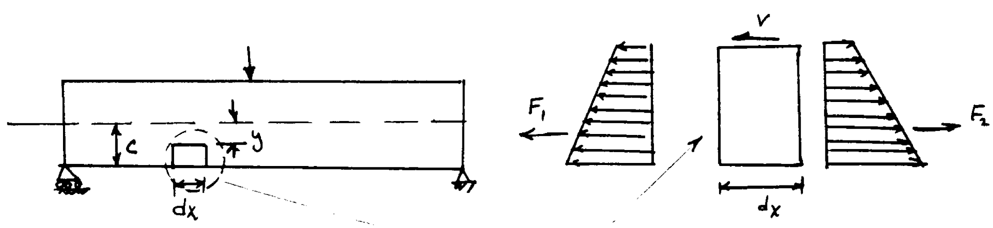


Figure 4.2 Shear in homogeneous rectangular beams.

Shearing Stresses are vital part of the beam load carrying capacity.

9.4. Background

Consider a small section of the beam with shear



$$F_1 = \frac{1}{2}\left(\frac{My}{I} + \frac{Mc}{I}\right)(b(c-y)) = \frac{M}{2}\left(\frac{c+y}{I}\right)b(c-y)$$

$$F_2 = \frac{1}{2}\left(M + \frac{\partial M}{\partial x}dx\right)\left(\frac{c+y}{I}\right)b(c-y) = \frac{M+dM}{2}\left(\frac{c+y}{I}\right)b(c-y)$$

$$(b)(dx)v = F_2 - F_1 = \frac{1}{2}(M + dM - M)\left(\frac{c+y}{I}\right)b(c-y)$$

$$v = \frac{dM}{dx}\left(\frac{b}{2}\right)\frac{(c+y)(c-y)}{I}b = \frac{V}{Ib}\left(\frac{b}{2}(c+y)(c-y)\right)$$

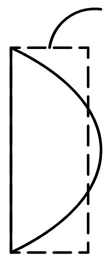
$$\underset{area}{(c-y)b} \qquad \underset{arm}{\frac{(c+y)}{2}}$$

$$v = \frac{VQ}{Ib}$$

1st moment of area below y is called Q

9.5. BACKGROUND

For a homogenous, rectangular beam shear stress varies as:

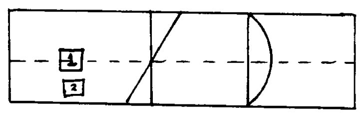


$$v_{\max} = \frac{V}{bd}$$

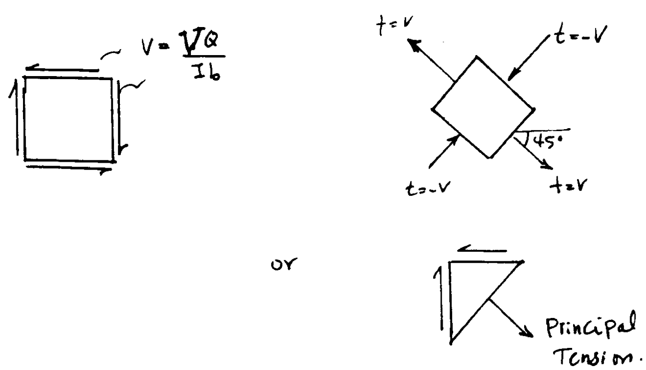
$$v_{\max} = \frac{3}{2}\frac{V}{bd}$$

Average stress is suitable
for concrete analysis

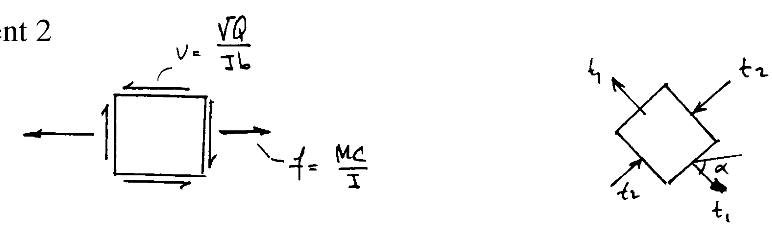
How will beam stresses vary?



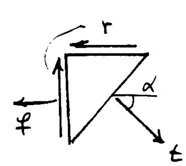
Element 1 at N.A.



Element 2

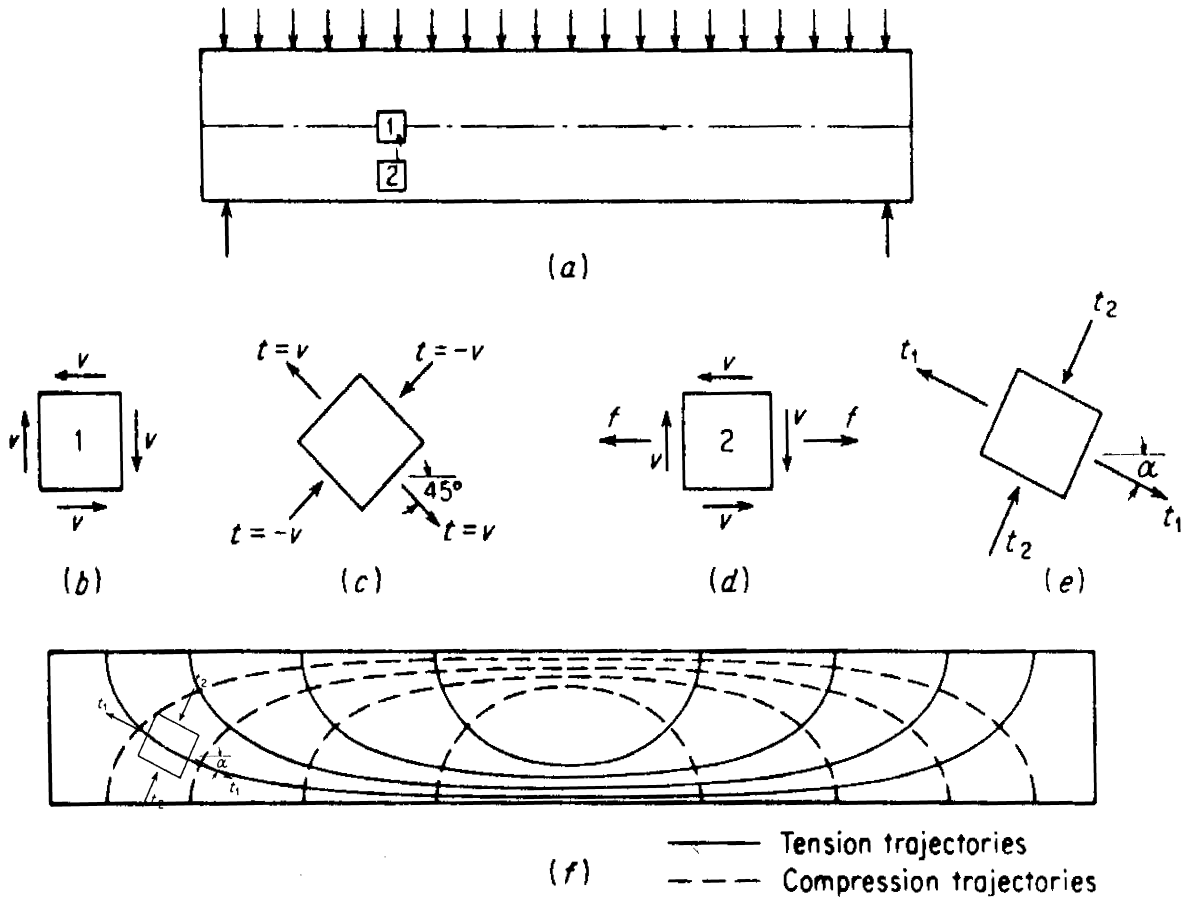


Principal Stresses



$$t \stackrel{\pm}{=} \frac{f}{2} \pm \sqrt{\frac{f^2}{4} + r^2}$$

with $\tan 2\alpha = \frac{2v}{f}$



Stress trajectories in homogeneous rectangular beam.

Tension stresses, which are of particular concern in the view of the low tensile capacity of the concrete are not confined only due to the horizontal bending stresses *f* which are caused by bending alone.

Tension stresses of various magnitude and inclinations, resulting from

- shear alone (at the neutral axis); or
- the combined action of shear and bending

exist in all parts of a beam and if not taken care of appropriately will result in failure of the beam. It is for this reason that the inclined tension stresses, known as **diagonal tension**, must be carefully considered in reinforced concrete design.

Two types of inclined cracking occur in concrete beams: web-shear cracking and flexure-shear cracking. These two types of inclined cracking are illustrated in Fig. 11.1.1.

Web-shear cracking begins from an interior point in a member when the principal tensile stresses exceed the tensile strength of the concrete. Flexure-shear cracking is initiated by flexural cracking. When flexural cracking occurs, the shear stresses in the concrete above the crack are increased. The flexure-shear crack develops when the combined shear and tensile stress exceeds the tensile strength of the concrete.

When inclined cracking occurs in a nonprestressed concrete member, it is generally of the flexure-shear type. Web-shear cracking generally occurs near the supports of deep flexural members with thin webs, or near the inflection point or bar cutoff points of continuous beams, particularly if the beam is subjected to axial tension.

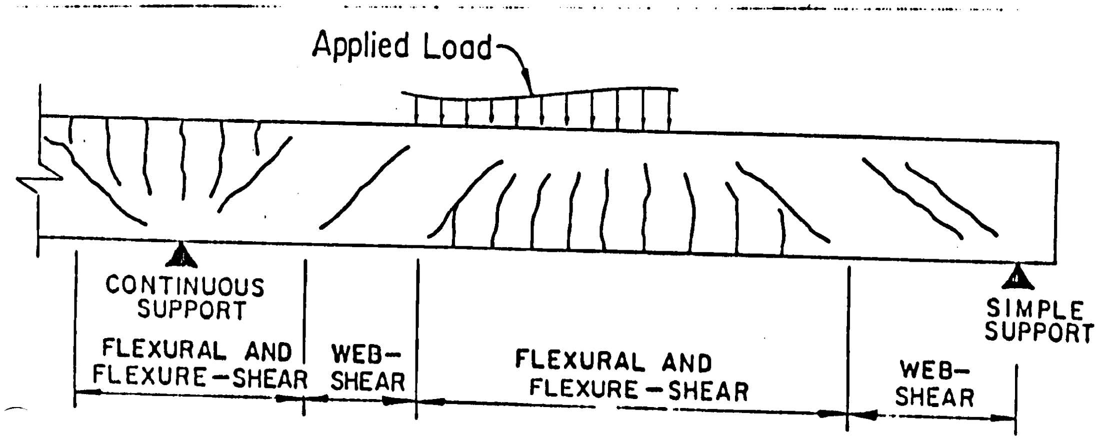


Figure R 11.4.2 Types of cracking in concrete beams

9.6. Criteria for formation of diagonal cracks in concrete beams

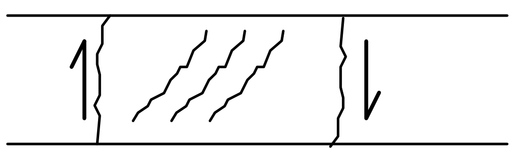
Large V (shear force), Small M (bending moment)

Little flexural cracking prior to formation of diagonal cracks.

$$v_{ave} = \frac{V}{bd}$$

- can be regarded as rough measure of stress
- Distribution of "*V*" is not known exactly, as reinforced concrete is non-homogeneous.
- Shear near N.A. will be largest

Crack from N.A. propagates toward edges:



called **web shear cracks**

From diagonal tension: Concrete tensile strength is given as:

$$v_{cr} = \frac{V}{bd} = 3\sqrt{f'_c} \Leftrightarrow 5\sqrt{f'_c}$$

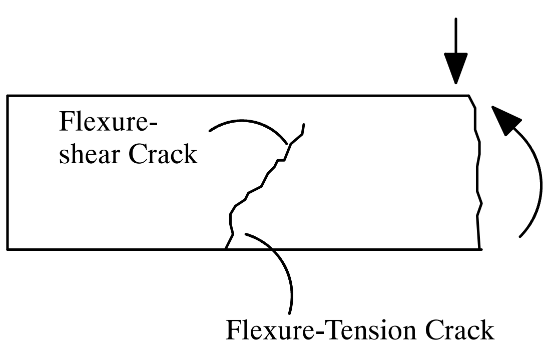
tests shown that the best estimate of cracking stress is

$$v_{cr} = \frac{V}{bd} = 3.5 \sqrt{f'_c}$$

Note: The most common type of shear crack occurs only under high shear; with thin webs.

Large V (shear force), Large M (bending moment)

Formation of flexure cracks precedes formation of shear cracks.



v at formation of shear cracks is actually larger than for web shear cracks. Presence of tension crack reduces effective shear area

Formation of flexure shear crack is unpredictable. Nominal shear stress at which diagonal tension cracks form and propagate is given as

$$v_{cr} = \frac{V_{cr}}{bd} = 1.9\sqrt{f_c'} \quad \textit{from many tests.} \tag{52}$$

It was also found that the reinforcement ratio ρ has an effect on diagonal crack formation for the following reason:

"As ρ is increased, tension crack depth decreases; area to resist shear increases."

Based on many tests, ACI-ASCE committee justified the following equation

$$\frac{V_c}{bd\sqrt{f_c'}} = 1.9 + 2500\rho\frac{Vd}{M\sqrt{f_c'}} < 3.5 \qquad \textit{ACI Equation} \quad 11\text{-}5$$

Vd/M term tells that the diagonal crack formation depends on *v* and *f* at the tip of the flexural crack. We can write shear stress as

$$v = k_1\frac{V}{bd} \tag{53}$$

where k_1 depends on depth of penetration of flexural cracks. Flexural stress *f* can be expressed as

$$f = \frac{Mc}{I} = k_2\frac{M}{bd^2} \tag{54}$$

where k_2 also depends on crack configuration. If we divide (53) by (54) we get

$$\frac{v}{f} = \frac{k_1}{k_2}\frac{V}{bd} \times \frac{bd^2}{M} = \overline{K}\frac{Vd}{M} \tag{55}$$

where $\overline{K}$ is determined from experiments.

ACI allows us to use an alternate form of Eq. (52) for concrete shear stress

$$\frac{V_c}{bd} = 2\sqrt{f_c'} \qquad \textit{ACI Eq.}\ 11-3 \tag{56}$$

Shear cracks in beams without shear reinforcement cannot be tolerated, can propagate into compression face, reducing effective compression area, area to resist shear.

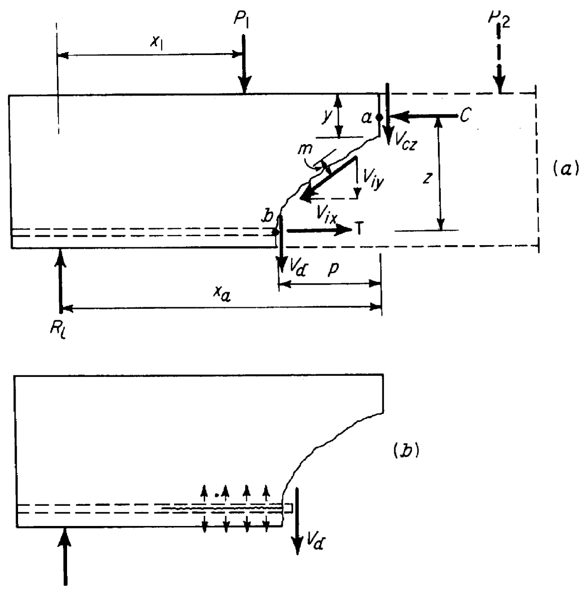
9.7. What Actions Contribute to Total Shear Resisting Force - No Shear Reinforcements

Cracked Beam without any shear reinforcement

1 Force resulting from aggregate interlock at crack.

2. Concrete shear stress in compression zone

3. Dowel shear from longitudinal flexural reinforcement.



Conservatively, we may neglect all but concrete stress. Nature of failure offers very little reserve capacity if any. As a result, design strength in shear (without shear reinforcement) is governed by strength which present before formation of diagonal cracks.

Web Reinforcement

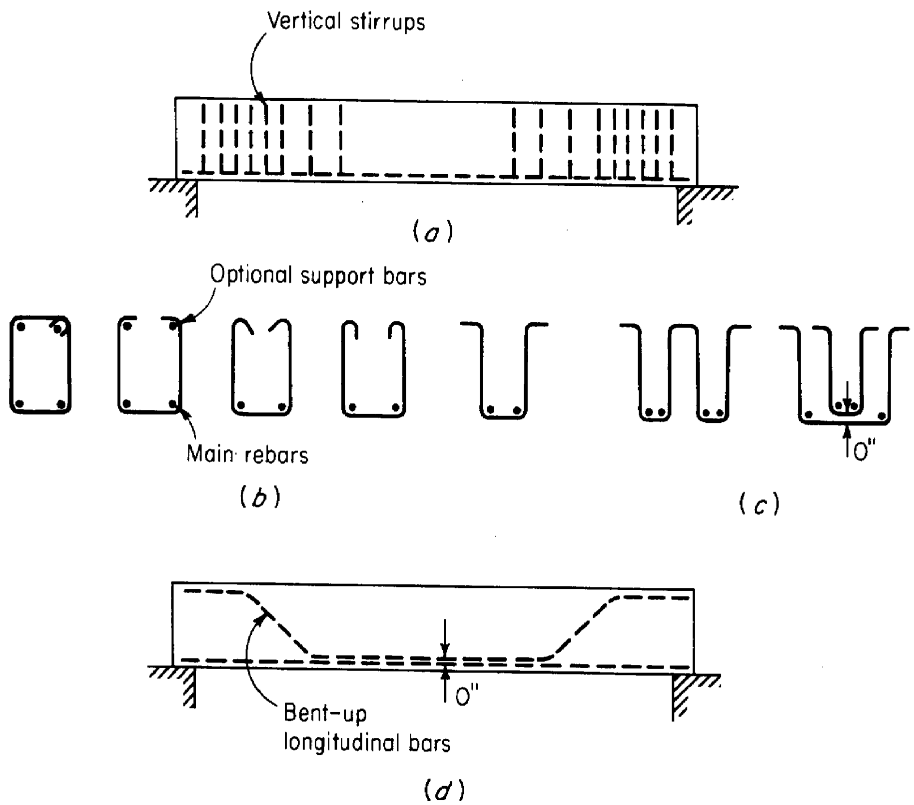
Shear reinforcement allows for

- Maximum utility of tension steel - Section capacity is not limited by shear
- Ductile failure mode - Shear failure is not ductile, it is sudden and dangerous.

9.8. Possible Configuration of Shear Reinforcement

- Vertical stirrups, also called "ties" or "hoops"
- Inclined stirrups
- Bend up bars

Generally #3, #4, and #5 bars are used for stirrups and are formed to fit around main longitudinal rebars with a hook at end to provide enough anchorage against pullout of the bars.



9.9. Effect of Stirrups

1. Before shear cracking - No effect (web steel is free of stress)

2. After shear cracking
   - Resist shear across crack;
   - Reduce shear cracking propagation;
   - Confines longitudinal steel - resists steel bond loss, splitting along steel, increase dowel actions;
   - Increase aggregate interlock by keeping cracks small.

3. Behavior of members with shear reinforcement is somewhat unpredictable - Current design procedures are based on:
   - Rational analysis;
   - Test results;
   - Success with previous designs.

9.10. Design of Shear Reinforcement - A Rational (!) Approach

1. Before cracking - Cracking load given as before:

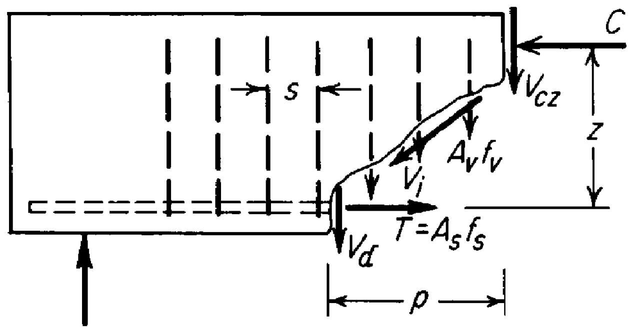
$$V_c = bd\left(1.9\sqrt{f'_c} + 2500\rho_w\frac{Vd}{M}\right) \leq 3.5\sqrt{f'_c}\,bd$$

2. After cracking

   Assuming V_c equals to that at cracking - This is conservative due to the effect of compression and diagonal tension in the remaining uncracked, compression zone of the beam.

9.11. Beams with Vertical Stirrups (or Beams with Shear Reinforcement)

Forces at diagonal crack in a beam with vertical stirrups can be shown as



$$V_N = total\ \ internal\ \ shear\ \ force \quad = \quad V_{cz} \quad + \quad \sum A_v f_v + V_d \quad + \quad V_{iy}$$

where
- $V_{cz} =$ Internal vertical force in the uncracked portion of concrete
- $V_d =$ Force across the longitudinal steel, acting as a dowel
- $V_{iy} =$ Aggregate interlock force in vertical direction
- $\Sigma A_v f_v =$ Vertical force in stirrups.

If horizontal projection of the crack is “p”, and the stirrup spacing is “s”, then the number of stirrups crossed by a random crack will be:

$$n \quad = \quad \frac{p}{s}$$

and total force contributed by stirrups will be:

$$V_s \quad = \quad nA_v f_s$$

which near failure will be

$$V_s \quad = \quad nA_v f_y \qquad f_s \quad = \quad f_y$$

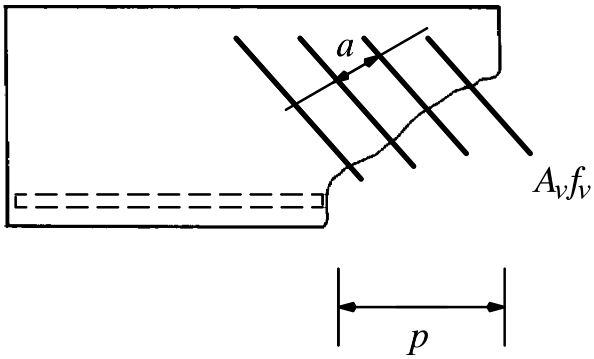
Also, we can conservatively neglect forces due to dowel and aggregate interlock. Therefore

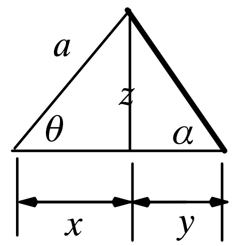
$$V_n \quad = \quad V_c \quad + \quad V_s \quad = \quad V_c \quad + \quad nA_v f_y$$

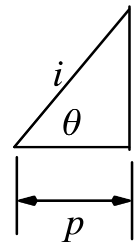
The only question remaining is that: What is the horizontal projection of the crack? Test shown that p=d is a good approximation: $p/s = d/s$ or

$$V_s \quad = \quad nA_v f_y \quad = \quad \frac{d}{s} A_v f_y \qquad \textit{This is Eq. } 11\text{–}15 \textit{ of } \boldsymbol{ACI}$$

9.12. Beams with Inclined Bars







$$\tan\theta = \frac{Z}{x} \quad \rightarrow \quad x = \frac{Z}{\tan\theta}$$

$$\tan\alpha = \frac{Z}{y} \quad \rightarrow \quad y = \frac{Z}{\tan\alpha}$$

$$\left.\begin{aligned} S &= x + y = \frac{Z}{\tan\theta} + \frac{Z}{\tan\alpha} \\ Z &= S\left(\frac{1}{\tan\theta} + \frac{1}{\tan\alpha}\right) \\ \sin\theta &= \frac{Z}{a} \quad \rightarrow \quad Z = a\sin\theta \end{aligned}\right\} \quad a = \frac{S}{\sin\theta\left(\frac{1}{\tan\theta} + \frac{1}{\tan\alpha}\right)}$$

$$n = \frac{i}{a} \;\; \& \;\; \cos\theta = \frac{p}{i} \quad \rightarrow \quad n = \frac{p}{a\cos\theta}$$

$$n = \frac{p}{\cos\theta}\frac{\sin\theta\left(\frac{1}{\tan\theta} + \frac{1}{\tan\alpha}\right)}{S} = \frac{p}{S}\tan\theta\left(\frac{1}{\tan\theta} + \frac{1}{\tan\alpha}\right) = \frac{p}{S}\left(1 + \frac{\tan\theta}{\tan\alpha}\right)$$

$$\text{if} \quad \theta = 45^o \quad \rightarrow \quad \tan(45) = 1 \quad \rightarrow \quad n = \frac{p}{S}\left(1 + \frac{1}{\tan\alpha}\right)$$

$$V_s = nA_v f_y \sin\alpha \; < \; 3\sqrt{f_c'}\, b_w d \qquad \textit{Eq. 11-17}$$

$$V_s = A_v f_y \sin\alpha\left[\frac{p}{S}\left(1 + \frac{\cos\alpha}{\sin\alpha}\right)\right] = A_v f_y \frac{d}{S}(\sin\alpha + \cos\alpha) \qquad \textit{Eq. 11-16}$$

9.13. ACI Code Provisions For Shear Design

According to ACI code procedures

$V_u \leq \phi V_n$ (Required strength ≤ Provided strength)

V_u = total shear force applied at a given section due to factored loads. ($1.2\, w_d + 1.6\, w_L$, etc.)

V_n = nominal shear strength, which is the sum of contributions of the concrete and the web steel if present

$$V_n = V_c + V_s$$

ϕ = strength reduction factor (ϕ=0.75 for shear) - Compare to the strength reduction factor for bending which is 0.9. The reason for the difference is:

- Sudden nature of failure for shear
- Imperfect understanding of the failure mode

ACI provisions:

Vertical stirrups

$$V_u \leq \phi V_c + \frac{\phi A_v f_y d}{s} \qquad \text{Sect 11.4.7.2 Eq. 11-15}$$

Inclined stirrups

$$V_u \leq \phi V_c + \frac{\phi A_v f_y d}{s}(\sin\alpha + \cos\alpha) \qquad \text{Sect 11.4.7.2 Eq. 11-16}$$

For design:

$$V_u = \phi V_c + \frac{\phi A_v f_y d}{s}$$

or

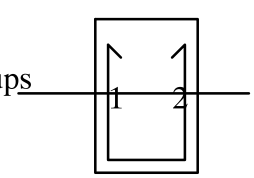
$$s = \frac{A_v f_y d}{\frac{V_u}{\phi} - V_c} \quad \text{or} \quad \boxed{s = \frac{\phi A_v f_y d}{V_u - \phi V_c}}$$

similarly one can find s for inclined bars

Note A_v is for 2 bars.

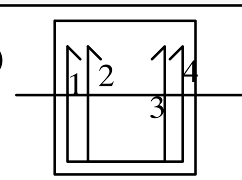
For example for #3 U-stirrups

$A_v = 2\,(0.11) = 0.22\ \text{in}^2$



$A_v = 4\,(0.11)$

$= 0.44\ \text{in}^2$



9.14. WHERE DOSE CODE REQUIRE SHEAR REINFORCEMENT?

According to ACI code section 11.5.5, we need to provide shear reinforcement when

$$V_u \geq \frac{\phi V_c}{2}$$

Exception are:

- Slabs and footings
- Concrete joist construction
- Special configuration beam (shallow)
- Special case when test to destruction shows adequate capacity

When V_u (the factored shear force) is no larger than ϕV_c then theoretically no web reinforcement is required. Even in such cases, the code requires at least a minimum area of web reinforcement equal to

$$A_{v,min} = 0.75\sqrt{f_c'}\frac{b_w S}{f_{yt}} \qquad Eq.(11-13) \qquad \text{for} \qquad \frac{1}{2}V_u \leq \phi V_c \leq V_u$$

$$s_{\max} = \frac{A_v f_y}{50 b_w}$$

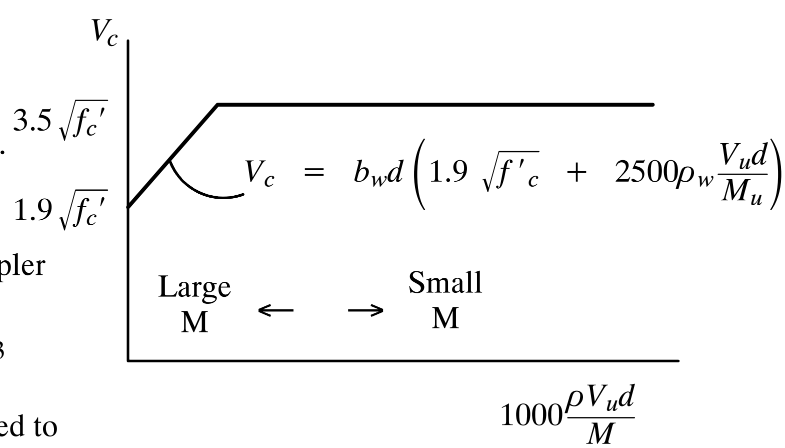
9.15. SHEAR STRENGTH PROVIDED BY CONCRETE

For members subjected to shear and flexure only

$$V_c = b_w d\left(1.9\sqrt{f'_c} + 2500\rho_w\frac{V_u d}{M_u}\right) \leq 3.5\sqrt{f'_c}\, b_w d \quad Eq.11-5 \;\; Sect \;\; 11.3.2$$

the second term in the parenthesis should be

$$\left|\frac{V_u d}{M_u}\right| \leq 1$$



where M_u is the factored moment occurring simultaneously with V_u at section considered.

Alternate form of Eq. 11-6 is the Eq. 11-3 of the ACI code which is much simpler

$$V_c = 2\sqrt{f'_c}\, b_w d \qquad Eq.\ 11-3$$

This gives more conservative values compared to Eq. 11-6 resulting in slightly more expensive design.

9.16. Maximum Stirrups Spacing

if $V_s \leq 4\sqrt{f_c'}\, b_w d$ the maximum spacing is the smallest of

$$A_{v,min} = 0.75\sqrt{f_c'}\,\frac{b_w S}{f_{yt}} \qquad S_{\max} = \frac{A_v f_y}{50 b_w} \qquad \textit{Eq. 11-13 of ACI}$$

$$S_{\max} = d/2 \qquad \text{ACI 11.4.5}$$

$$S_{\max} = 24 \;\; inches$$

if $V_s > 4\sqrt{f_c'}\, b_w d$ the maximum spacing is the smallest of

$$A_{v,min} = 0.75\sqrt{f_c'}\,\frac{b_w S}{f_{yt}} \qquad S_{\max} = \frac{A_v f_y}{50 b_w} \qquad \textit{Eq. 11-13 of ACI}$$

$$S_{\max} = d/4 \qquad \text{ACI 114.5}$$

$$S_{\max} = 12 \;\; inches$$

In no case V_s can exceed $V_s \leq 8\sqrt{f_c'}\, b_w d$ ACI 11.4.7.9

9.17. Example of Shear Reinforcement

Select the spacing of U-shaped stirrups made of No. 3 bars for the beam shown below using both Eqs. 11-3 and 11-5 of ACI 318 code to obtain V_c. Compare the resulting space using two formulas.