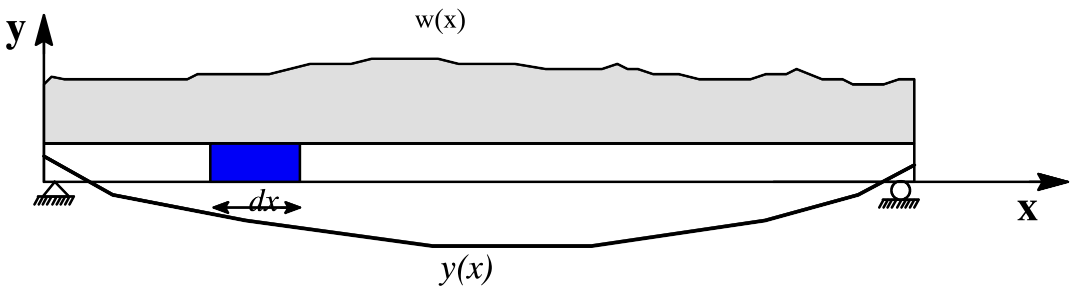
CHAPTER 13.
DEFLECTION

13.1. Reading Assignment

Text: Sect 6.4 through 6.7 and 6.9
ACI 318: Chap 9.

13.2. Calculation of Deflection of R/C beams

Review of theory of deflection of homogeneous beams in elastic flexure:



It is possible to make the following observations from geometry

Deflection = $y(x)$
Slope = dy/dx
Curvature = $d^2y/dx^2 = \phi = 1/\rho$

$$y = \int\int \phi \, dx \, dx$$

and, with similar observations based on equilibrium for

Moment; $M =$ $EI\, d^2y/dx^2 =$ $EI\phi$
Shear; $V =$ $EI\, d^3y/dx^3 =$ dM/dx
Load; $w =$ $EI\, d^4y/dx^4 =$ dV/dx

$$M = \int\int w dx dx$$

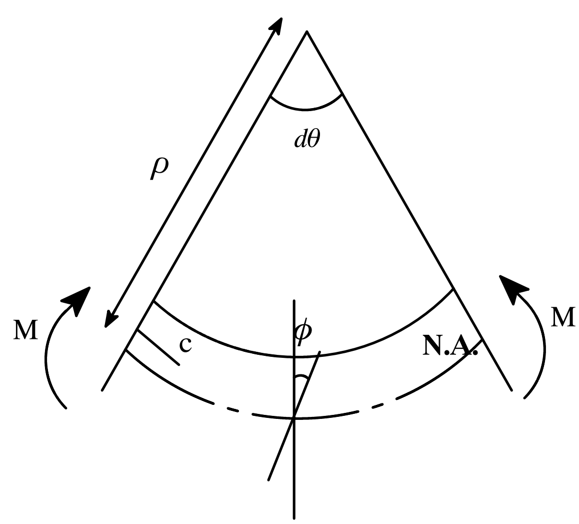
For a homogeneous beam under constant moment at location c:

$\varepsilon_x = c\, d\theta/dx$
$c/\rho = c\, d\theta/dx$

therefore

$$\frac{d\theta}{dx} = \frac{1}{\rho}$$

so



$$\epsilon_x = c/\rho \quad and \quad \sigma_x = Ec/\rho$$

and for equilibrium

$$M = \int (Ec^2/\rho)dA = (E/\rho)\int c^2 dA$$

or

$$M = \frac{EI}{\rho} \rightarrow \frac{M}{EI} = \frac{1}{\rho} = \phi$$

where ϕ becomes a **link** between geometry and equilibrium.

Coming back to the real world, we see that the relationships developed for homogeneous members are not applicable to concrete members; new relationships must be developed.

Two approaches are common:

1) Develop a "synthetic" *EI* for the beam and use the relationships developed for homogeneous beams – ACI 318 endorsed this approach for calculation of service load deflections.

2) Calculate a relationship between moment and curvature which considers all levels of moment. This can be used when a more accurate estimate of deflection is desired or when loads larger than service loads are considered.

13.3. ACI Code Method

Consider only service loads and service load deflections. Cannot handle ultimate loads.

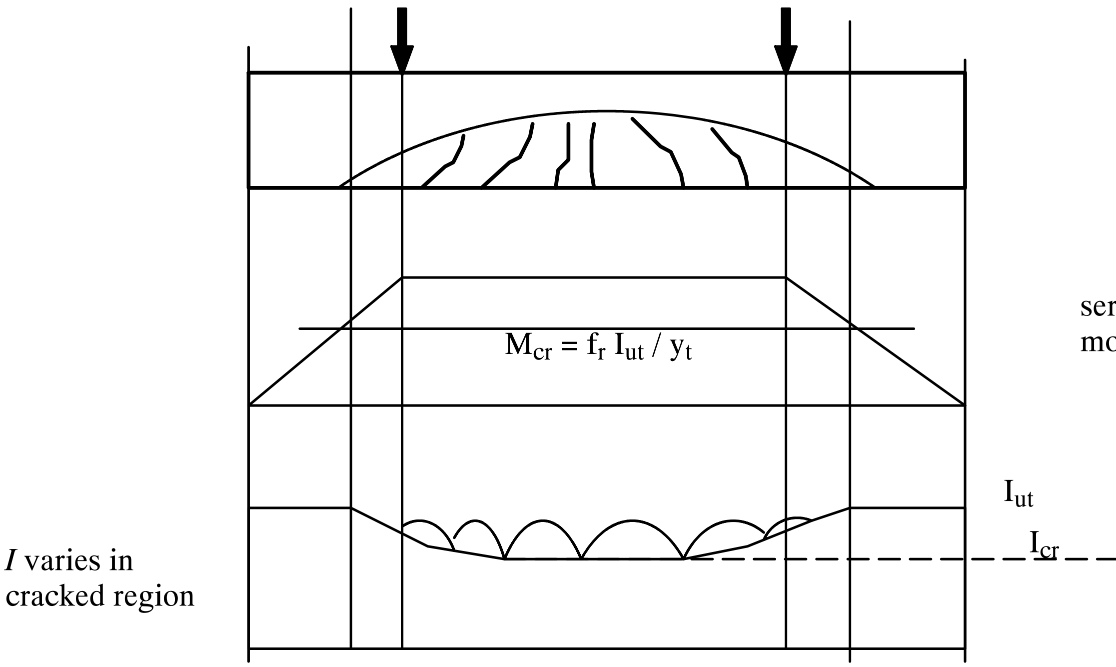
Total deflection is composed of two components:

1) Instantaneous Deflection – when loads applied

2) Additional deflections which occur over time due to creep and shrinkage

Consider first the instantaneous deflection. For moments at or below the cracking moment, the moment of inertia is that of the uncracked transformed section (I_{ut}); $E = E_c$. Assume $f_r = 7.5sqrt(f'_c)$

At moment larger than the cracking moment, behavior is complex, not entirely predictable.



The effective moment of inertia of the beam (I_e) depends on:

a) Shape of the moment diagram – Depends on loading

b) Crack pattern and Spacing (not predictable)

c) Amount of reinforcing, location of bar cut offs, and changes in section

The results have shown that the following approximation gives reasonable results: **ACI 9.5.2.3**

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \leq I_g \qquad \text{ACI Eq. (9-7)}$$

Where

$$M_{cr} = \frac{f_r I_g}{y_t} \quad and \quad f_r = 7.5\sqrt{f_c'} \qquad \text{ACI Eq. (9-8)}$$

M_a = Maximum moment in member at stage of deflection is computed

I_{cr} = Moment of inertia of cracked, transformed section (at steel yield)

I_g = Moment of inertia of gross concrete section – neglect reinforcement

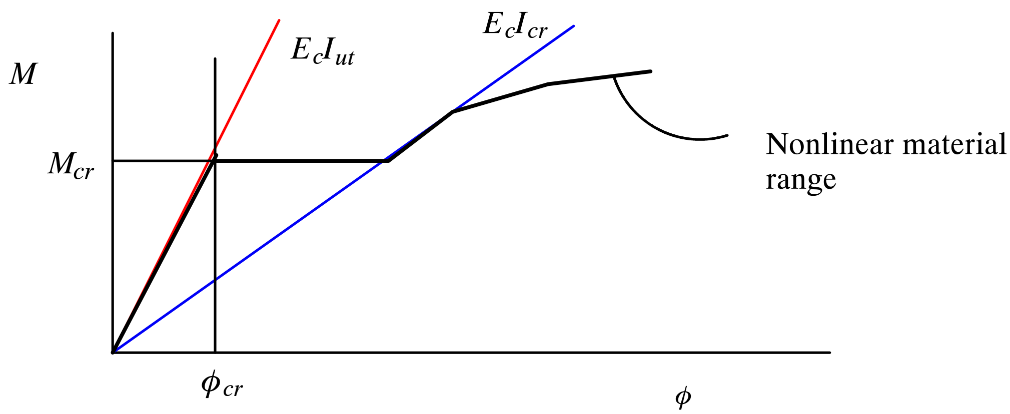
y_t = distance from N.A. to tension face

The effective moment of inertia is somewhere between I_g and I_{cr}; is assumed constant for entire span. For continuous spans, take average of maximum positive and negative moment sections.

Note the limiting values of the equation:

when $M_a = M_{cr}$, $I_e = I_g$

when $M_a >> M_{cr}$, $I_e \rightarrow I_{cr}$



13.4. Consideration of Long-Term Deflections – Creep and Shrinkage

- Deflection due to shrinkage comes soon after casting (majority) with long term shrinkage dependent on environment.
- Deflection due to creep is proportional to stress level and concrete characteristics.

Code method for calculating long term deflections: (ACI 9.5.2.5)

$$\delta_{total} = \delta_{instantaneous} + \lambda\delta_{instantaneous}$$

where

$$\lambda = \frac{\xi}{1 + 50\rho'} \qquad \text{Eq. 9-10 of ACI}$$

Why is ρ' is used?

- Primary creep effect in compression zone.
- Steel does not creep – takes load from concrete
- concrete stress reduced – creep decreased

based on cornell studies, "Variability and Analysis of Data for 318–71 method" ACI journal, January 1972.

- T is a time dependent coefficient which a material property depending on shrinkage and creep. It is given in commentary Fig. 9.5.2.5 page 98 of ACI Code.
- ρ' should be taken at midspan for simple and continuous spans and at support for cantilever.
- Values of T are satisfactory for beams and one way slabs but underestimates time dependent deflection of 2-way slabs.
- For $f'_c > 6000$ psi lower values of T should be used.

13.5. Permissible Deflection in Beams and One-Way Slabs

Permissible deflections in a structural system are governed primarily by the amount that can be sustained by the interacting components of a structure without loss of aesthetic appearance and without detriment to the deflecting member. The level of acceptability of deflection values is a function of such factors as the type of building, the use or nonuse of partitions, the presence of plastered ceilings, or the sensitivity of equipment or vehicular systems that are being supported by the floor. Since deflection limitations have to be placed at service load levels, structures designed conservatively for low concrete and steel stresses would normally have no deflection problems. Present-day structures; however, are designed by ultimate load procedures efficiently utilizing high-strength concretes and steels. More slender members resulting from such designs would have to be better controlled for serviceability deflection performance, immediate and long-term.

13.6. Empirical Method of Minimum Thickness Evaluation for Deflection Control

The ACI Code recommends in Table 9.5(a) minimum thickness for beams as a function of the span length, where no deflection computations are necessary if the member is not supporting or attached to construction likely to be damaged by large deflections. Other deflections would have to be calculated and controlled as in Table 9.5(b) if the total beam thickness is less than required by the table, the designer should verify the deflection serviceability performance of the beam through detailed computations of the immediate and long-term deflections.

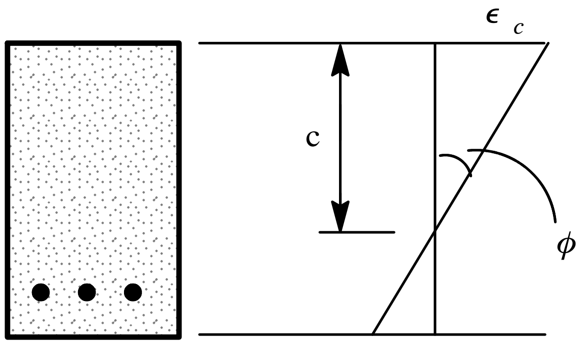
13.7. Permissible Limits of Calculated Deflection

the ACI Code requires that the calculated deflection for a beam or one-way slab has to satisfy the serviceability requirement of minimum permissible deflection for the various structural conditions listed in Table 9.5(b) if Table 9.5(a) is *not* used. However, long-term effects cause measurable increases in deflection with time and result sometimes in excessive overstress in the steel and concrete. Hence, it is always advisable to calculate the total time-dependent deflection and design the beam size based on the permissible span/deflection ratios of Table 9.5(b)

13.8. Second Approach to Deflection Calculation (Sect. 6.9 of text)

Determine a relationship between moment and curvature for entire range of beam action.

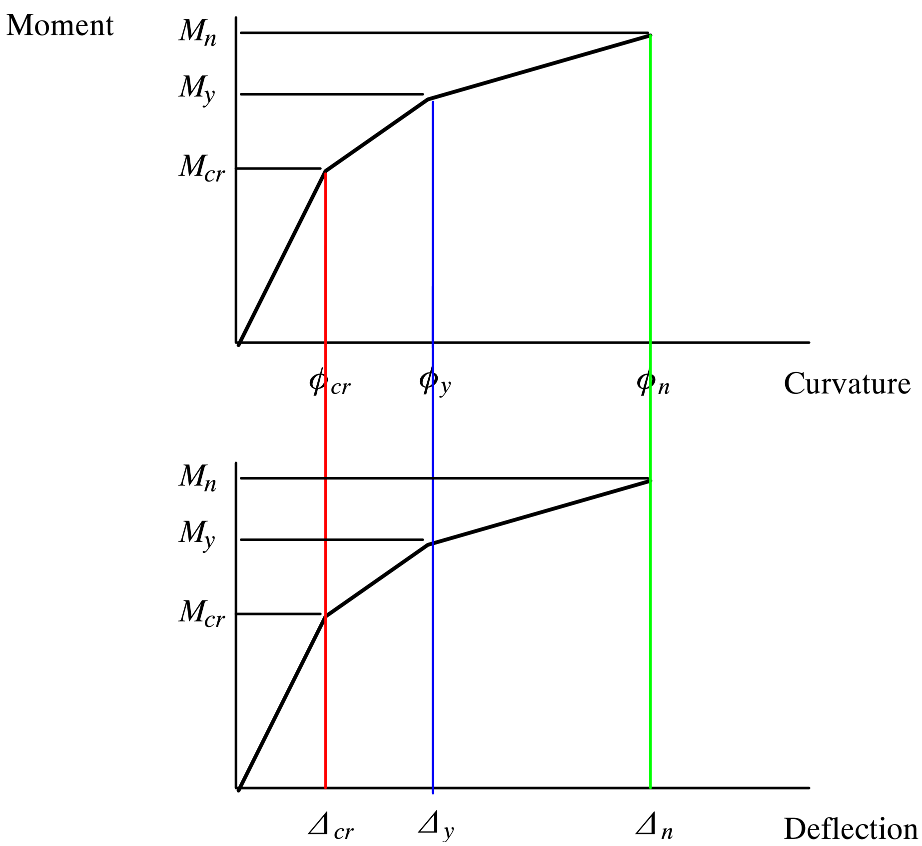
$$curvature = \phi = \frac{1}{\rho} = \frac{\epsilon_c}{c}$$



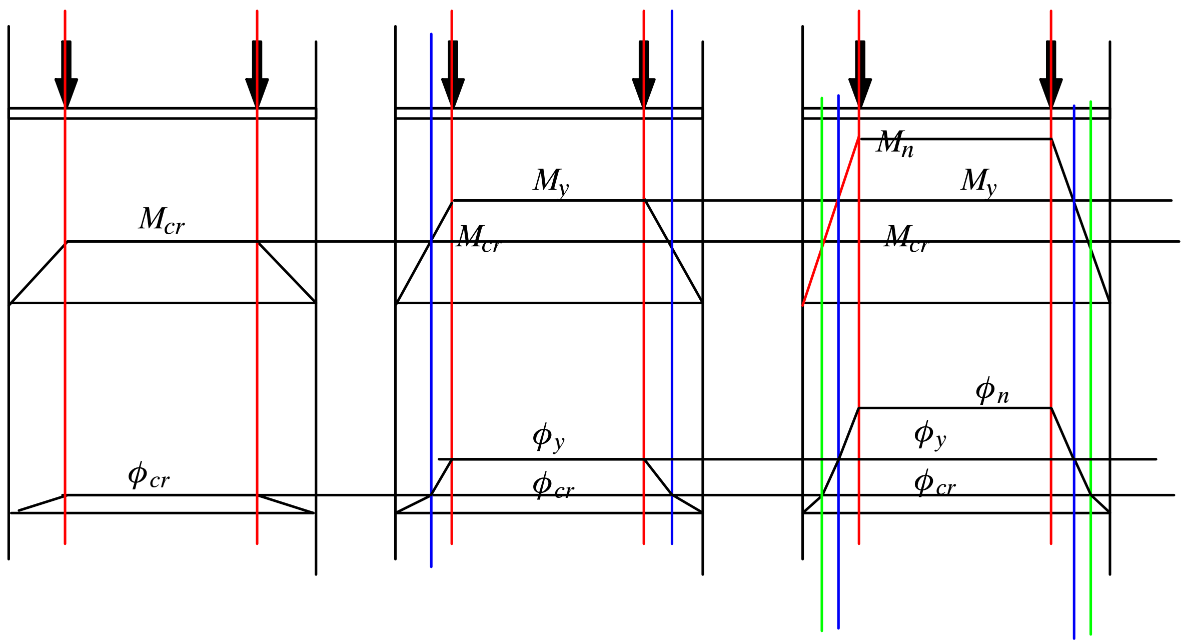
we know “c”, and ε_c for three particular moment conditions:

- cracking
- yield
- nominal

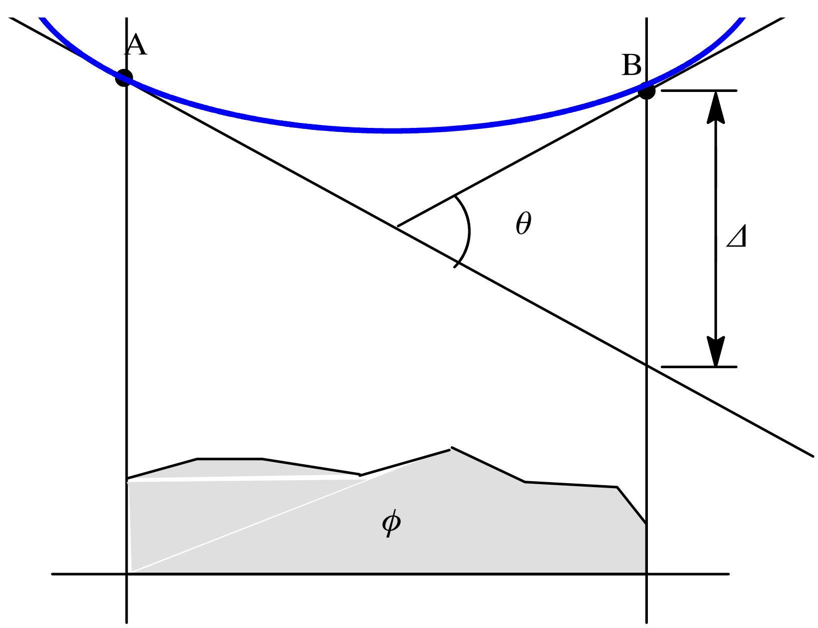
The M–ϕ curve can be constructed with these three points:



By being able to relate ϕ to M, curvature distribution for any loading can be plotted



Use this curvature diagram as we would for an elastic homogeneous member: Moment area is a simple way to obtain deflection using this method.



13.9. Three rules for using moment-area method:

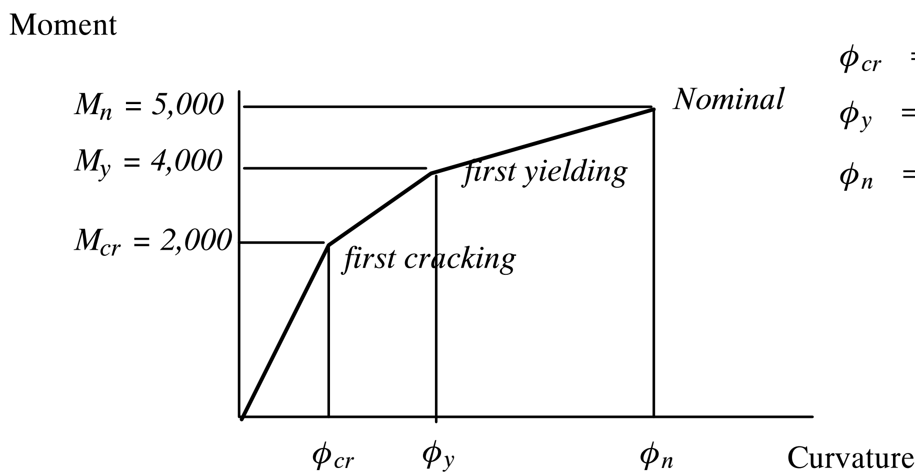
Rule 1) The change in slope between A and B (θ) is equal to the area of the ϕ diagram between A and B.

Rule 2) The deflection of B from the tangent at A (Δ) is equal to the moment of the ϕ diagram between A and B about B.

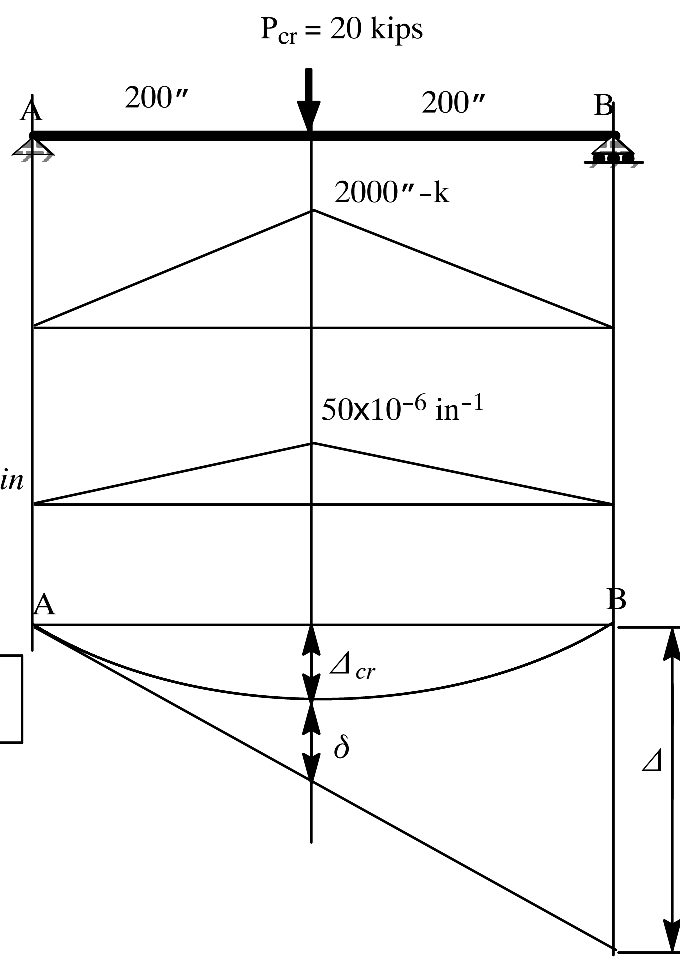
Rule 3) Two points on the elastic curve, or one point and the direction of the tangent at the point are required to locate a curve in space.

13.10. Example of deflection calculation using M–ϕ curve and Moment–Area Method:

Consider a beam section for which the following M–ϕ curve has been developed. Find deflection at point of load for cracking, yield, and ultimate moment.



$$\phi_{cr} = 50 \times 10^{-6}$$
$$\phi_y = 200 \times 10^{-6}$$
$$\phi_n = 500 \times 10^{-6}$$



Δ = Moment of ϕ diagram about B

area of ϕ diagram =
= (400 in)(50x10^{-6} in^{-1})(1/2)
= 0.01

$$\Delta = 1 \times 10^{-2} \times 200 = 2.0 \ in$$

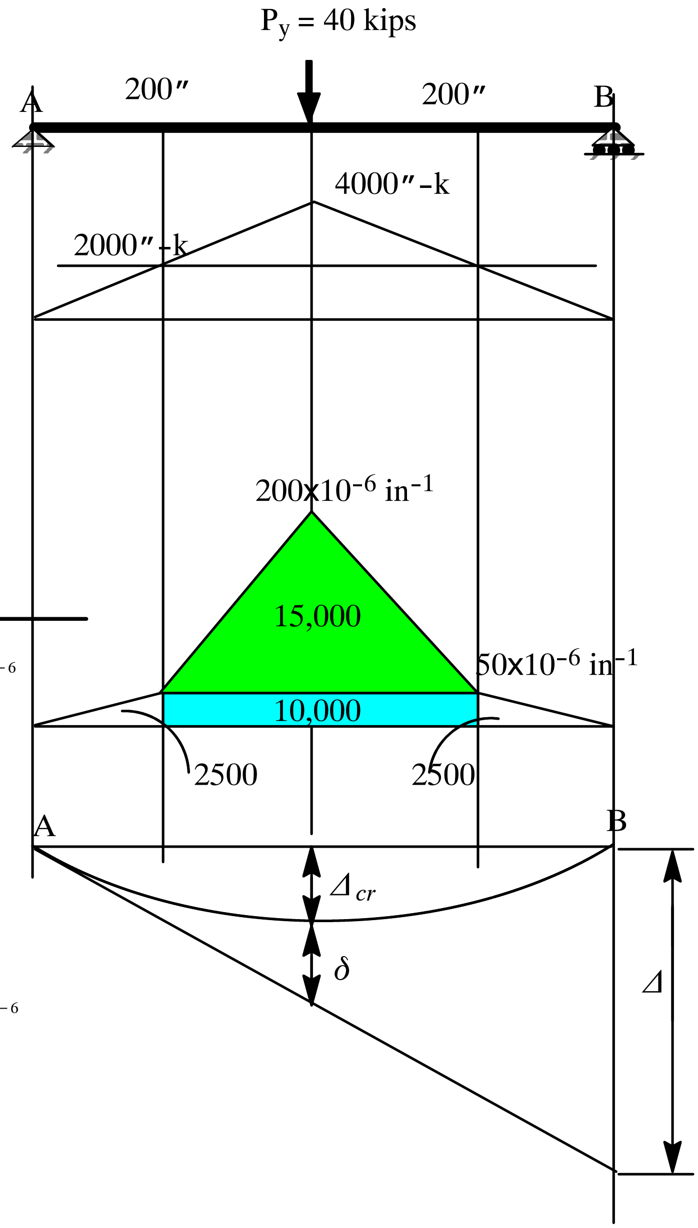
$$\Delta_{cr} = \frac{\Delta}{2} - \delta = 1 - \delta$$

$$\delta = (50 \times 10^{-6})(200)(\tfrac{1}{2})(200)(\tfrac{1}{3}) = 0.33 \ in$$

therefore,

$$\boxed{\Delta_{cr} = 1 - \delta = 1 - 0.33 = 0.67 \ in}$$

At Yielding



$\Delta \;=$

$$2,500\left(300 + \frac{100}{3}\right) = 833,333 \times 10^{-6}$$

$$(15,000 + 10,000)200 = 5,000,000 \times 10^{-6}$$

$$2,500\left(\frac{200}{3}\right) = 166,666 \times 10^{-6}$$

$$\Delta = 6,000,000 \times 10^{-6}$$

$$\Delta = 6 \; in$$

$\delta \;=$

$$2,500\left(100 + \frac{100}{3}\right) = 333,333 \times 10^{-6}$$

$$5,000(50) = 250,000 \times 10^{-6}$$

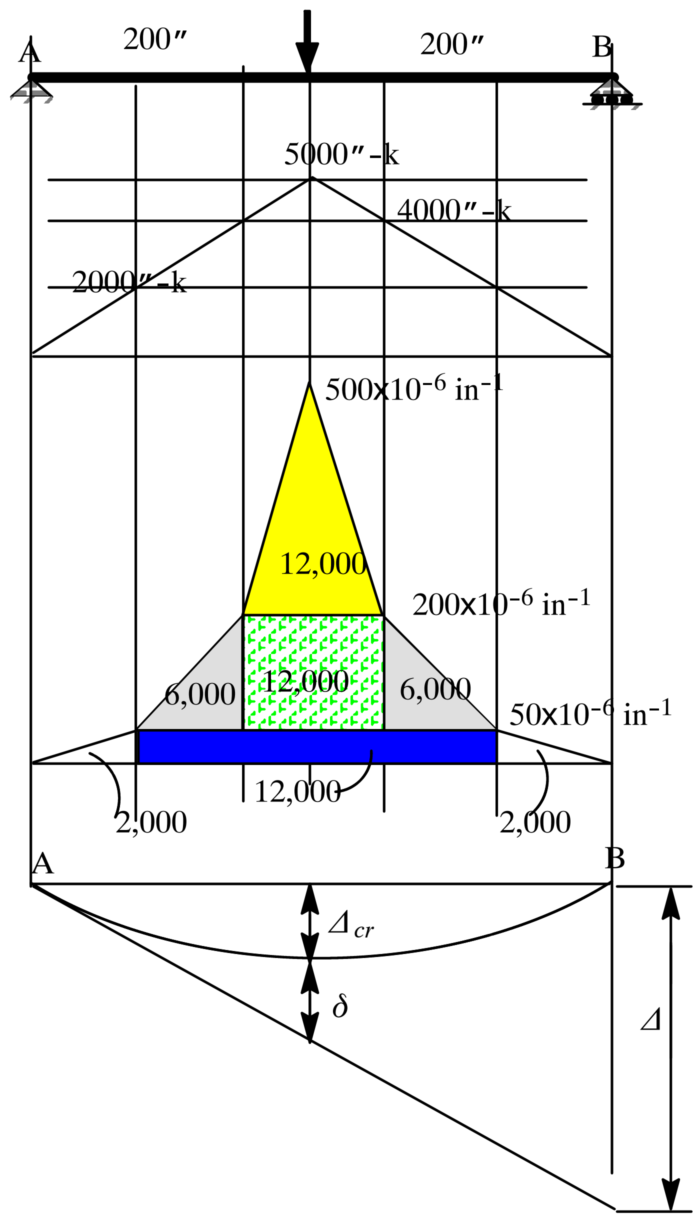
$$7,500\left(\frac{100}{3}\right) = 250,000 \times 10^{-6}$$

$$\delta = 833,333 \times 10^{-6}$$

$$\delta = 0.83 \; in$$

$$\Delta_y = \frac{\Delta}{2} - \delta = \frac{6}{2} - 0.83 = 2.17 \; in$$

At Nominal Load



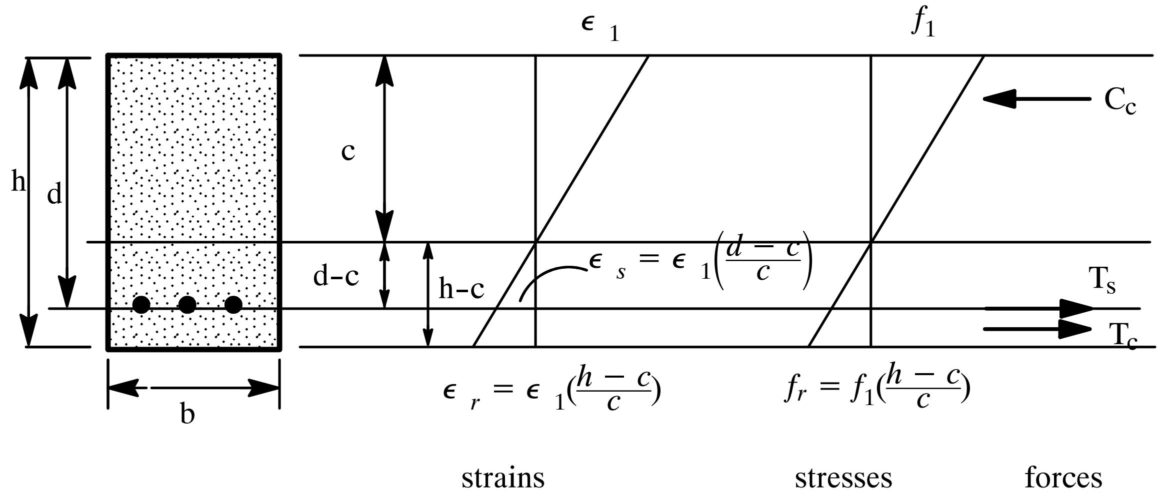
$$\Delta = \left[(3)(12,000) + (2)(6,000) + 2(2,000)\right] \times 200 \times 10^{-6} = 10.4 \ in$$

$$\delta = \left[(2,000)(120 + \frac{80}{3}) + (6,000)(6) + (6,000)(40 + \frac{80}{3}) + (6,000)(20) + (6,000)(\frac{40}{3})\right] \times 10^{-6} = 1.25 \ in$$

$$\Delta_n = \frac{\Delta}{2} - \delta = \frac{10.2}{2} - 1.25 = 3.95 \ in$$

13.11. How to Find a Moment-Curvature for a Beam

Crack Moment



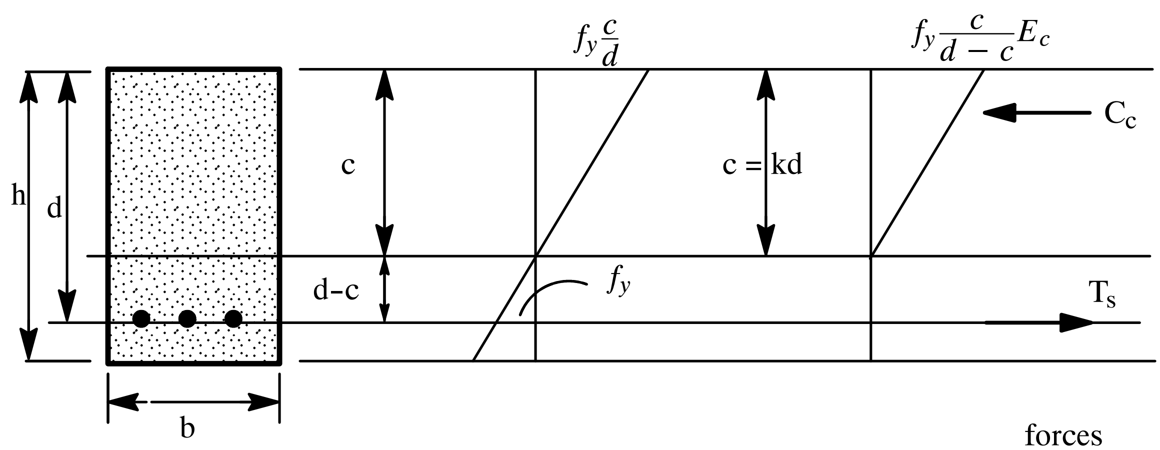
$$\rho = \frac{A_s}{bd} \qquad n = \frac{E_s}{E_c}$$

solve for "c" and find M_{cr}.

$$\frac{c}{d} = \frac{2\rho(n - 1) + (h/d)^2}{2\rho(n - 1) + 2(h/d)}$$

and curvature will be = $\phi = \frac{f_1/E_c}{c}$

Yield Moment



$$\rho = \frac{A_s}{bd} \qquad n = \frac{E_s}{E_c}$$

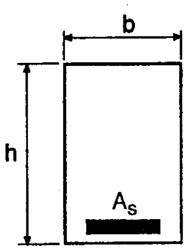
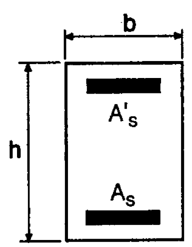
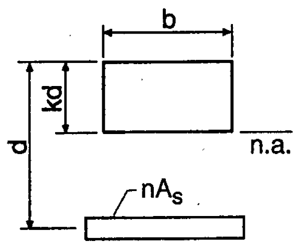
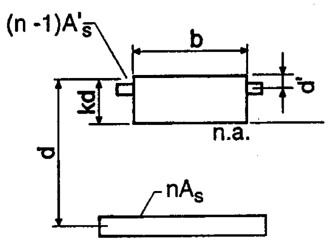
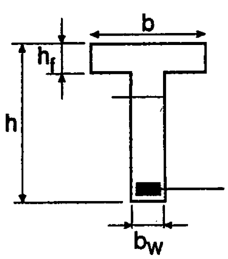
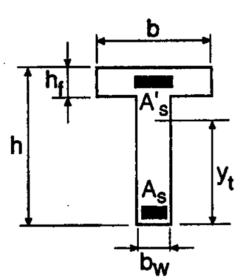
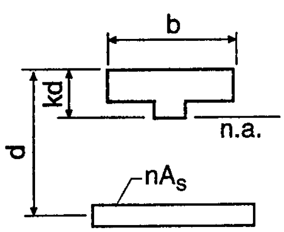
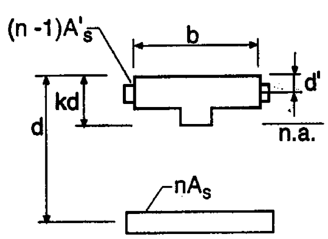
solve for "c" and find M_y.

$$k = -\rho n + \sqrt{(\rho n)^2 + 2\rho n}$$

$$c = kd$$

and curvature will be =

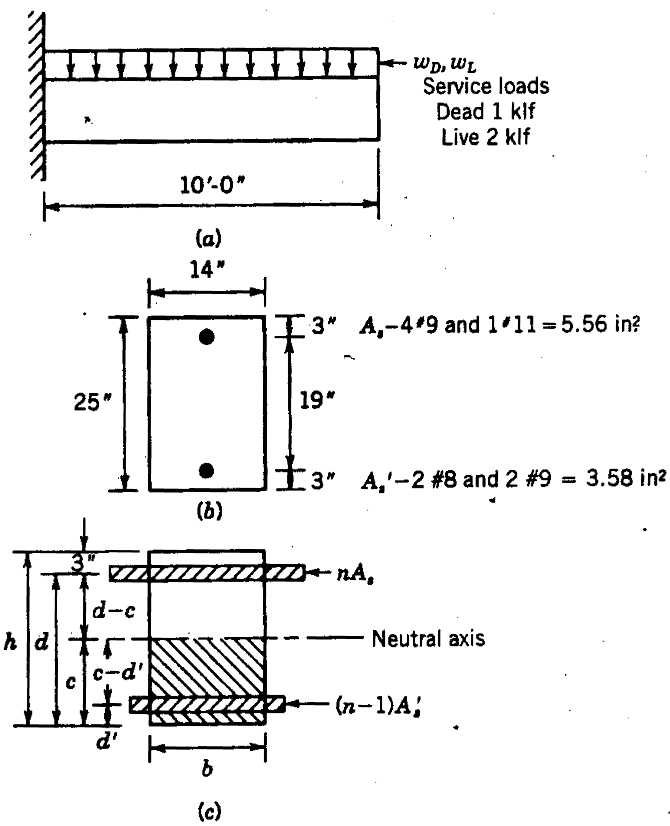
$$\phi = \frac{\epsilon_y \frac{c}{d - c}}{c} = \frac{f_y/E_C}{d - c}$$

Gross Section	Cracked Transformed Section	Gross and Cracked Moment of Inertia
 	 Without compression steel  With compression steel	$n = \frac{E_s}{E_c}$ $B = \frac{b}{nA_s}$ $I_g = \frac{bh^3}{12}$ *Without Compression Steel* $kd = (\sqrt{2dB + 1} - 1)/B$ $I_{cr} = bk^3d^3/3 + nA_s(d - kd)^2$ *With Compression Steel* $r = \frac{(n-1)A_s'}{nA_s}$ $kd = \left[\sqrt{2dB(1 + rd'/d) + (1 + r)^2} - (1 + r)\right]/B$ $I_{cr} = bk^3d^3/3 + nA_s(d - kd)^2 + (n - 1)A_s'(kd - d')^2$
 	 Without compression steel  With compression steel	$n = \frac{E_s}{E_c}$ $C = \frac{b_w}{nA_s}$ $\quad f = \frac{h_f(b - b_w)}{nA_s}$ $y_t = h - \frac{1}{2}\frac{\left[(b - b_w)h_f^2 + b_wh^2\right]}{\left[(b - b_w)h_f + b_wh\right]}$ $I_g = (b - b_w)h_f^3/12 + b_wh^3/12 +$ $+ (b - b_w)h_f(h - h_f/2 - y_t)^2 + b_wh(y_t - h/2)^2$ *Without Compression Steel* $kd = \left[\sqrt{C(2d + h_ff) + (1 + f)^2} - (1 + f)\right]/C$ $I_{cr} = (b - b_w)h_f^3/12 + b_wk^3d^3/3 +$ $+ (b-b_w)h_f(kd-h_f/2)^2 + nA_s(d-kd)^2$ *With Compression Steel* $kd = \left[\sqrt{C(2d + h_ff + 2rd') + (f + r + 1)^2} - (f + r + 1)\right]/C$ $I_{cr} = (b-b_w)h_f^3/12 + b_wk^3d^3/3 + (b-b_w)h_f(kd-h_f/2)^2 +$ $+ nA_s(d-kd)^2 + (n-1)A_s'(kd-d')^2$

Exanple.

Consider the beam and cross section shown below

1. Determine I_{cr}, I_g, *and* I_{cr}
2. Assuming 20% of the live load is a long term sustained load, calculate the long-term sustained load deflection of the free end tip.
3. Assuming that the full live load has been previously repeated numerous times, compute the instantaneous tip deflection as the load is increased from $1.0DL + 0.2LL$ to $1.0DL + 1.0LL$.
4. Assuming that $1.0DL + 0.2LL$ is in place before attachment of fragile partitions that might be damaged by large deflections, are such partitions likely to be damaged when the member is fully loaded with design live load and creep effects?



$f'_c = 3.0$ ksi

$f_y = 40$ ksi

Figure 13.1. Details considered in deflection of cantilever beam, (a) Support and loading, (b) Cross section, and (c) Transformed cracked section.

Solution

(1) $M_{cr} = f_r I_g / y_t$ (Code Eq. 9.8)

$f_r = 7.5\sqrt{f'_c} = 7.5\sqrt{3000} = 411$ psi

$I_g = bh^3/12$ (neglecting the reinforcement as permitted in the Code Sec. 9.0 definition of I_g and y_t)

$= 14 \times 25^3/12 = 18{,}230 \text{ in.}^4$ Say, $I_g = 18{,}200 \text{ in.}^4$

$y_t = h/2 = 25/2 = 12.5$

$M_{cr} = 411 \times 18\,230/12.5 = 599{,}400$ lb-in. $= 599.4$ k-in.

$= 49.9$ k-ft

I_{cr} is the moment of inertia of the cracked, transformed section shown in Fig. 3.25*c*. The modular ratio n is used to transform the reinforcement into equivalent concrete.

$$n = E_s/E_c = 29 \times 10^6/(57{,}000\sqrt{f'_c}) = 29 \times 10^6/(3.122 \times 10^6) = 9.3$$

The transformed tension reinforcement $nA_s = 9.3 \times 5.56 = 51.71 \text{ in.}^2$ The transformed compression reinforcement corrected for the displaced concrete $(n-1)A'_s = 8.3 \times 3.58 = 29.71 \text{ in.}^2$ Summing moments of the shaded areas about the neutral axis in Fig. 3.25*c* allows the depth to the neutral axis c to be determined.

$$nA_s(d-c) = (n-1)A'_s(c-d') = (bc \cdot c/2)$$

$$(51.71)(22-c) = (29.71)(c-3) + (14/2)c^2$$

Solving the quadratic equation by completing the square

$$c^2 + 11.63(+5.82^2) = 175.25(+5.82^2)$$

$$c = 8.64 \text{ in.}$$

$$I_{cr} = bc^3/12 + bc(c/2)^2 + (n-1)(A'_s)(c-d')^2 + nA_s(d-c)^2$$

$$= 14 \times 8.64^3/12 + 14 \times 8.64 \times 4.32^2 + 29.71 \times (8.64-3)^2 + 51.71 \times (22-8.64)^2$$

$$= 752 + 2257 + 945 + 9230 = 13{,}184 \text{ in.}^4$$

Say $I_{cr} = 13{,}200 \text{ in.}^4$

(2) If 20% of the live load is sustained load, the effective sustained load is $w_s = w_D + 0.2w_L = 1 + 0.2(2) = 1.4$ klf. The moment at the support owing to this load is $M_s = w_s\ell^2/2 = 1.4 \times 10^2/2 = 70$ k-ft $= 840$ k-in.

Thus $M_s > M_{cr} = 599.4$ k-in. This means that the beam will have cracked under the sustained loading and the short-time cantilever tip deflection $\Delta = w\ell^4/(8EI)$ must be determined based on I_e corresponding to $M_a = M_s$. From Code Eq. 9.7

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr}$$

$$I_e = \left(\frac{599.4}{840}\right)^3 18{,}200 + \left[1 - \left(\frac{599.4}{840}\right)^3\right] 13{,}200 = 15{,}000 \text{ in.}^4$$

Short time $\Delta_{inst} = w_s \ell^4/(8EI)$. Choosing all units in kips and inches, $\Delta_{inst} = (1.4/12)(10 \times 12)^4/(8 \times 3.122 \times 10^3 \times 15{,}000) = 0.065$ in. This short-term deflection is supplemented by the long-term deflection resulting from creep and shrinkage. This additional deflection is taken as the multiplier $\lambda = \xi/(1 + 50\rho')$ times the deflection under the sustained load. For this case, which has a very large amount of compression reinforcement, $\rho' = A_s'/bd = 3.58/14 \times 22 = 0.0116$. The values of ξ are tabulated in Code Sec. 9.5.2.5 and shown in Fig. 3.26. For very long-term loading ξ may be taken as 2.0 so that $\lambda = 2.0/(1 + 50 \times 0.0116) = 1.27$. Thus, time adds $\Delta_t = 1.27\ \Delta_{inst} = 1.27 \times 0.065 = 0.082$ in. This *adds* to the initial or instantaneous deflection computed previously to give $\Delta = 0.065 + 0.082 = 0.147$ in. as the tip deflection under sustained load. In this very stiff beam the total sustained load deflection is only $\ell/816$ so that there is no need to further check the much larger deflection limits of Code Table 9.5b for this loading case.

(3) The repeated application of design live load $w_L = 2$ klf means that the beam must be well cracked. M_a for maximum load of $w_D + w_L$ is $(1.0 + 2.0)(10)^2/2 = 150$ k-ft $= 1800$ k-in. To find the instantaneous tip deflection as the load is increased from the sustained load level, $1DL + 0.2LL$ (A) to full load $1DL + 1LL$ (B), the differential $\Delta = \Delta_B - \Delta_A$ must be determined. Δ_A is the instantaneous deflection found in part (2), that is, $\Delta_A = 0.065$ in. Δ_B will depend on I_e for $M_a = 1800$ k-in. Using Code Eq. 9.7

$$I_e = \left(\frac{599.4}{1800}\right)^3 18{,}200 + \left[1 - \left(\frac{599.4}{1800}\right)^3\right] 13{,}200 = 13{,}400 \text{ in.}^4$$

At this service load level, for all practical purposes, $I_e = I_{cr}$ in this example. Code Eq. 9.7 has been frequently criticized as being very complex for members with substantial load or shrinkage cracks when I_{cr} is more realistic and considerably simpler.

$$\Delta_B = [(1.0 + 2.0)/12](10 \times 12)^4/(8 \times 3.122 \times 10^3 \times 13{,}400)$$
$$= 0.155 \text{ in.}$$

The instantaneous deflection when the live load is increased from $0.2LL$ to $1.0LL$ is found as

$$\Delta = \Delta_B - \Delta_A = 0.155 - 0.065 = 0.09 \text{ in.}$$

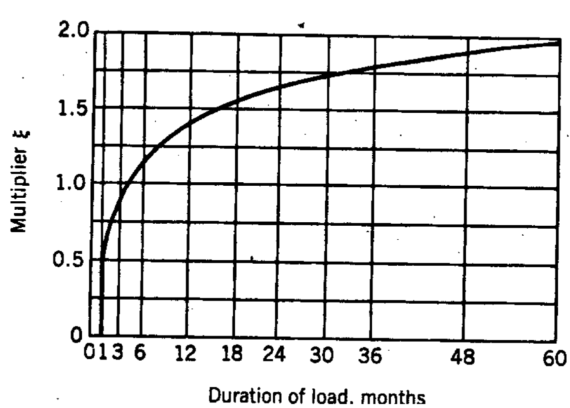


Figure 3.26 Factor for long-term deflection. (From Code Commentary.)