Chapter 16

Plane Frame Analysis Using the **Stiffness Method**

1

Stiffness method of analysis: **frames**

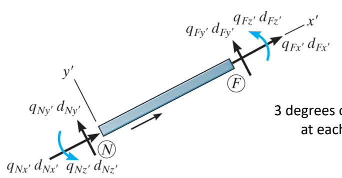
- To show how to apply the stiffness method to determine the displacements and reactions at points on a plane frame.

2

Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

- In this section we will develop the stiffness matrix for a prismatic frame member referenced from the local *x*′, *y*′, *z*′ coordinate system.



3 degrees of freedom
at each node

- Here the member is subjected to axial loads $q_{Nx'}$, $q_{Fx'}$, shear loads $q_{Ny'}$, $q_{Fy'}$, and bending moments $q_{Nz'}$, $q_{Fz'}$, at its near and far ends, respectively.

3

Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

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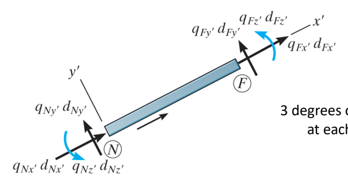

3 degrees of freedom
at each node

- These loadings all act in the positive coordinate directions along with their associated displacements.

4

Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

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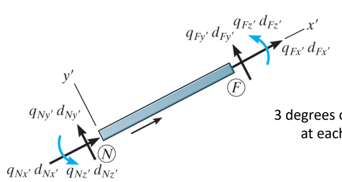

3 degrees of freedom
at each node

- The moments $q_{Nz'}$ and $q_{Fz'}$ are ***positive counterclockwise***, since by the right-hand rule the moment vectors are then directed along the positive *z*′ axis, which is out of the page.

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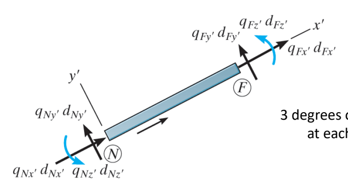
Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

- In this section we will develop the stiffness matrix for a prismatic frame member referenced from the local *x*′, *y*′, *z*′ coordinate system.



3 degrees of freedom
at each node

- We have established each of the load-displacement relationships caused by these loadings in the previous chapters.

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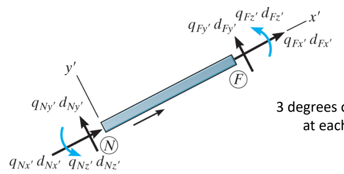
Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

- In this section we will develop the stiffness matrix for a prismatic frame member referenced from the local *x*′, *y*′, *z*′ coordinate system.



- The axial load was discussed in Chapter 14, the shear load and the bending moment in Chapter 15.

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Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

- By superposition, the resulting six load-displacement relations for the member in local coordinates in matrix form are:

$$
\begin{Bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nx'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix}
=
\begin{array}{c}
\begin{array}{cccccc} N_{x'} & N_{y'} & N_{z'} & F_{x'} & F_{y'} & F_{z'} \end{array} \\
\begin{bmatrix}
C_1 & 0 & 0 & -C_1 & 0 & 0 \\
0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\
0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\
-C_1 & 0 & 0 & C_1 & 0 & 0 \\
0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\
0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2
\end{bmatrix}
\end{array}
\begin{Bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nx'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix}
$$

$C_1 = \dfrac{AE}{L}$ $\quad C_2 = \dfrac{EI}{L^3}$ $\quad \boxed{\mathbf{q} = \mathbf{k'd}}$

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Stiffness method of analysis: **frames**

Frame-Member Stiffness Matrix

- The ***member stiffness matrix*** consists of 36 influence coefficients that physically represent the load on the member when the member undergoes a specified unit displacement.

- Specifically, each column in the matrix represents the member loadings for unit displacements identified by the code number listed above the columns.

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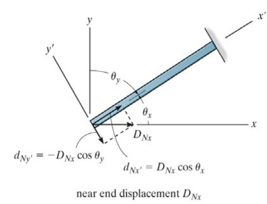
Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- As in the case for trusses, we must be able to transform the internal member loads **q** and deformations **d** from local *x*′, *y*′, *z*′ coordinates to global *x*, *y*, *z* coordinates.



$$d_{Nx'} = D_{Nx}\cos\theta_x$$

$$d_{Ny'} = -D_{Nx}\cos\theta_y$$

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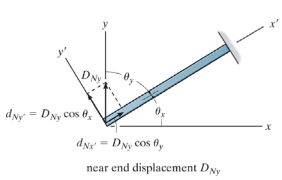
Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- As in the case for trusses, we must be able to transform the internal member loads **q** and deformations **d** from local *x*′, *y*′, *z*′ coordinates to global *x*, *y*, *z* coordinates.



$$d_{Nx'} = D_{Ny}\cos\theta_y$$

$$d_{Ny'} = -D_{Ny}\cos\theta_x$$

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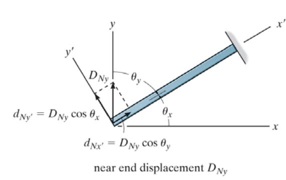
Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- Finally, since the z' and z axes are coincident, a rotation D_{Nz} about z causes a corresponding rotation about $d_{Nz'}$ about z'.



$$d_{Nz'} = D_{Nz}$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- In a similar manner, if global displacements D_{Fx} in the x direction, D_{Fy} in the y direction, and a rotation D_{Fz} are imposed on the far end of the member, the resulting transformation equations become:

$$d_{Fx'} = D_{Fx}\cos\theta_x \qquad d_{Fx'} = D_{Fy}\cos\theta_y \qquad d_{Fz'} = D_{Fz}$$

$$d_{Fy'} = -D_{Fx}\cos\theta_y \qquad d_{Fy'} = -D_{Fy}\cos\theta_x$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- Letting $\lambda_x = \cos\theta_x$ and $\lambda_y = \cos\theta_y$ represent the direction cosines of the member; we can write the superposition of displacements in matrix form as:

$$\underbrace{\begin{Bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nz'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix}}_{\{\mathbf{d}\}} = \frac{EI}{L^3}\underbrace{\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[\mathbf{T}]}\underbrace{\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Nz} \\ D_{Fx} \\ D_{Fy} \\ D_{Fz} \end{Bmatrix}}_{\{\mathbf{D}\}} \qquad \boxed{\mathbf{d} = \mathbf{TD}}$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- By inspection, **T** transforms the six global x, y, z displacements **D** into the six local x', y', z' displacements **d**.

$$\underbrace{\begin{Bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nz'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix}}_{\{\mathbf{d}\}} = \frac{EI}{L^3}\underbrace{\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[\mathbf{T}]}\underbrace{\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Nz} \\ D_{Fx} \\ D_{Fy} \\ D_{Fz} \end{Bmatrix}}_{\{\mathbf{D}\}} \qquad \boxed{\mathbf{d} = \mathbf{TD}}$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- **T** is referred to as the ***displacement transformation matrix***.

$$\mathbf{T} = \frac{EI}{L^3}\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boxed{\mathbf{d} = \mathbf{TD}}$$

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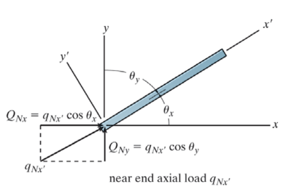
Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- If we now apply each component of load to the ***near*** end of the member, we can transform the load components from local to global coordinates. Consider the ***axial load*** $q_{Nx'}$:



$$Q_{Nx} = q_{Nx'}\cos\theta_x$$

$$Q_{Ny} = q_{Nx'}\cos\theta_y$$

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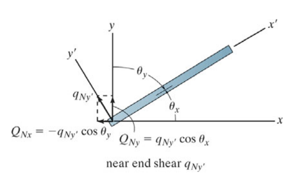
Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- If the ***shear load*** $q_{Ny'}$ is applied, then its components are:



$$Q_{Nx} = -q_{Ny'}\cos\theta_y$$

$$Q_{Ny} = q_{Ny'}\cos\theta_x$$

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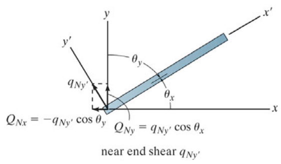
Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- Finally, the bending moment $q_{Nz'}$ is collinear with Q_{Nz}, and so:



$$Q_{Nz} = q_{Nz'}$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- In a similar manner, the far end loads $q_{Fx'}$, $q_{Fy'}$, $q_{Fz'}$ give the following components:

$$Q_{Fx} = q_{Fx'} \cos\theta_x \qquad Q_{Fx} = -q_{Fy'} \cos\theta_y \qquad Q_{Fz} = q_{Fz'}$$

$$Q_{Fy} = q_{Fx'} \cos\theta_y \qquad Q_{Fy} = q_{Fy'} \cos\theta_x$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- Letting $\lambda_x = \cos\theta_x$ and $\lambda_y = \cos\theta_y$ represent the direction cosines of the member; we can write the superposition of displacements in matrix form as:

$$\underbrace{\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Nz} \\ Q_{Fx} \\ Q_{Fy} \\ Q_{Fz} \end{Bmatrix}}_{\{\mathbf{Q}\}} = \frac{EI}{L^3} \underbrace{\left[\begin{array}{ccc:ccc} \lambda_x & -\lambda_y & 0 & 0 & 0 & 0 \\ \lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \lambda_x & -\lambda_y & 0 \\ 0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathbf{T}^T]} \underbrace{\begin{Bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix}}_{\{\mathbf{q}\}} \qquad \boxed{\mathbf{Q} = \mathbf{T}^T\mathbf{q}}$$

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Stiffness method of analysis: **frames**

Displacement and Force Transformation Matrices

- Here the ***force transformation matrix*** $\mathbf{T}^T$ transforms the six member loads expressed in local coordinates into the six loadings expressed in global coordinates.

$$\underbrace{\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Nz} \\ Q_{Fx} \\ Q_{Fy} \\ Q_{Fz} \end{Bmatrix}}_{\{\mathbf{Q}\}} = \frac{EI}{L^3} \underbrace{\left[\begin{array}{ccc:ccc} \lambda_x & -\lambda_y & 0 & 0 & 0 & 0 \\ \lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \lambda_x & -\lambda_y & 0 \\ 0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathbf{T}^T]} \underbrace{\begin{Bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix}}_{\{\mathbf{q}\}} \qquad \boxed{\mathbf{Q} = \mathbf{T}^T\mathbf{q}}$$

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Stiffness method of analysis: **frames**

Frame-Member Global Stiffness Matrix

- The results of the previous sections will now be combined to determine the stiffness matrix for a member that relates the global loadings **Q** to the global displacements **D**.

- To do this, substitute **d** = **TD** into **q** = **k'd** to get **q** = **k'TD.**

- Substituting this result into **Q** = **T**^{T}**q** give: $\mathbf{Q} = \mathbf{T}^T\mathbf{k}'\mathbf{T}\mathbf{D}$

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{D}} \qquad \boxed{\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T}}$$

- Here **k** represents the ***global stiffness matrix*** for the member.

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Stiffness method of analysis: **frames**

Frame-Member Global Stiffness Matrix

- The ***global stiffness matrix*** is:

$$\mathbf{k} = \begin{array}{c} \begin{array}{cccccc} N_x & N_y & N_z & F_x & F_y & F_z \end{array} \\ \left[\begin{array}{ccc:ccc} \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y \\ \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x \\ -\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} \\ \hdashline -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y \\ -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x \\ -\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} \end{array}\right] \end{array} \begin{array}{c} \\ N_x \\ N_y \\ N_z \\ F_x \\ F_y \\ F_z \end{array}$$

- As expected, this 6×6 matrix is *symmetric*.

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Stiffness method of analysis: **frames**

Frame-Member Global Stiffness Matrix

- The ***global stiffness matrix*** is:

$$\mathbf{k} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|} N_x & N_y & N_z & F_x & F_y & F_z \end{array}$$

$$\mathbf{k} = \left[\begin{array}{ccc|ccc} \left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y \\ \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x \\ -\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} \\ \hline -\left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2+\frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y \\ -\left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L}-\frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2+\frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x \\ -\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} \end{array}\right]\begin{array}{c} N_x \\ N_y \\ N_z \\ F_x \\ F_y \\ F_z \end{array}$$

- The location of each element in the matrix is defined by the code number at the ***near*** end, N_x, N_y, and N_z, and the ***far*** end, F_x, F_y, and F_z

Stiffness method of analysis: **frames**

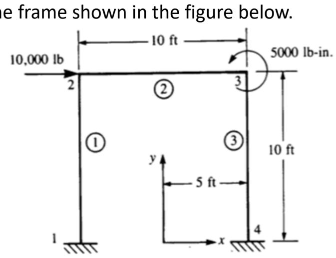
Application of the Stiffness Method for Frame Analysis

- Once the member stiffness matrices are established, they may be assembled into the structure stiffness matrix in the usual manner.
- By writing the structure stiffness equation **Q** = **KD**, the matrices can be partitioned and the displacements at the unconstrained nodes can then be determined, followed by the reactions and internal loadings at the nodes.
- Distributed loads acting on a member, fabrication errors, temperature changes, inclined supports, and internal supports are handled in the same manner as was outlined for trusses and beams.

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- Consider the frame shown in the figure below.

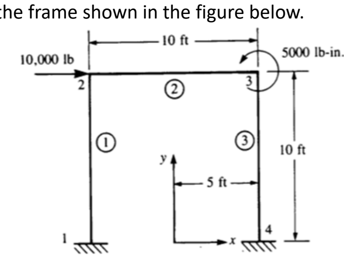


- The frame is fixed at nodes 1 and 4 and subjected to a positive horizontal force of 10,000 *lb.* applied at node 2 and to a positive moment of 5,000 *lb-in* applied at node 3.

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- Consider the frame shown in the figure below.

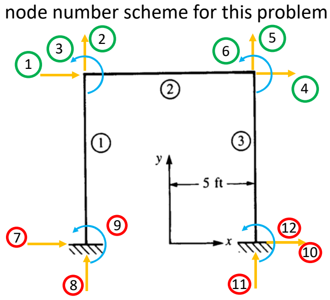


- Let E = 30 x 10^6 *psi* and A = 10 in^2 for all elements, and let I = 200 in^4 for elements 1 and 3, and I = 100 in^4 for element 2.

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

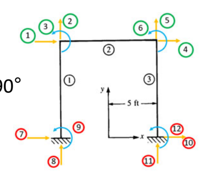
- Here is the node number scheme for this problem



- Code numbers 1 through 6 represent the unknowns, and numbers 7 through 12 represent the knowns.

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem



- **Element 1:** The angle between x and x' is 90°

$$\theta_x = 90^\circ \qquad \theta_y = 0^\circ$$

$$\lambda_x = 0 \qquad \lambda_y = 1$$

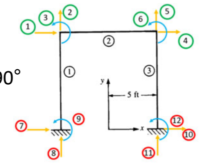
$$\frac{12I}{L^2} = \frac{12(200\,in^4)}{(120\,in)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200\,in^4)}{120\,in} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30\times10^6\,psi}{120\,in} = 250{,}000\ {}^{lb}\!/\!{}_{in^3}$$

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

➢ **Element 1:** The angle between x and x' is 90°



$$\mathbf{k}^{(1)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} 7 & 8 & 9 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc|ccc} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ \hline -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{array}\right] \end{array} \, {}^{lb}\!/_{in}$$

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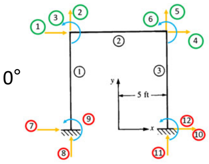
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

➢ **Element 2:** The angle between x and x' is 0°



$$\theta_x = 0^\circ \qquad \theta_y = 90^\circ$$

$$\lambda_x = 1 \qquad \lambda_y = 0$$

$$\frac{12I}{L^2} = \frac{12(100\,in^4)}{(120\,in)^2} = 0.0835\ in^2 \qquad \frac{6I}{L} = \frac{6(100\,in^4)}{120\,in} = 5.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6\,psi}{120\,in} = 250{,}000\ {}^{lb}\!/_{in^3}$$

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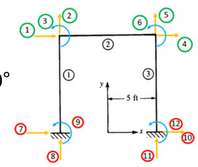
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

➢ **Element 2:** The angle between x and x' is 0°



$$\mathbf{k}^{(2)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & 0.0835 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ \hline -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0.0835 & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{array}\right] \end{array} \, {}^{lb}\!/_{in}$$

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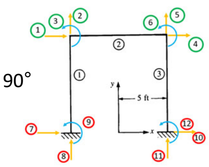
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

➢ **Element 3:** The angle between x and x' is 90°



$$\theta_x = 90^\circ \qquad \theta_y = 180^\circ$$

$$\lambda_x = 0 \qquad \lambda_y = -1$$

$$\frac{12I}{L^2} = \frac{12(200\,in^4)}{(120\,in)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200\,in^4)}{120\,in} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6\,psi}{120\,in} = 250{,}000\ {}^{lb}\!/_{in^3}$$

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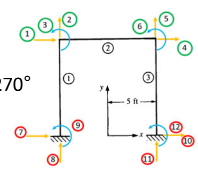
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

➢ **Element 3:** The angle between x and x' is 270°



$$\mathbf{k}^{(3)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 10 & 11 & 12 \end{array} \\ \left[\begin{array}{ccc|ccc} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ \hline -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{array}\right] \end{array} \, {}^{lb}\!/_{in}$$

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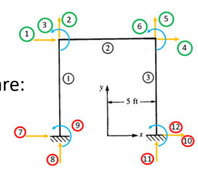
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

➢ The boundary conditions for this problem are:

$$D_7 = D_8 = D_9 = D_{10} = D_{11} = D_{12} = 0$$



➢ After applying the boundary conditions the global beam equations reduce to:

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5{,}000 \end{Bmatrix} = 2.5 \times 10^5 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ \hline -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{array}\right] \end{array} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

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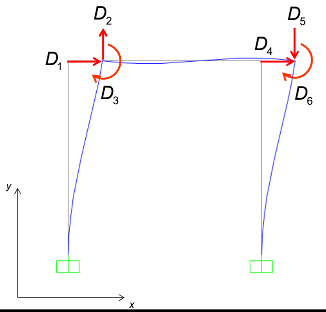
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- Solving the above equations gives:

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

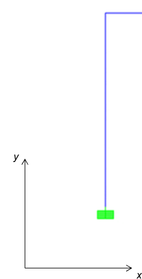


Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- Solving the above equations gives:

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

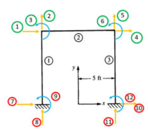


Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 1:** The element force-displacement equations can be obtained using **q** = **k'TD**. Therefore, **TD** is:

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \lambda_x = 0 \qquad \lambda_y = 1$$



$$\mathbf{TD} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_7 = 0 \\ D_8 = 0 \\ D_9 = 0 \\ D_1 = 0.211\ in \\ D_2 = 0.00148\ in \\ D_3 = -0.00153\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\ in \\ -0.211\ in \\ -0.00153\ rad \end{Bmatrix}$$

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 1:** Recall the elemental stiffness matrix is:

$$\mathbf{k}' = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right]$$

$$C_1 = \frac{AE}{L} = \frac{10\ in^2 \left(30 \times 10^6\ psi\right)}{120\ in} = 2.5 \times 10^6\ {}^{lb}\!/_{in}$$

$$C_2 = \frac{EI}{L^3} = \frac{200\ in^4 \left(30 \times 10^6\ psi\right)}{\left(120\ in\right)^3} = 3{,}472.22\ {}^{lb}\!/_{in}$$

- The local force-displacement equations are:

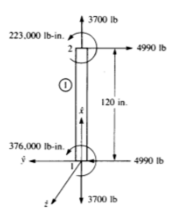
$$\mathbf{q}^{(1)} = \mathbf{k}'\mathbf{TD} = 2.5 \times 10^5 \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ \hline -10 & 0 & 0 & 10 & 0 & 10 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\ in \\ -0.211\ in \\ -0.00153\ rad \end{Bmatrix}$$

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 1:** Simplifying the above equations gives:

$$\begin{Bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix}_{\boxed{1}} = \begin{Bmatrix} -3{,}700\ lb \\ 4{,}990\ lb \\ 376\ k \cdot in \\ 3{,}700\ lb \\ -4{,}990\ lb \\ 223\ k \cdot in \end{Bmatrix}$$

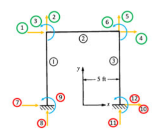


Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 2:** The element force-displacement equations can be obtained using **f'** = **k'TD**. Therefore, **TD** is:

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \lambda_x = 1 \qquad \lambda_y = 0$$



$$\mathbf{TD} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_1 = 0.211\ in \\ D_2 = 0.00148\ in \\ D_3 = -0.00153\ rad \\ D_4 = 0.209\ in \\ D_5 = -0.00148\ in \\ D_6 = -0.00149\ rad \end{Bmatrix} = \begin{Bmatrix} 0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 2:** The local force-displacement equations are:

$$\mathbf{k}'=\left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0\\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2\\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2\\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0\\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2\\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2\end{array}\right]$$

$$C_1=\frac{AE}{L}=\frac{10\,in^2\left(30\times10^6\,psi\right)}{120\,in}=2.5\times10^6\ {}^{lb}/_{in}$$

$$C_2=\frac{EI}{L^3}=\frac{100\,in^4\left(30\times10^6\,psi\right)}{\left(120\,in\right)^3}=1{,}736.11\ {}^{lb}/_{in}$$

- The local force-displacement equations are:

$$\mathbf{q}^{(2)}=\mathbf{k}'\mathbf{TD}=2.5\times10^5\left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0\\ 0 & 0.0833 & 5 & 0 & -0.0833 & 5\\ 0 & 5 & 400 & 0 & -5 & 200\\ \hline -10 & 0 & 0 & 10 & 0 & 0\\ 0 & -0.0833 & -5 & 0 & 0.0833 & -5\\ 0 & 5 & 200 & 0 & -5 & 400\end{array}\right]\begin{Bmatrix} 0.211\,in\\ 0.00148\,in\\ -0.00153\,rad\\ 0.209\,in\\ -0.00148\,in\\ -0.00149\,rad\end{Bmatrix}$$

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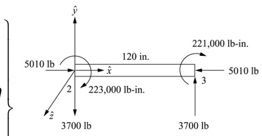
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 2:** Simplifying the above equations gives:

$$\begin{Bmatrix} q_{Nx'}\\ q_{Ny'}\\ q_{Nz'}\\ q_{Fx'}\\ q_{Fy'}\\ q_{Fz'}\end{Bmatrix}_{\boxed{2}}=\begin{Bmatrix} 5{,}010\,lb\\ -3{,}700\,lb\\ -223\,k\cdot in\\ -5{,}010\,lb\\ 3{,}700\,lb\\ -221\,k\cdot in\end{Bmatrix}$$



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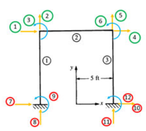
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 3:** The element force-displacement equations can be obtained using **f'** = **k'TD**. Therefore, **TD** is:

$$\mathbf{T}=\left[\begin{array}{ccc|ccc} \lambda_x & \lambda_y & 0 & 0 & 0 & 0\\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & \lambda_x & \lambda_y & 0\\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{array}\right]\qquad \begin{aligned}\lambda_x&=0\\ \lambda_y&=-1\end{aligned}$$



$$\mathbf{TD}=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix} D_4=0.209\,in\\ D_5=-0.00148\,in\\ D_6=-0.00149\,rad\\ D_{10}=0\\ D_{11}=0\\ D_{12}=0\end{Bmatrix}=\begin{Bmatrix} 0.00148\,in\\ 0.209\,in\\ -0.00149\,rad\\ 0\\ 0\\ 0\end{Bmatrix}$$

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Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 3:** The local force-displacement equations are:

$$\mathbf{k}'=\left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0\\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2\\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2\\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0\\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2\\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2\end{array}\right]$$

$$C_1=\frac{AE}{L}=\frac{10\,in^2\left(30\times10^6\,psi\right)}{120\,in}=2.5\times10^6\ {}^{lb}/_{in}$$

$$C_2=\frac{EI}{L^3}=\frac{200\,in^4\left(30\times10^6\,psi\right)}{\left(120\,in\right)^3}=3{,}472.22\ {}^{lb}/_{in}$$

- The local force-displacement equations are:

$$\mathbf{q}^{(3)}=\mathbf{k}'\mathbf{TD}=2.5\times10^5\left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -10 & 0 & 0\\ 0 & 0.167 & 10 & 0 & -0.167 & 10\\ 0 & 10 & 800 & 0 & -10 & 400\\ \hline -10 & 0 & 0 & 10 & 0 & 10\\ 0 & -0.167 & -10 & 0 & 0.167 & -10\\ 0 & 10 & 400 & 0 & -10 & 800\end{array}\right]\begin{Bmatrix} 0.00148\,in\\ 0.209\,in\\ -0.00149\,rad\\ 0\\ 0\\ 0\end{Bmatrix}$$

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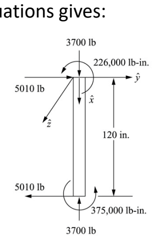
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- **Element 3:** Simplifying the above equations gives:

$$\begin{Bmatrix} q_{Nx'}\\ q_{Ny'}\\ q_{Nz'}\\ q_{Fx'}\\ q_{Fy'}\\ q_{Fz'}\end{Bmatrix}_{\boxed{3}}=\begin{Bmatrix} 3{,}700\,lb\\ 5{,}010\,lb\\ 226\,k\cdot in\\ -3{,}700\,lb\\ -5{,}010\,lb\\ 375\,k\cdot in\end{Bmatrix}$$



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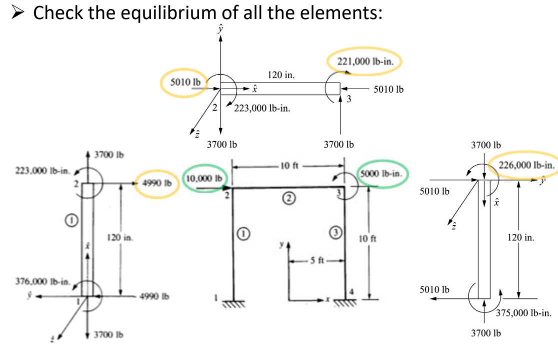
Stiffness method of analysis: **frames**

Example 16.1 – Beam Problem

- Check the equilibrium of all the elements:



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Stiffness method of analysis: **beams**

Let's work some problems

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Displacement method of analysis: **slope-defection method**

Any questions?

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