Chapter 15

Beam Analysis Using the **Stiffness Method**

Stiffness method of analysis: **beams**

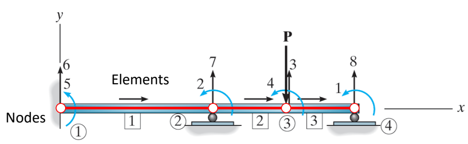
Member and Node Identification

- To apply the stiffness method to beams, we must first determine how to ***subdivide the beam*** into its component ***finite elements***.
- In general, each element must be ***free from load and have a prismatic cross section***.
- The nodes of each element are located at a support or at points where:
  1. members are connected,
  2. an external force is applied,
  3. the cross-sectional area suddenly changes, or
  4. the displacement or rotation is to be determined.

Stiffness method of analysis: **beams**

Member and Node Identification

- For example, consider the beam below:

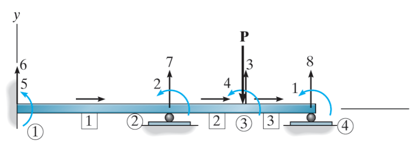


- Using the same scheme as that for trusses, four nodes are specified numerically within a circle , and the three elements are identified numerically within a square.

Stiffness method of analysis: **beams**

Member and Node Identification

- For example, consider the beam below:

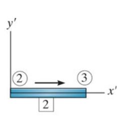


- Finally, the "**near**" and "**far**" ends of each element are identified by the arrows written alongside each element.

Stiffness method of analysis: **beams**

Global and Member Coordinates

- The global coordinate system will be identified using *x*, *y*, *z* axes that generally have their origin at a node and are positioned so that the nodes at other points on the beam all have positive coordinates.

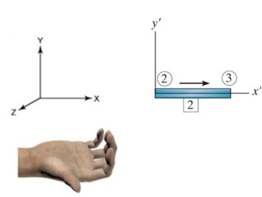


- The local or member *x′*, *y′*, *z′* coordinates have their origin at the "**near**" end of each element, and the positive axis is directed towards the "**far**" end.

Stiffness method of analysis: **beams**

Global and Member Coordinates

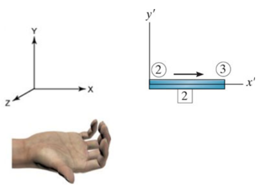
- In both cases we have used a ***right-handed coordinate system***, so that when the fingers of the right hand are curled from the *x* (*x′*) axis towards the *y* (*y′*) axis, the thumb points in the positive direction of the *z* (*z′*) axis, which is directed out of the page.



Stiffness method of analysis: **beams**

Global and Member Coordinates

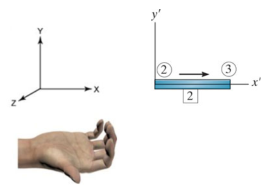
- Notice that for each beam element the *x* and *x'* axes will be collinear, and the global and member coordinates will all be parallel.



Stiffness method of analysis: **beams**

Global and Member Coordinates

- Therefore, unlike the case for trusses, here we ***will not*** need to develop ***transformation matrices*** between these coordinate systems.



Stiffness method of analysis: **beams**

Code Numbers

- Once the elements and nodes have been identified, and the global coordinate system has been established, the ***degrees of freedom*** for the beam and its kinematic determinacy can be determined.

- If we consider the effects of both bending and shear and neglect axial deformation, then ***each node*** on a beam can have two degrees of freedom, namely, a ***vertical displacement and a rotation.***

Stiffness method of analysis: **beams**

Code Numbers

- As in the case of trusses, these linear and rotational displacements will be identified by code numbers.

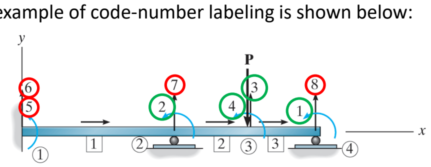
- The ***lowest code numbers will be used to identify the unknown displacements*** (unconstrained degrees of freedom), and the highest numbers to identify the known displacements (constrained degrees of freedom).

- Recall that the reason for choosing this method of identification has to do with the convenience of later ***partitioning the structure stiffness matrix***, so that the unknown displacements can be found in the most direct manner.

Stiffness method of analysis: **beams**

Code Numbers

- An example of code-number labeling is shown below:



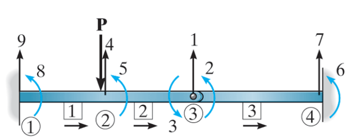
- Here there are eight degrees of freedom.

- Code numbers 1 through 4 represent the unknowns, and numbers 5 through 8 represent the knowns, which in this case are all zero.

Stiffness method of analysis: **beams**

Code Numbers

- Consider, another example beam shown below:



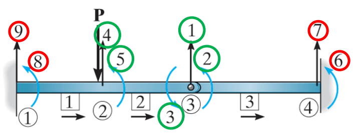
- Notice that the internal hinge at node 3 deflects the same for both elements 2 and 3; however, the rotation at the end of each element is different.

- For this reason, ***three code numbers*** are used to show these deflections.

Stiffness method of analysis: **beams**

Code Numbers

- Consider, another example beam shown below:

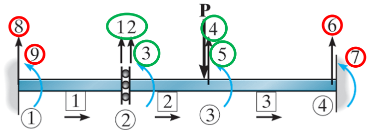


- Here there are nine degrees of freedom, five of which are unknown.
- Code numbers **1** through **5** represent the unknowns, and numbers **6** through **9** represent the knowns, which in this case are all zero.

Stiffness method of analysis: **beams**

Code Numbers

- Consider the slider mechanism used on the beam below:

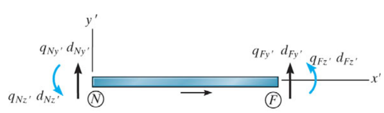


- There are five unknown deflection components labeled with the lowest code numbers.
- Code numbers **1** through **5** represent the unknowns, and numbers **6** through **9** represent the knowns, which in this case are all zero.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- We will now develop the ***stiffness matrix for a beam element*** or member having a constant cross-sectional area and referenced from the local *x′, y′, z′* coordinate system.



positive sign convention

- The origin of the coordinates is placed at the "**near**" end *N*, and the positive axis extends toward the "**far**" end *F*.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

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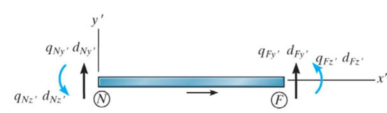


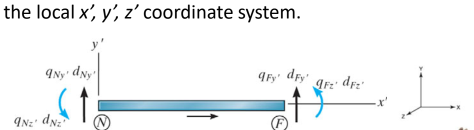
positive sign convention

- There are two reactions at each end of the element, consisting of shear forces $q_{Ny'}$ and $q_{Fy'}$ and bending moments $q_{Nz'}$ and $q_{Fz'}$.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- We will now develop the stiffness matrix for a beam element or member having a constant cross-sectional area and referenced from the local *x′, y′, z′* coordinate system.



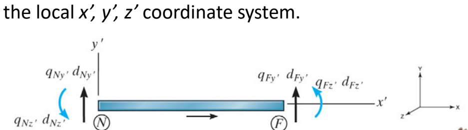
positive sign convention

- Notice the moments $q_{Nz'}$ and $q_{Fz'}$ are positive ***counterclockwise***, since by the ***right-hand rule*** the moment vectors are then directed along the positive axis.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- We will now develop the stiffness matrix for a beam element or member having a constant cross-sectional area and referenced from the local *x′, y′, z′* coordinate system.



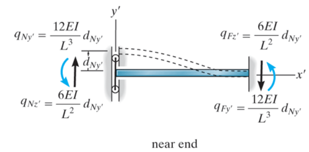
positive sign convention

- Linear and angular displacements associated with these loadings also follow this same ***positive sign convention***.

Stiffness method of analysis: **beams**

y' Displacements

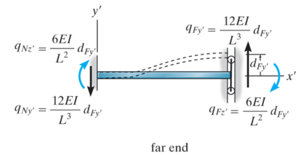
- When a positive displacement $d_{Ny'}$ is imposed on the element, while other possible displacements are ***prevented***, the required shear forces and bending moments are:



Stiffness method of analysis: **beams**

y' Displacements

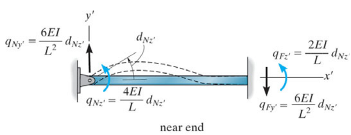
- Likewise, when $d_{Fy'}$ is imposed, the necessary shear forces and bending moments are:



Stiffness method of analysis: **beams**

z' Rotations

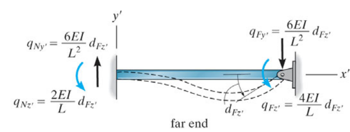
- To impose a positive rotation $d_{Nz'}$ while all other displacements are prevented, the required shear forces and bending moments are:



Stiffness method of analysis: **beams**

z' Rotations

- Likewise, when $d_{Fz'}$ is imposed, the resultant loadings are



Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- If the above results are added, the resulting ***four load-displacement relations*** for the member can then be expressed in matrix form as:

$$\begin{Bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \end{array} \\ \left[\begin{array}{cc:cc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ \hdashline -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right] \end{array} \begin{Bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix} \qquad \mathbf{q} = \mathbf{k}\mathbf{d}$$

- The matrix **k** is referred to as the ***member stiffness matrix***.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- If the above results are added, the resulting ***four load-displacement relations*** for the member can then be expressed in matrix form as:

$$\begin{Bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \end{array} \\ \left[\begin{array}{cc:cc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ \hdashline -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right] \end{array} \begin{Bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix} \qquad \mathbf{q} = \mathbf{k}\mathbf{d}$$

- The 16 influence coefficients k_{ij} account for the load on the member when the member undergoes a specified unit displacement.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- If the above results are added, the resulting ***four load-displacement relations*** for the member can then be expressed in matrix form as:

$$\begin{array}{c} \\ \begin{Bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix} \end{array} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{Bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix} \end{array} \qquad \mathbf{q} = \mathbf{kd}$$

- For example, if $d_{Ny'} = 1$ *while all other displacements are zero,* the member will be subjected only to the four loadings indicated in the first column of the **k** matrix.

Stiffness method of analysis: **beams**

Beam-Member Stiffness Matrix

- If the above results are added, the resulting ***four load-displacement relations*** for the member can then be expressed in matrix form as:

$$\begin{array}{c} \\ \begin{Bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix} \end{array} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{Bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{Bmatrix} \end{array} \qquad \mathbf{q} = \mathbf{kd}$$

- In a similar manner, the other columns are the member loadings for unit displacements identified by code numbers listed above the columns.

Stiffness method of analysis: **beams**

Beam-Structure Stiffness Matrix

- Once all the ***member stiffness matrices*** have been found, we can then assemble them into the ***structure stiffness matrix*** **K**.
- Each **k** matrix are identified by the two code numbers at the near end of the member ($N_{y'}$, $N_{z'}$) followed by those at the far end ($F_{y'}$, $F_{z'}$)
- Like in the analysis of a truss, when assembling the matrices, each matrix element must be placed in the same location of the **K** matrix.

Stiffness method of analysis: **beams**

Beam-Structure Stiffness Matrix

- Once all the ***member stiffness matrices*** have been found, we can then assemble them into the ***structure stiffness matrix*** **K**.
- In this way, **K** will have an order that will be equal to the highest code number assigned to the beam.
- This represents the total number of ***degrees of freedom***.

Stiffness method of analysis: **beams**

Application of the Stiffness Method for Beam Analysis

- After the structure stiffness matrix is determined, the loads **Q** at the nodes of the beam can then be related to the displacements **D** using the structure stiffness equation:

$$\mathbf{Q} = \mathbf{KD}$$

- Partitioning the matrices into the known and unknown elements of load and displacement, we have:

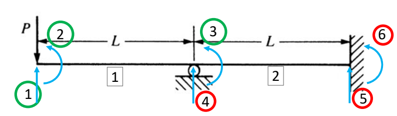
$$\begin{Bmatrix} \mathbf{Q}_k \\ \mathbf{Q}_u \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_u \\ \mathbf{D}_k \end{Bmatrix} \qquad \begin{aligned} \mathbf{Q}_k &= \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k \\ \mathbf{Q}_u &= \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k \end{aligned}$$

- The unknown displacements $\mathbf{D}_u$ are determined from the first of these equations. Then using these values, the support reactions $\mathbf{Q}_u$ are calculated from the second equation.

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- Consider the beam shown below. Assume that EI is constant, and the length is $2L$ (no shear deformation).



- The beam element stiffness matrices are:

$$\mathbf{k}^{(1)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 3 \end{array} \qquad \mathbf{k}^{(2)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} 4 & 3 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 3 \\ 5 \\ 6 \end{array}$$

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).
- The total stiffness matrix can be assembled as:

$$
\begin{array}{cccc|c}
1 & 2 & 4 & 3 & \\
\hline
12 & 6L & -12 & 6L & 1\\
6L & 4L^2 & -6L & 2L^2 & 2\\
-12 & -6L & 12 & -6L & 4\\
6L & 2L^2 & -6L & 4L^2 & 3
\end{array}
\qquad
\begin{array}{cccc|c}
4 & 3 & 5 & 6 & \\
\hline
12 & 6L & -12 & 6L & 4\\
6L & 4L^2 & -6L & 2L^2 & 3\\
-12 & -6L & 12 & -6L & 5\\
6L & 2L^2 & -6L & 4L^2 & 6
\end{array}
$$

Element 1 Element 2

Stiffness method of analysis: **beams**

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$$
\begin{array}{cccc|c}
1 & 2 & 4 & 3 & \\
\hline
12 & 6L & -12 & 6L & 1\\
6L & 4L^2 & -6L & 2L^2 & 2\\
-12 & -6L & 12 & -6L & 4\\
6L & 2L^2 & -6L & 4L^2 & 3
\end{array}
\qquad
\begin{array}{cccc|c}
4 & 3 & 5 & 6 & \\
\hline
12 & 6L & -12 & 6L & 4\\
6L & 4L^2 & -6L & 2L^2 & 3\\
-12 & -6L & 12 & -6L & 5\\
6L & 2L^2 & -6L & 4L^2 & 6
\end{array}
$$

Element 1 Element 2

Stiffness method of analysis: **beams**

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- The total stiffness matrix can be assembled as:

$$
\begin{array}{cccccc|c}
1 & 2 & 4 & 3 & 5 & 6 & \\
\hline
12 & 6L & -12 & 6L & & & 1\\
6L & 4L^2 & -6L & 2L^2 & & & 2\\
-12 & -6L & 24 & 0 & -12 & 6L & 4\\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 3\\
 & & -12 & -6L & 12 & -6L & 5\\
 & & 6L & 2L^2 & -6L & 4L^2 & 6
\end{array}
$$

Element 1 Element 2

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$$
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1 & 2 & 4 & 3 & 5 & 6 & \\
\hline
12 & 6L & -12 & 6L & & & 1\\
6L & 4L^2 & -6L & 2L^2 & & & 2\\
-12 & -6L & 24 & 0 & -12 & 6L & 4\\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 3\\
 & & -12 & -6L & 12 & -6L & 5\\
 & & 6L & 2L^2 & -6L & 4L^2 & 6
\end{array}
$$

Element 1 Element 2

Stiffness method of analysis: **beams**

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- The total stiffness matrix can be assembled as:

$$
\begin{array}{cccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
12 & 6L & 6L & -12 & & & 1\\
6L & 4L^2 & 2L^2 & -6L & & & 2\\
-12 & -6L & 0 & 24 & -12 & 6L & 4\\
6L & 2L^2 & 8L^2 & 0 & -6L & 2L^2 & 3\\
 & & -6L & -12 & 12 & -6L & 5\\
 & & 2L^2 & 6L & -6L & 4L^2 & 6
\end{array}
$$

Element 1 Element 2

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).
- The total stiffness matrix can be assembled as:

$$
\begin{array}{cccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
12 & 6L & 6L & -12 & & & 1\\
6L & 4L^2 & 2L^2 & -6L & & & 2\\
6L & 2L^2 & 8L^2 & 0 & -6L & 2L^2 & 3\\
-12 & -6L & 0 & 24 & -12 & 6L & 4\\
 & & -6L & -12 & 12 & -6L & 5\\
 & & 2L^2 & 6L & -6L & 4L^2 & 6
\end{array}
$$

Element 1 Element 2

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).
- The total stiffness matrix can be assembled as:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6L & 6L & -12 & 0 & 0 \\ 6L & 4L^2 & 2L^2 & -6L & 0 & 0 \\ 6L & 2L^2 & 8L^2 & 0 & -6L & 2L^2 \\ -12 & -6L & 0 & 24 & -12 & -6L \\ 0 & 0 & -6L & -12 & 12 & -6L \\ 0 & 0 & 2L^2 & 6L & -6L & 4L^2 \end{bmatrix} \end{array} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

Element 1 Element 2

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The boundary conditions are: $D_4 = D_5 = D_6 = 0$

$$\begin{Bmatrix} P \\ 0 \\ 0 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6L & 6L & -12 & 0 & 0 \\ 6L & 4L^2 & 2L^2 & -6L & 0 & 0 \\ 6L & 2L^2 & 8L^2 & 0 & -6L & 2L^2 \\ -12 & -6L & 0 & 24 & -12 & -6L \\ 0 & 0 & -6L & -12 & 12 & -6L \\ 0 & 0 & 2L^2 & 6L & -6L & 4L^2 \end{bmatrix} \end{array} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stiffness method of analysis: **beams**

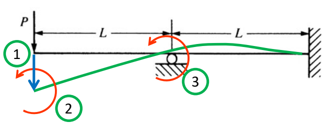
Example 15.1 – Beam Problem

- The equation reduce to three equation in three unknowns:

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$$

- Solving the above equations gives:

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \frac{PL^2}{4EI} \begin{Bmatrix} -\frac{7L}{3} \\ 3 \\ 1 \end{Bmatrix}$$

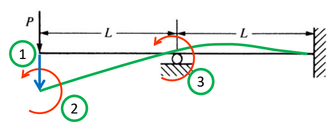


Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The positive signs for the rotations indicate that both are in the counterclockwise direction.
- The negative sign on the displacement indicates a deformation in the -*y* direction.
- Solving the above equations gives:

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \frac{PL^2}{4EI} \begin{Bmatrix} -\frac{7L}{3} \\ 3 \\ 1 \end{Bmatrix}$$



Stiffness method of analysis: **beams**

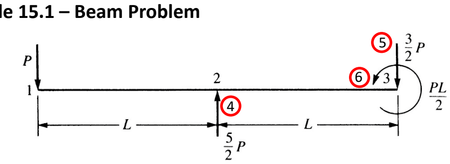
Example 15.1 – Beam Problem

- Multiplying the stiffness matrix by the displacement give the reaction forces

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \frac{P}{4L} \begin{bmatrix} 12 & 6L & 6L & -12 & 0 & 0 \\ 6L & 4L^2 & 2L^2 & -6L & 0 & 0 \\ 6L & 2L^2 & 8L^2 & 0 & -6L & 2L^2 \\ -12 & -6L & 0 & 24 & -12 & -6L \\ 0 & 0 & -6L & -12 & 12 & -6L \\ 0 & 0 & 2L^2 & 6L & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -7L/3 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ 0 \\ 5P/2 \\ -3P/2 \\ PL/2 \end{Bmatrix}$$

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem



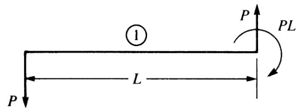
$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \frac{P}{4L} \begin{bmatrix} 12 & 6L & 6L & -12 & 0 & 0 \\ 6L & 4L^2 & 2L^2 & -6L & 0 & 0 \\ 6L & 2L^2 & 8L^2 & 0 & -6L & 2L^2 \\ -12 & -6L & 0 & 24 & -12 & -6L \\ 0 & 0 & -6L & -12 & 12 & -6L \\ 0 & 0 & 2L^2 & 6L & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -7L/3 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ 0 \\ 5P/2 \\ -3P/2 \\ PL/2 \end{Bmatrix}$$

Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The local nodal forces for **Element 1**:

$$
\begin{Bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix}_{\boxed{1}} = \frac{P}{4L}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 4 & 3 \end{array} \\
\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
\end{array}
\begin{Bmatrix} -\frac{7L}{3} \\ 3 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ P \\ -PL \end{Bmatrix}
$$

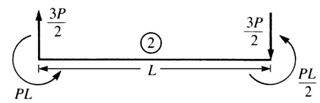


Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The local nodal forces for **Element 2**:

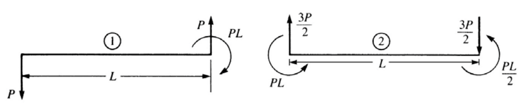
$$
\begin{Bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{Bmatrix}_{\boxed{2}} = \frac{P}{4L}
\begin{array}{c}
\begin{array}{cccc} 4 & 3 & 5 & 6 \end{array} \\
\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
\end{array}
\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.5P \\ PL \\ -1.5P \\ 0.5PL \end{Bmatrix}
$$



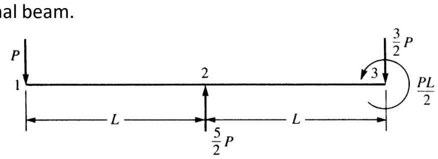
Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- The free-body diagrams for each element are shown below.



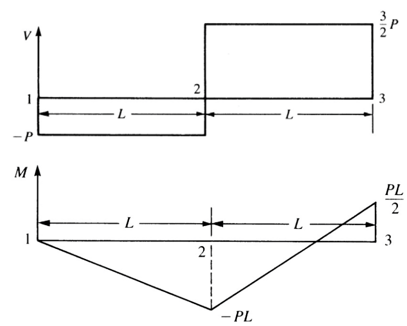
- Combining the elements gives the forces and moments for the original beam.



Stiffness method of analysis: **beams**

Example 15.1 – Beam Problem

- Therefore, the shear force and bending moment diagrams are:



Stiffness method of analysis: **beams**

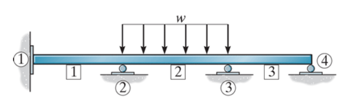
Intermediate Loadings

- For application, it is important that all the finite elements of the beam be ***free of loading along their length***.
- This is necessary since the stiffness matrix for each element was developed for ***loadings applied only at its ends***.
- Oftentimes, however, beams will support a distributed loading, and this condition will require modification to perform the matrix analysis.

Stiffness method of analysis: **beams**

Intermediate Loadings

- To handle this case, we will use the principle of superposition in a manner like that used for trusses.
- Consider the beam below, which is subjected to the uniform distributed load *w* acting on member 2.

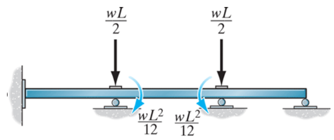


actual loading

Stiffness method of analysis: **beams**

Intermediate Loadings

- First, we will apply its fixed-end moments **FEMs** and reactions to the nodes which will be used as external loadings in the stiffness method.



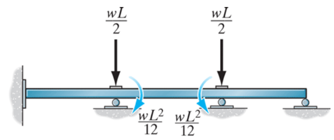
for matrix analysis

- These loadings will give the correct displacements and internal reactions (shear and moment) at the ends of members 1 and 3.

Stiffness method of analysis: **beams**

Intermediate Loadings

- First, we will apply its fixed-end moments **FEMs** and reactions to the nodes which will be used as external loadings in the stiffness method.



for matrix analysis

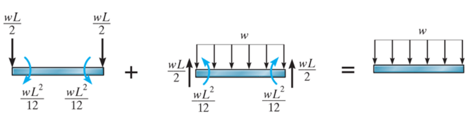
- To obtain the correct internal reactions for member 2 we must add the **reverse fixed-end loadings** back on this member.

Stiffness method of analysis: **beams**

Intermediate Loadings

- For example, if the matrix analysis produces shear forces $q_{Ny'}$ and $q_{Fy'}$ moments $d_{Nz'}$ and $d_{Fz'}$ for member 2.

- Then the loadings must be added to these loadings to determine the results.



fixed-end element loading on nodes

actual loading and reactions on fixed-supported element

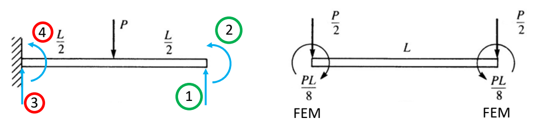
actual loading

$$\mathbf{q} = \mathbf{k}\mathbf{d} + \mathbf{q}_0$$

Stiffness method of analysis: **beams**

Example 15.2 – Cantilever Beam Problem

- Consider the beam below, determine the vertical displacement and rotation at the free-end and the nodal forces, including reactions. Assume ***EI*** is constant throughout the beam.



- We will use one element and replace the concentrated load with the appropriate nodal forces.

Stiffness method of analysis: **beams**

Example 15.2 – Cantilever Beam Problem

- The beam stiffness equations are:

$$\begin{Bmatrix} Q_3 \\ Q_4 \\ Q_1 \\ Q_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{Bmatrix} D_3 \\ D_4 \\ D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} -P/2 \\ -PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix}$$

- Apply global stiffness equations are:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \end{array} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} -P/2 \\ PL/8 \\ -P/2 \\ -PL/8 \end{Bmatrix}$$

Stiffness method of analysis: **beams**

Example 15.2 – Cantilever Beam Problem

- Apply the boundary conditions: $D_3 = D_4 = 0$

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -P/2 \\ PL/8 \\ -P/2 \\ -PL/8 \end{Bmatrix}$$

- The beam stiffness equations become:

$$\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} -P/2 \\ PL/8 \end{Bmatrix} \qquad \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} -5PL^3/48EI \\ -PL^2/8EI \end{Bmatrix}$$

Stiffness method of analysis: **beams**

Example 15.2 – Cantilever Beam Problem

- To obtain the global nodal forces, we begin by evaluating the effective nodal forces.

$$\begin{Bmatrix} Q_3 \\ Q_4 \\ Q_1 \\ Q_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -5PL^3/48EI \\ -PL^2/8EI \end{Bmatrix} = \begin{Bmatrix} P/2 \\ 3PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix}$$

$$\begin{Bmatrix} q_{Nx} \\ q_{Nz} \\ q_{Fx} \\ q_{Fxz} \end{Bmatrix} = \begin{Bmatrix} P/2 \\ 3PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix}$$

Stiffness method of analysis: **beams**

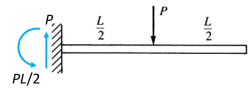
Example 15.2 – Cantilever Beam Problem

- Using the above expression in the following equation, gives:

reactions at fixed-support

$$\mathbf{q} = \mathbf{kd} - \mathbf{q_0}$$

$$\underbrace{\begin{Bmatrix} q_{Nx} \\ q_{Nz} \\ q_{Fx} \\ q_{Fxz} \end{Bmatrix}}_{\mathbf{q}} = \underbrace{\begin{Bmatrix} P/2 \\ 3PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix}}_{\mathbf{kd}} - \underbrace{\begin{Bmatrix} -P/2 \\ -PL/8 \\ -P/2 \\ PL/8 \end{Bmatrix}}_{\mathbf{q_0}} = \underbrace{\begin{Bmatrix} \boxed{P} \\ \boxed{PL/2} \\ 0 \\ 0 \end{Bmatrix}}_{\mathbf{q}}$$



Stiffness method of analysis: **beams**

Let’s work some problems

Displacement method of analysis: **slope-defection method**

Any questions?