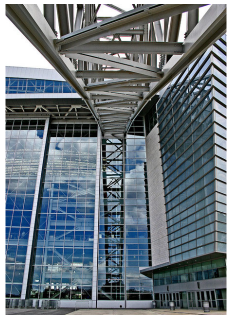
Chapter 14

Truss Analysis Using the **Stiffness Method**



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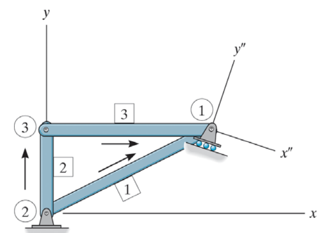
Stiffness method of analysis: **trusses**

Nodal Coordinates

- A truss can be supported by a roller placed on an *incline*
- When this occurs the constraint of zero deflection at the support (node) *cannot* be directly defined using a single horizontal and vertical global coordinate system.



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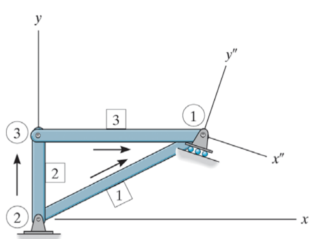
Stiffness method of analysis: **trusses**

Nodal Coordinates

- For example, consider the truss below; the condition of zero displacements at the node is 1 defined only along the y’’ axis, and because the roller can displace along the x’’ axis



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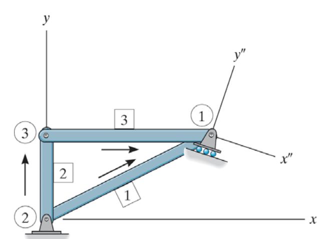
Stiffness method of analysis: **trusses**

Nodal Coordinates

- This node will have displacement components along both global coordinates axes *x* and *y*.



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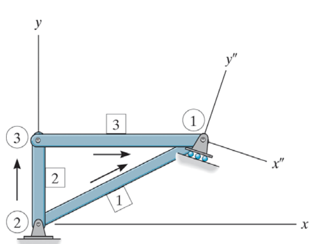
Stiffness method of analysis: **trusses**

Nodal Coordinates

- For this reason, we cannot include the zero-displacement condition at this node without making some modifications to the matrix analysis procedure.



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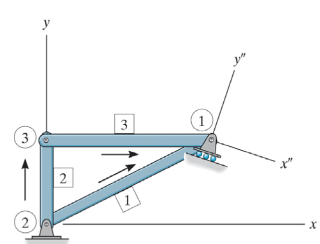
Stiffness method of analysis: **trusses**

Nodal Coordinates

- To solve this problem, we will use a set of ***nodal coordinates*** *x*’’ and *y*’’ located at the inclined support.



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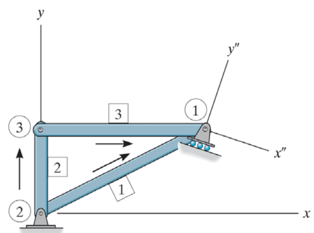
Stiffness method of analysis: **trusses**

Nodal Coordinates

- These axes are oriented such that the reaction and support displacement are along each of these coordinate axes.



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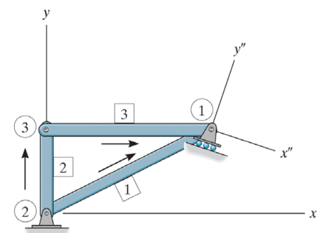
Stiffness method of analysis: **trusses**

Nodal Coordinates

- To establish the global stiffness equation for the truss, it now becomes necessary to ***transform*** the force and displacement for each of the connecting members at this support to the global *x*, *y* coordinate system.



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Stiffness method of analysis: **trusses**

Nodal Coordinates

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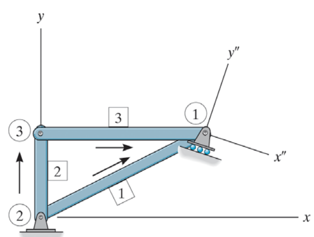

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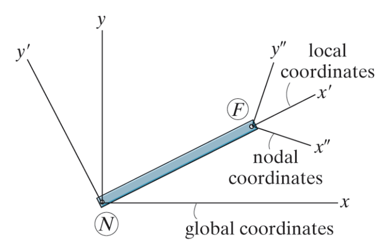
Stiffness method of analysis: **trusses**

Nodal Coordinates

- For example, consider the truss member below, which has a global coordinate system x, y at the near node Ⓝ and a nodal coordinate system at the far node Ⓕ.



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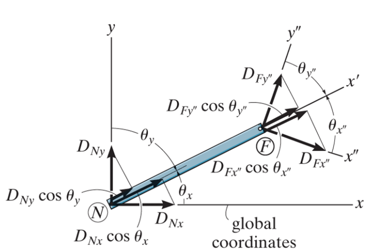
Stiffness method of analysis: **trusses**

Nodal Coordinates

- When global and nodal displacements occur at both the near and far nodes, then they will have components along the *x*′ axis as shown below:



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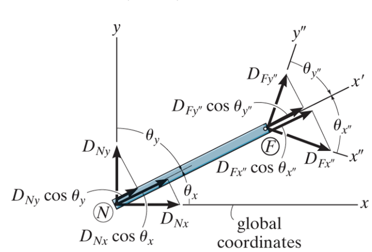
Stiffness method of analysis: **trusses**

Nodal Coordinates

- The displacements in the direction along the ends of the member become:

$$d_N = D_{Nx}\cos\theta_x + D_{Ny}\cos\theta_y \qquad d_F = D_{Nx''}\cos\theta_{x''} + D_{Ny''}\cos\theta_{y''}$$



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Stiffness method of analysis: **trusses**

Nodal Coordinates

- These equations can be written in matrix form as

$$\begin{Bmatrix} d_N \\ d_F \end{Bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_{x''} & \lambda_{y''} \end{bmatrix} \begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx''} \\ D_{Fy''} \end{Bmatrix}$$

- Likewise, member forces at the near and far ends of the member, have global and nodal components of

$$\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx''} \\ Q_{Fy''} \end{Bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_{x''} \\ 0 & \lambda_{y''} \end{bmatrix} \begin{Bmatrix} q_N \\ q_F \end{Bmatrix}$$

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Stiffness method of analysis: **trusses**

Nodal Coordinates

- These displacement and force transformation matrices are now used to develop the member stiffness matrix for this situation.

$$\boxed{\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T}} \qquad \mathbf{k} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_{x''} \\ 0 & \lambda_{y''} \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_{x''} & \lambda_{y''} \end{bmatrix}$$

$$\boxed{\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}}$$

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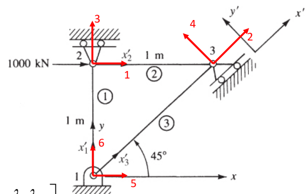
Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

- Consider the plane truss shown below. Assume E = 210 GPa, A = 6 x 10^{-4} m^2 for element 1 and 2, and $A = \sqrt{2}$(6 x 10^{-4})m^2 for element 3.
- Determine the stiffness matrix for each element.



$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}$$

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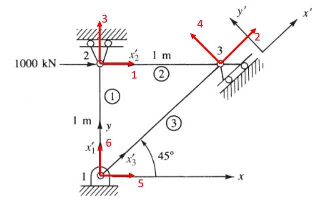
Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

- The global elemental stiffness matrix for ***element 1*** is:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}$$



$$\theta_x = 90^\circ \qquad \lambda_x = 0$$
$$\theta_y = 0^\circ \qquad \lambda_y = 1$$
$$\theta_{x''} = 90^\circ \qquad \lambda_{x''} = 0$$
$$\theta_{y''} = 0^\circ \qquad \lambda_{y''} = 1$$

$$\mathbf{k}^{(1)} = \frac{(210\times10^6 kN/m^2)(6\times10^{-4} m^{-2})}{1\,m} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 3 \end{array}$$

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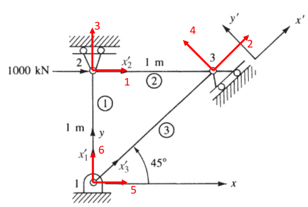
Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

- The global elemental stiffness matrix for ***element 2*** is:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}$$



$$\theta_x = 0^\circ \qquad \lambda_x = 1$$
$$\theta_y = 90^\circ \qquad \lambda_y = 0$$
$$\theta_{x''} = 45^\circ \qquad \lambda_{x''} = \sqrt{2}/2$$
$$\theta_{y''} = 135^\circ \qquad \lambda_{y''} = -\sqrt{2}/2$$

$$\mathbf{k}^{(2)} = \frac{(210\times10^6 kN/m^2)(6\times10^{-4} m^2)}{1\,m} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 \\ -\sqrt{2}/2 & 0 & 0.5 & -0.5 \\ \sqrt{2}/2 & 0 & -0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \end{array}$$

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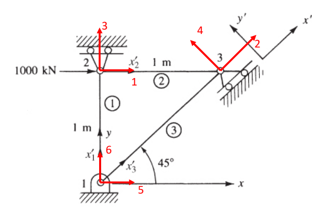
Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

- The global elemental stiffness matrix for ***element 3*** is:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x\lambda_{x''} & -\lambda_x\lambda_{y''} \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_y\lambda_{x''} & -\lambda_y\lambda_{y''} \\ -\lambda_x\lambda_{x''} & -\lambda_y\lambda_{x''} & \lambda_{x''}^2 & \lambda_{x''}\lambda_{y''} \\ -\lambda_x\lambda_{y''} & -\lambda_y\lambda_{y''} & \lambda_{x''}\lambda_{y''} & \lambda_{y''}^2 \end{bmatrix}$$



$$\theta_x = 45^\circ \qquad \lambda_x = \sqrt{2}/2$$
$$\theta_y = 45^\circ \qquad \lambda_y = \sqrt{2}/2$$
$$\theta_{x''} = 0^\circ \qquad \lambda_{x''} = 1$$
$$\theta_{y''} = 90^\circ \qquad \lambda_{y''} = 0$$

$$\mathbf{k}^{(3)} = \frac{(210\times10^6 kN/m^2)\sqrt{2}(6\times10^{-4} m^{-2})}{\sqrt{2}\ m} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 2 & 4 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -\sqrt{2}/2 & 0 \\ 0.5 & 0.5 & -\sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 2 \\ 4 \end{array}$$

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Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

- Using the stiffness method, the global stiffness matrix is:

$$\mathbf{K} = 1{,}260 \times 10^5 \,{}^{N}\!/_{m} \begin{bmatrix} 1.0000 & -0.7071 & 0.0000 & 0.7071 & 0.0000 & 0.0000 \\ -0.7071 & 1.5000 & 0.0000 & -0.5000 & -0.7071 & -0.7071 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & -1.0000 \\ 0.7071 & -0.5000 & 0.0000 & 0.5000 & 0.0000 & 0.0000 \\ 0.0000 & -0.7071 & 0.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & -0.7071 & -1.0000 & 0.0000 & 0.5000 & 1.5000 \end{bmatrix}$$

- The displacement boundary conditions are: $D_3 = D_4 = D_5 = D_6 = 0$

$$\begin{Bmatrix} 1{,}000 \\ 0 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = 1{,}260 \times 10^5 \,{}^{N}\!/_{m} \begin{bmatrix} 1.0000 & -0.7071 & 0.0000 & 0.7071 & 0.0000 & 0.0000 \\ -0.7071 & 1.5000 & 0.0000 & -0.5000 & -0.7071 & -0.7071 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & -1.0000 \\ 0.7071 & -0.5000 & 0.0000 & 0.5000 & 0.0000 & 0.0000 \\ 0.0000 & -0.7071 & 0.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & -0.7071 & -1.0000 & 0.0000 & 0.5000 & 1.5000 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

- By applying the boundary conditions the global force-displacement equations are:

$$1{,}260 \times 10^5 \,{}^{N}\!/_{m} \begin{bmatrix} 1 & -0.707 \\ -0.707 & 1.5 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 = 1{,}000\ kN \\ Q_2 = 0 \end{Bmatrix}$$

- Solving the equation gives: $D_3 = 11.91\,mm$ $\quad D''_5 = 5.61\,mm$

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Stiffness method of analysis: **trusses**

Example 14.3 – Plane Truss Problem

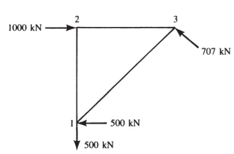
- The global nodal forces are calculated as:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = 1{,}260 \times 10^2 \,{}^{N}\!/_{mm} \begin{bmatrix} 1.0000 & -0.7071 & 0.0000 & 0.7071 & 0.0000 & 0.0000 \\ -0.7071 & 1.5000 & 0.0000 & -0.5000 & -0.7071 & -0.7071 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & -1.0000 \\ 0.7071 & -0.5000 & 0.0000 & 0.5000 & 0.0000 & 0.0000 \\ 0.0000 & -0.7071 & 0.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & -0.7071 & -1.0000 & 0.0000 & 0.5000 & 1.5000 \end{bmatrix} \begin{Bmatrix} 11.91 \\ 5.61 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Therefore:

$Q_3 = 0$ $\qquad Q_4 = 707\ kN$

$Q_5 = -500\ kN$ $\qquad Q_6 = -500\ kN$



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Stiffness method of analysis: **trusses**

Trusses Having Thermal Changes and Fabrication Errors

- If members of a truss are subjected to an increase or decrease in length due to thermal changes or fabrication errors, then it is necessary to use the method of superposition to obtain the solution.
- This requires three steps. ***First***, the fixed-end forces necessary to *prevent* the movement of the nodes as caused by temperature or fabrication are calculated.
- ***Second***, equal but opposite forces are placed on the truss at the nodes, and the displacements of the nodes are calculated using the matrix analysis.

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Stiffness method of analysis: **trusses**

Trusses Having Thermal Changes and Fabrication Errors

- ***Third***, the actual forces in the members and the reactions on the truss are determined by superposing these two results.
- This procedure, of course, is only necessary if the truss is statically indeterminate.
- If the truss is statically determinate, the displacements at the nodes can be found by this method; however, the temperature changes and fabrication errors will not affect the reactions and the member forces since the truss is free to adjust to these changes in length.

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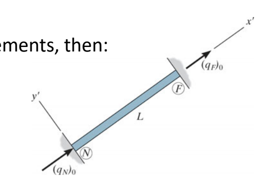
Stiffness method of analysis: **trusses**

Trusses Having Thermal Effects

- If a truss member of length L is subjected to a temperature increase ΔT, the member will undergo an increase in length of $\Delta L = \alpha \Delta T L$, where α is the coefficient of thermal expansion.
- A compressive force q_0 applied to the ends of the member will cause a decrease in the member's length of $\Delta L' = q_0 L/AE$.
- If we equate these two displacements, then:

$$(q_N)_0 = AE\alpha\Delta T$$

$$(q_F)_0 = -AE\alpha\Delta T$$



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Stiffness method of analysis: **trusses**

Trusses Having Thermal Effects

- If a temperature decrease occurs, then ΔT becomes negative, and these forces reverse direction to hold the member rigid.
- We can transform these two forces into global coordinates

$$\begin{Bmatrix} (Q_{Nx})_0 \\ (Q_{Ny})_0 \\ (Q_{Fx})_0 \\ (Q_{Fy})_0 \end{Bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} AE\alpha\Delta T \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = AE\alpha\Delta T \begin{Bmatrix} \lambda_x \\ \lambda_y \\ -\lambda_x \\ -\lambda_y \end{Bmatrix}$$

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Stiffness method of analysis: **trusses**

Trusses Having Fabrication Errors

- If a truss member is made too long by an amount ΔL before it is fitted into a truss, then the force q_0 needed to keep the member at its design length L is $q_0 = AE\Delta L/L$.

$$(q_N)_0 = \frac{AE\Delta L}{L} \qquad (q_F)_0 = -\frac{AE\Delta L}{L}$$

$$\begin{Bmatrix} (Q_{Nx})_0 \\ (Q_{Ny})_0 \\ (Q_{Fx})_0 \\ (Q_{Fy})_0 \end{Bmatrix} = \frac{AE\Delta L}{L} \begin{Bmatrix} \lambda_x \\ \lambda_y \\ -\lambda_x \\ -\lambda_y \end{Bmatrix}$$

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Stiffness method of analysis: **trusses**

Application of the Stiffness Method for Truss Analysis

- In the general case, with the truss subjected to applied forces, temperature changes, and fabrication errors, the initial force-displacement relationship for the truss then becomes

$$\mathbf{Q} = \mathbf{K}\mathbf{D} + \mathbf{Q}_0$$

- Here $\mathbf{Q}_0$ is a column matrix for the entire truss of the fixed-end forces caused by the temperature changes and fabrication errors of the members.

$$\begin{Bmatrix} \mathbf{Q}_k \\ \mathbf{Q}_u \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_u \\ \mathbf{D}_k \end{Bmatrix} + \begin{Bmatrix} (\mathbf{Q}_k)_0 \\ (\mathbf{Q}_u)_0 \end{Bmatrix}$$

$\mathbf{Q}_k$, $\mathbf{D}_k$, $(\mathbf{Q}_k)_0$

known loads and displacements

$\mathbf{Q}_u$, $\mathbf{D}_u$, $(\mathbf{Q}_u)_0$

unknown loads and displacements

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Stiffness method of analysis: **trusses**

Application of the Stiffness Method for Truss Analysis

- Expanding the partitioned matrix yields:

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k + (\mathbf{Q}_k)_0 \qquad \mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k + (\mathbf{Q}_u)_0$$

- Solving for $\mathbf{D}_u$ gives: $\mathbf{D}_u = \mathbf{K}_{11}^{-1}\left(\mathbf{Q}_k - \mathbf{K}_{12}\mathbf{D}_k - (\mathbf{Q}_k)_0\right)$
- Once these displacements are obtained, the member forces are then determined by superposition:

$$\mathbf{q} = \mathbf{k}'\mathbf{T}\mathbf{D} + \mathbf{q}_0$$

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Stiffness method of analysis: **trusses**

Application of the Stiffness Method for Truss Analysis

- If this equation is expanded to determine the force at the far end of the member, we obtain

$$q_F = \frac{AE}{L}\begin{bmatrix} -\lambda_x & -\lambda_y & \lambda_x & \lambda_y \end{bmatrix} \begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix} + (q_F)_0$$

- This result is similar to what we had earlier, except here we have the additional term $(q_F)_0$, which represents the initial fixed-end member force due to temperature changes and/or fabrication errors.

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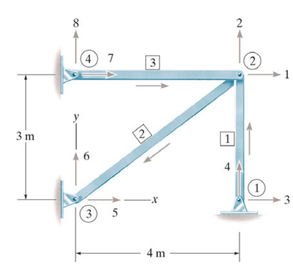
Stiffness method of analysis: **trusses**

Example 14.4 – Plane Truss Problem with Fabrication Error

- Determine the force in members 1 and 2 of the pin-connected assembly below if member 2 was made 0.01 m too short before it was fitted into place. Take AE = 8(10^3) kN.



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Stiffness method of analysis: **trusses**

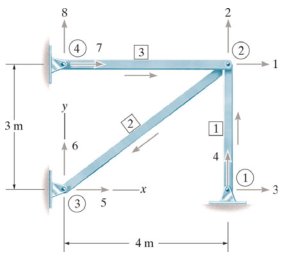
Example 14.4 – Plane Truss Problem with Fabrication Error

- Since the member is short, then ΔL = 0.01 m , and therefore applying global force equations to member 2 are:

$$\lambda_x = -0.8 \qquad \lambda_y = -0.6$$

$$\begin{Bmatrix} (Q_1)_0 \\ (Q_2)_0 \\ (Q_5)_0 \\ (Q_5)_0 \end{Bmatrix} = \frac{AE(-0.01)}{5} \begin{Bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{Bmatrix}$$

$$\begin{Bmatrix} (Q_1)_0 \\ (Q_2)_0 \\ (Q_5)_0 \\ (Q_5)_0 \end{Bmatrix} = AE \begin{Bmatrix} 0.0016 \\ 0.0012 \\ -0.0016 \\ -0.0012 \end{Bmatrix}$$



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Stiffness method of analysis: **trusses**

Example 14.4 – Plane Truss Problem with Fabrication Error

- The global elemental stiffness matrix are:

element 1: $\lambda_x = 0 \quad \lambda_y = 1 \qquad \Rightarrow \quad \mathbf{k}^{(1)} = \dfrac{AE}{3} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$

element 2: $\lambda_x = -0.8 \quad \lambda_y = -0.6 \qquad \Rightarrow \quad \mathbf{k}^{(2)} = \dfrac{AE}{5} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array}$

element 3: $\lambda_x = 1 \quad \lambda_y = 0 \qquad \Rightarrow \quad \mathbf{k}^{(3)} = \dfrac{AE}{4} \begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

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Stiffness method of analysis: **trusses**

Example 14.4 – Plane Truss Problem with Fabrication Error

- The total global stiffness matrix is:

$$\mathbf{K} = AE \begin{bmatrix} 0.378 & 0.096 & 0 & 0 & -0.128 & -0.096 & -0.25 & 0 \\ 0.096 & 0.405 & 0 & -0.333 & -0.096 & -0.072 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.333 & 0 & 0.333 & 0 & 0 & 0 & 0 \\ -0.128 & -0.096 & 0 & 0 & 0.128 & 0.096 & 0 & 0 \\ -0.096 & -0.072 & 0 & 0 & 0.096 & 0.072 & 0 & 0 \\ -0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

element 1

element 2

element 3

- The total global force-displacement equations are:

$$\begin{Bmatrix} 0 \\ 0 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = AE \begin{bmatrix} 0.378 & 0.096 & 0 & 0 & -0.128 & -0.096 & -0.25 & 0 \\ 0.096 & 0.405 & 0 & -0.333 & -0.096 & -0.072 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.333 & 0 & 0.333 & 0 & 0 & 0 & 0 \\ -0.128 & -0.096 & 0 & 0 & 0.128 & 0.096 & 0 & 0 \\ -0.096 & -0.072 & 0 & 0 & 0.096 & 0.072 & 0 & 0 \\ -0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + AE \begin{Bmatrix} 0.0016 \\ 0.0012 \\ 0 \\ 0 \\ -0.0016 \\ -0.0012 \\ 0 \\ 0 \end{Bmatrix}$$

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Stiffness method of analysis: **trusses**

Example 14.4 – Plane Truss Problem with Fabrication Error

- Applying the boundary conditions for the truss, the above equations reduce to:

$$\begin{Bmatrix} 0 \\ 0 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = AE \begin{bmatrix} 0.378 & 0.096 & 0 & 0 & -0.128 & -0.096 & -0.25 & 0 \\ 0.096 & 0.405 & 0 & -0.333 & -0.096 & -0.072 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.333 & 0 & 0.333 & 0 & 0 & 0 & 0 \\ -0.128 & -0.096 & 0 & 0 & 0.128 & 0.096 & 0 & 0 \\ -0.096 & -0.072 & 0 & 0 & 0.096 & 0.072 & 0 & 0 \\ -0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + AE \begin{Bmatrix} 0.0016 \\ 0.0012 \\ 0 \\ 0 \\ -0.0016 \\ -0.0012 \\ 0 \\ 0 \end{Bmatrix}$$

$$0 = AE\left[0.378D_1 + 0.096D_2\right] + AE\left[0.0016\right]$$

$$0 = AE\left[0.096D_1 + 0.405D_2\right] + AE\left[0.0012\right]$$

$$\boxed{\begin{aligned} D_1 &= -0.003704\,m \\ D_2 &= -0.002084\,m \end{aligned}}$$

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Stiffness method of analysis: **trusses**

Example 14.4 – Plane Truss Problem with Fabrication Error

- To determine the force in members 1 and 2 we must apply

$$q_F = \frac{AE}{L}\left[-\lambda_x \quad -\lambda_y \quad \lambda_x \quad \lambda_y\right] \begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix} + (q_F)_0$$

$$q_F = \frac{AE}{L}\left[-\lambda_x \quad -\lambda_y \quad \lambda_x \quad \lambda_y\right] \begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix} - \frac{AE}{L}(\Delta L)$$

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Stiffness method of analysis: **trusses**

Example 14.4 – Plane Truss Problem with Fabrication Error

- To determine the force in members 1 and 2 we must apply

element 1: $\lambda_x = 0 \quad \lambda_y = 1$

$$q_1 = \frac{8(10^3)}{3}\left[0 \quad -1 \quad 0 \quad 1\right] \begin{Bmatrix} 0 \\ 0 \\ -0.003704 \\ -0.002084 \end{Bmatrix} + (0) = \boxed{-5.56\,kN}$$

element 2: $\lambda_x = -0.8 \quad \lambda_y = -0.6$

$$q_2 = \frac{8(10^3)}{5}\left[0.8 \quad 0.6 \quad -0.8 \quad -0.6\right] \begin{Bmatrix} -0.003704 \\ -0.002084 \\ 0 \\ 0 \end{Bmatrix} - \frac{8(10^3)}{5}(-0.01)$$

$$= \boxed{9.26\,kN}$$

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Stiffness method of analysis: **trusses**

Space-Truss Analysis

- The analysis of both statically determinate and indeterminate space trusses can be performed by using the same procedure discussed previously.
- To account for the three-dimensional aspects of the problem, however, additional elements must be included in the transformation matrix **T**.

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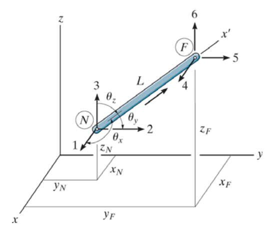
Stiffness method of analysis: **trusses**

Space-Truss Analysis

- To do this, consider the truss member shown below.
- By inspection, the direction cosines between the global and local coordinates can be found as:



$$\lambda_x = \cos\theta_x = \frac{x_F - x_N}{L_{NF}} = \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2 + (z_F - z_N)^2}}$$

$$\lambda_y = \cos\theta_y = \frac{y_F - y_N}{L_{NF}} = \frac{y_F - y_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2 + (z_F - z_N)^2}}$$

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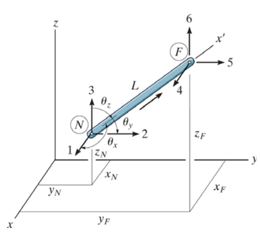
Stiffness method of analysis: **trusses**

Space-Truss Analysis

- To do this, consider the truss member shown below.
- By inspection, the direction cosines between the global and local coordinates can be found as:



$$\lambda_z = \cos\theta_z = \frac{z_F - z_N}{L_{NF}} = \frac{z_F - z_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2 + (z_F - z_N)^2}}$$

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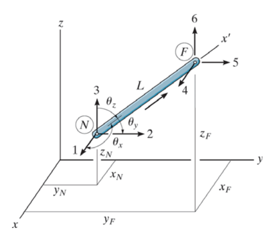
Stiffness method of analysis: **trusses**

Space-Truss Analysis

- To do this, consider the truss member shown below.
- By inspection, the direction cosines between the global and local coordinates can be found as:



$$\lambda_z = \cos\theta_z = \frac{z_F - z_N}{L_{NF}} = \frac{z_F - z_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2 + (z_F - z_N)^2}}$$

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Stiffness method of analysis: **trusses**

Space-Truss Analysis

- As a result of the third dimension, the transformation matrix becomes:

$$\mathbf{T} = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & \lambda_z \end{bmatrix}$$

$$\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T}$$

$$\mathbf{k} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ \lambda_z & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \\ 0 & \lambda_z \end{bmatrix} \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_x & \lambda_y & \lambda_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & \lambda_z \end{bmatrix}$$

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Stiffness method of analysis: **trusses**

Space-Truss Analysis

- As a result of the third dimension, the transformation matrix becomes:

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c} \begin{array}{cccccc} N_x & N_y & N_z & F_x & F_y & F_z \end{array} \\ \left[\begin{array}{ccc|ccc} \lambda_x^2 & \lambda_x\lambda_y & \lambda_x\lambda_z & -\lambda_x^2 & -\lambda_x\lambda_y & -\lambda_z\lambda_x \\ \lambda_y\lambda_x & \lambda_y^2 & \lambda_y\lambda_z & -\lambda_y\lambda_x & -\lambda_y^2 & -\lambda_y\lambda_z \\ \lambda_z\lambda_x & \lambda_z\lambda_y & \lambda_z^2 & -\lambda_z\lambda_x & -\lambda_z\lambda_y & -\lambda_z^2 \\ \hline -\lambda_x^2 & -\lambda_x\lambda_y & -\lambda_x\lambda_z & \lambda_x^2 & \lambda_x\lambda_y & \lambda_x\lambda_z \\ -\lambda_y\lambda_x & -\lambda_y^2 & -\lambda_y\lambda_z & \lambda_y\lambda_x & \lambda_y^2 & \lambda_y\lambda_z \\ -\lambda_z\lambda_x & -\lambda_z\lambda_y & -\lambda_z^2 & \lambda_z\lambda_x & \lambda_z\lambda_y & \lambda_z^2 \end{array}\right] \end{array} \begin{array}{c} N_x \\ N_y \\ N_z \\ F_x \\ F_y \\ F_z \end{array}$$

Member stiffness matrix in global coordinates

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Stiffness method of analysis: **trusses**

Space-Truss Analysis

- Here the code numbers along the rows and columns reference the x, y, z directions at the near end, N_x, N_y, N_z, followed by those at the far end, F_x, F_y, F_z.

$$
\mathbf{k} = \frac{AE}{L}
\begin{array}{c}
\begin{array}{cccccc} N_x & N_y & N_z & F_x & F_y & F_z \end{array} \\
\left[\begin{array}{ccc|ccc}
\lambda_x^2 & \lambda_x\lambda_y & \lambda_x\lambda_z & -\lambda_x^2 & -\lambda_x\lambda_y & -\lambda_z\lambda_x \\
\lambda_y\lambda_x & \lambda_y^2 & \lambda_y\lambda_z & -\lambda_y\lambda_x & -\lambda_y^2 & -\lambda_y\lambda_z \\
\lambda_z\lambda_x & \lambda_z\lambda_y & \lambda_z^2 & -\lambda_z\lambda_x & -\lambda_z\lambda_y & -\lambda_z^2 \\
\hline
-\lambda_x^2 & -\lambda_x\lambda_y & -\lambda_x\lambda_z & \lambda_x^2 & \lambda_x\lambda_y & \lambda_x\lambda_z \\
-\lambda_y\lambda_x & -\lambda_y^2 & -\lambda_y\lambda_z & \lambda_y\lambda_x & \lambda_y^2 & \lambda_y\lambda_z \\
-\lambda_z\lambda_x & -\lambda_z\lambda_y & -\lambda_z^2 & \lambda_z\lambda_x & \lambda_z\lambda_y & \lambda_z^2
\end{array}\right]
\end{array}
\begin{array}{c} \\ N_x \\ N_y \\ N_z \\ F_x \\ F_y \\ F_z \end{array}
$$

Member stiffness matrix in global coordinates

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Stiffness method of analysis: **trusses**

Space-Truss Analysis

- An easier form for remembering the components of the stiffness matrix in global coordinates is:

$$
\lambda = \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & \lambda_x\lambda_z \\ \lambda_x\lambda_y & \lambda_y^2 & \lambda_y\lambda_z \\ \lambda_x\lambda_z & \lambda_y\lambda_z & \lambda_z^2 \end{bmatrix}
\qquad
\mathbf{k} = \frac{AE}{L}\begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}
$$

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Stiffness method of analysis: **trusses**

Space-Truss Analysis

- For computer programming, it is generally more efficient to use **k** than to carry out the matrix multiplication for each member.
- Computer storage space is saved if the structure stiffness matrix **K** is first initialized with all zero elements; then as the elements of each member stiffness matrix are generated, they are placed directly into their respective positions in **K**.
- After the structure stiffness matrix has been developed, the same procedure outlined for 2D trusses can be followed to determine the joint displacements, support reactions, and internal member forces.

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Displacement method of analysis: **moment distribution for beams**

Let's work some problems

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Displacement method of analysis: **slope-defection method**

Any questions?

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