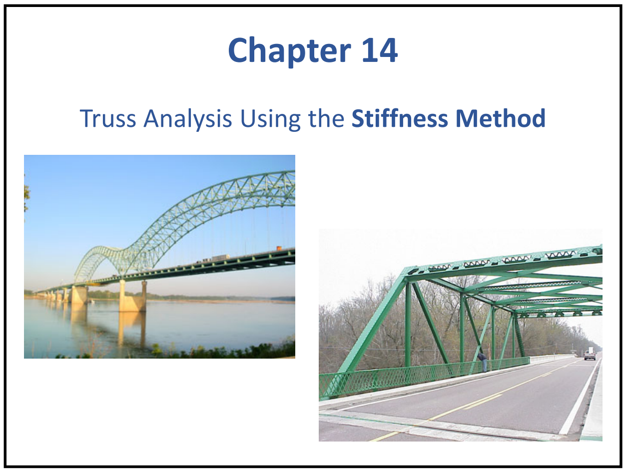
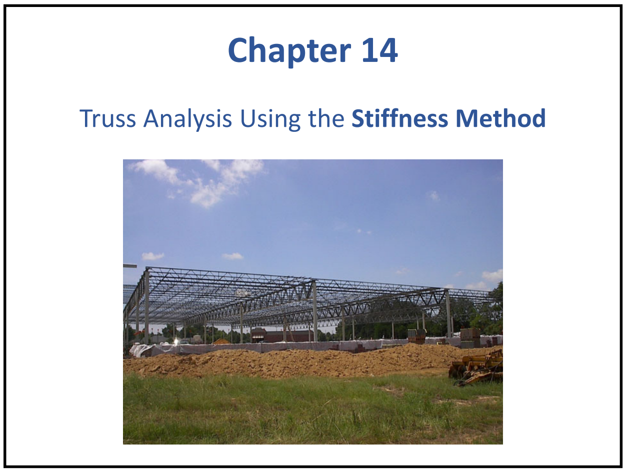


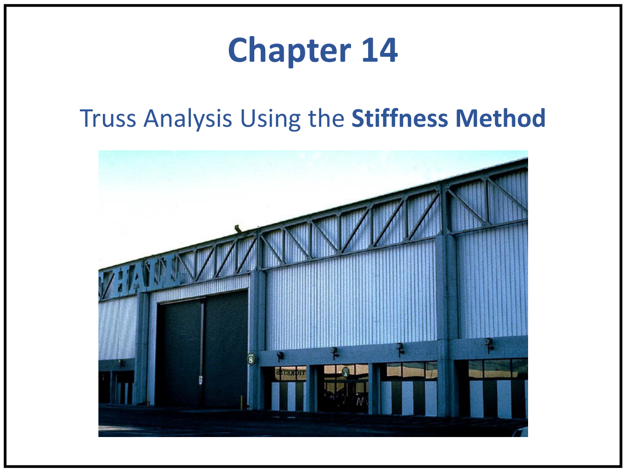
Chapter 14

Truss Analysis Using the **Stiffness Method**



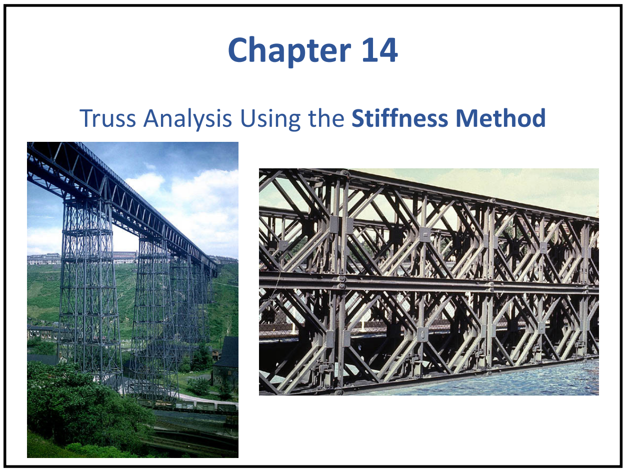
Chapter 14

Truss Analysis Using the **Stiffness Method**



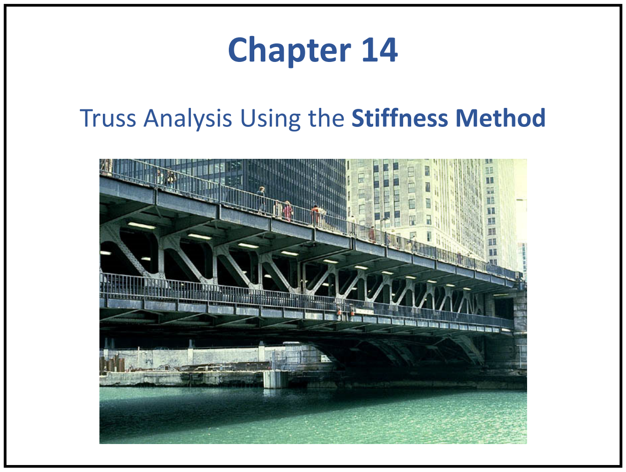
Chapter 14

Truss Analysis Using the **Stiffness Method**



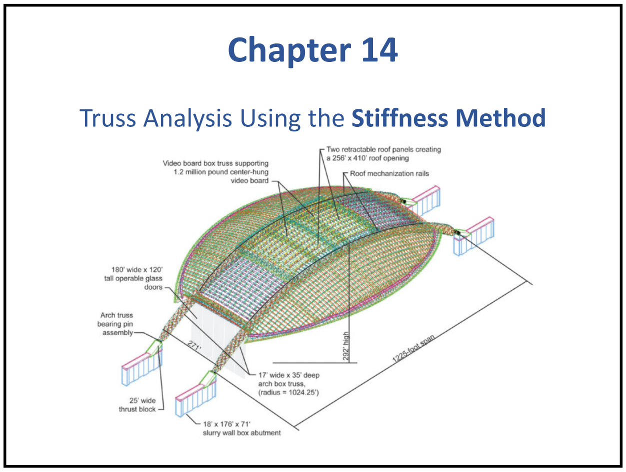
Chapter 14

Truss Analysis Using the **Stiffness Method**



Chapter 14

Truss Analysis Using the **Stiffness Method**



Chapter 14

Truss Analysis Using the **Stiffness Method**

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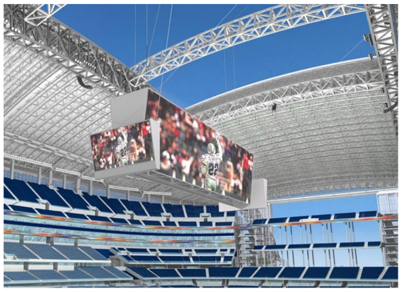


Chapter 14

Truss Analysis Using the **Stiffness Method**

Stiffness method of analysis: **trusses**

Fundamentals of the Stiffness Method

- There are essentially two ways in which structures can be analyzed using matrix methods.
- The stiffness method, to be used in this and the following chapters, is a displacement method.
- A force method, called the flexibility method, as outlined in Sec. 9.2, can also be used; however, this method will not be presented here.

Stiffness method of analysis: **trusses**

Fundamentals of the Stiffness Method

- There are several reasons for this.
- Most important, the stiffness method can be used to analyze both ***statically determinate*** and ***statically indeterminate*** structures.
- The flexibility method requires a different procedure for each of these two cases.
- Also, it is generally easier to formulate the necessary matrices for computer operations using the stiffness method; and once this is done, the calculations can be performed efficiently.

Stiffness method of analysis: **trusses**

Fundamentals of the Stiffness Method

- Application of the stiffness method requires subdividing the structure into a series of discrete ***finite elements***, and then identifying their end points as ***nodes***.
- For truss analysis, the ***finite elements*** are represented by each of the members of the truss, and the ***nodes*** represent the joints.
- The force-displacement relationships for each element are determined and then these are related to one another using the force equilibrium equations written at the ***nodes***.

Stiffness method of analysis: **trusses**

Fundamentals of the Stiffness Method

- These relationships, for the entire structure, are then grouped together into what is called the ***structure stiffness matrix*** **K**.
- Once it is established, the unknown displacements of the nodes can then be determined for any given loading on the truss.
- When these displacements are known, the external and internal forces in the truss can be calculated using the ***force-displacement relations*** for each member.

Stiffness method of analysis: **trusses**

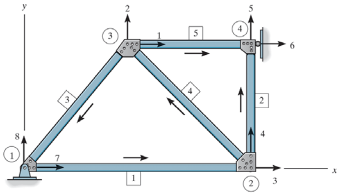
Member and Node Identification

- One of the first steps when applying the stiffness method is to identify the elements or members of the truss and their nodes.
- We will specify each member by a number enclosed within a ***square*** □, and use a number enclosed within a ***circle*** ○ to identify the nodes.
- Also, the “near” and “far” ends of the member must be identified.
- This will be done using an ***arrow*** written along the member, with the head of the arrow directed toward the far end.

Stiffness method of analysis: **trusses**

Member and Node Identification

- Examples of member, node, and “direction” identification for a truss are shown below.
- These assignments have all been done ***arbitrarily***.*



Stiffness method of analysis: **trusses**

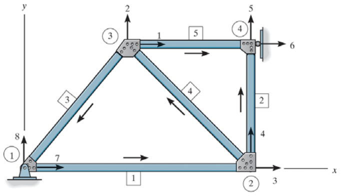
Global and Member Coordinates

- Since loads and displacements are vector quantities, it is necessary to establish a ***coordinate system*** to specify their sense of direction.
- Here we will use two different types of coordinate systems.

Stiffness method of analysis: **trusses**

Global and Member Coordinates

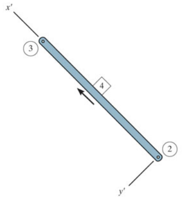
- A single ***global*** or ***structure coordinate system***, *x*, *y*, will be used to specify the sense of each of the *external* force and displacement components at the nodes.



Stiffness method of analysis: **trusses**

Global and Member Coordinates

- A ***local*** or ***member coordinate system*** will be used for each member to specify the sense of direction of its displacements and *internal* loadings.
- This system will be identified using axes with the origin at the “near” node and the axis extending toward the “far” node.



Stiffness method of analysis: **trusses**

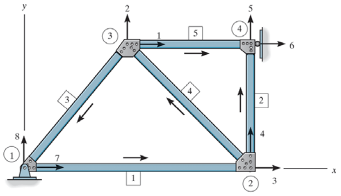
Kinematic Indeterminacy

- The unconstrained displacements for a truss joint represent the primary unknowns of any displacement method, and therefore these must be identified.
- There are two degrees of freedom, or two displacements, for each joint (***node***).
- For application, each degree of freedom will be specified on the truss using a ***code number***, shown at the joint or node, and referenced to its positive global coordinate direction using an associated arrow.

Stiffness method of analysis: **trusses**

Kinematic Indeterminacy

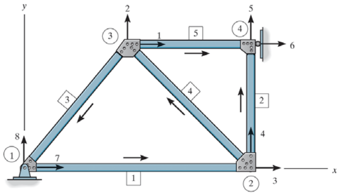
- For example, the truss below has eight degrees of freedom, which have been identified by the ***code numbers*** 1 through 8.



Stiffness method of analysis: **trusses**

Kinematic Indeterminacy

- The truss is kinematically indeterminate to the fifth degree because of these eight possible displacements: 1 through 5 represent unknown or ***unconstrained degrees of freedom***, and 6 through 8 represent ***constrained degrees of freedom***.

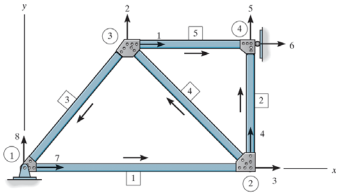


- Due to the pin and roller, these displacements are zero

Stiffness method of analysis: **trusses**

Kinematic Indeterminacy

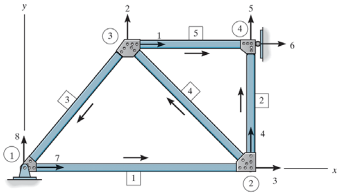
- For later application, *the lowest code numbers will always be used to identify the unknown displacements (unconstrained degrees of freedom) and the highest code numbers will be used to identify the known displacements (constrained degrees of freedom)*.



Stiffness method of analysis: **trusses**

Kinematic Indeterminacy

- The reason for choosing this method of identification has to do with the convenience of later partitioning the structure stiffness matrix **K** so that the unknown displacements can be found in the most direct manner.

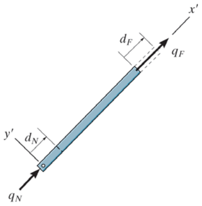


Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- In this section we will establish the stiffness matrix for a single truss member using local coordinates, oriented as shown below.

- The terms in this matrix represent the load-displacement relations for the member.

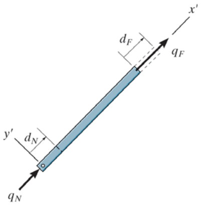


Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- Since the loads on a truss member only act along the member, then the displacements of the nodes are only along the axis.

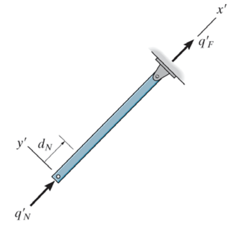
- To obtain the load-displacement relations we will apply two independent displacements to the member.



Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- When a positive displacement occurs on the near end of the member, while the far end is held pinned (fixed), the forces developed at the ends of the member are:



near end displacement d_N

$$q'_N = \frac{AE}{L} d_N$$

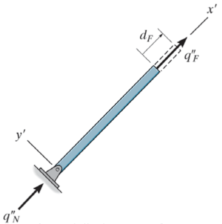
$$q'_F = -\frac{AE}{L} d_N$$

- Here ***q'***$_F$ is negative since for equilibrium it must act in the negative direction.

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- Likewise, a positive displacement at the far end, keeping the near end pinned (fixed)



far end displacement d_F

$$q''_F = \frac{AE}{L} d_F$$

$$q''_N = -\frac{AE}{L} d_F$$

- Here ***q''***$_N$ is negative since for equilibrium it must act in the negative direction.

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

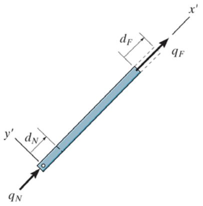
- By superposition, the resultant forces caused by both displacements are

$$q_N = q'_N + q''_N$$

$$q_N = \frac{AE}{L} d_N - \frac{AE}{L} d_F$$

$$q_N = q'_F + q''_F$$

$$q_F = -\frac{AE}{L} d_N + \frac{AE}{L} d_F$$

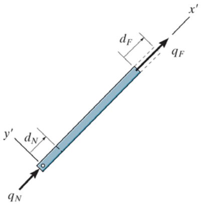


near and far end displacements d_N and d_F

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- These load-displacement equations may be written in ***matrix form*** as:



near and far end displacements d_N and d_F

$$\begin{Bmatrix} q_N \\ q_F \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_N \\ d_F \end{Bmatrix}$$

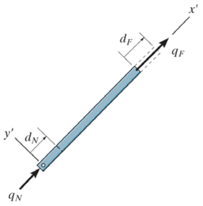
$$\mathbf{q} = \mathbf{k'd}$$

$$\mathbf{k'} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- Please review **Appendix A** in the textbook for an introduction to matrix algebra.



near and far end displacements d_N and d_F

$$\begin{Bmatrix} q_N \\ q_F \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_N \\ d_F \end{Bmatrix}$$

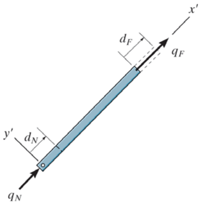
$$\mathbf{q} = \mathbf{k'd}$$

$$\mathbf{k'} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- This matrix **k'** is called the ***member stiffness matrix***, and it has the same form for each member of the truss.



near and far end displacements d_N and d_F

$$\begin{Bmatrix} q_N \\ q_F \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_N \\ d_F \end{Bmatrix}$$

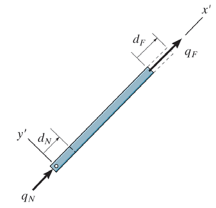
$$\mathbf{q} = \mathbf{k'd}$$

$$\mathbf{k'} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- The four elements that comprise it are called ***member stiffness influence coefficients***, k'_{ij}.



near and far end displacements d_N and d_F

$$\begin{Bmatrix} q_N \\ q_F \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_N \\ d_F \end{Bmatrix}$$

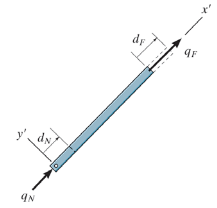
$$\mathbf{q} = \mathbf{k'd}$$

$$\mathbf{k'} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness method of analysis: **trusses**

Member Stiffness Matrix

- Physically, $\mathbf{k'}_{ij}$ represents the force at joint *i* when a *unit displacement* is imposed at joint *j*.



near and far end displacements d_N and d_F

$$\begin{Bmatrix} q_N \\ q_F \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_N \\ d_F \end{Bmatrix}$$

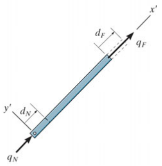
$$\mathbf{q} = \mathbf{k'd}$$

$$\mathbf{k'} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

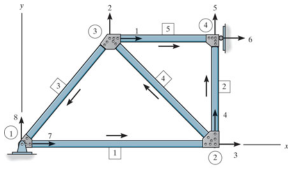
Stiffness method of analysis: **trusses**

Displacement and Force Transformation Matrices

- Since a truss is composed of many members (elements), we must now develop a method for transforming the member forces **q** and displacements **d** defined in local coordinates, to global coordinates.



Local coordinates

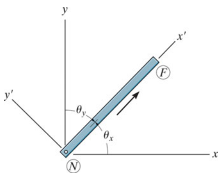


Global coordinates

Stiffness method of analysis: **trusses**

Displacement and Force Transformation Matrices

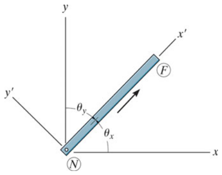
- To do this, we will define the direction of each member using the ***smallest angles*** between the ***positive x, y*** global axes and the ***positive* x'** local axis.
- These angles are θ_x and θ_y as shown below:



Stiffness method of analysis: **trusses**

Displacement and Force Transformation Matrices

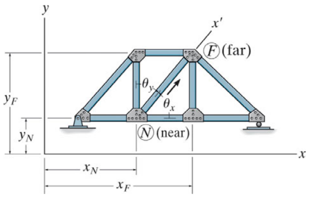
- The cosines of these angles will be used in the matrix analysis that follows, where $\lambda_x = \cos\theta_x$ and $\lambda_y = \cos\theta_y$
- Numerical values for λ_x and λ_y can easily be generated by a computer once the *x*, *y* coordinates of the near end *N* and far end *F* of the member have been specified.



Stiffness method of analysis: **trusses**

Displacement and Force Transformation Matrices

- For example, consider member *NF* of the truss shown below.
- Here the coordinates of *N* and *F* are and respectively.



$$\lambda_x = \cos\theta_x = \frac{x_F - x_N}{L_{NF}}$$

$$= \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

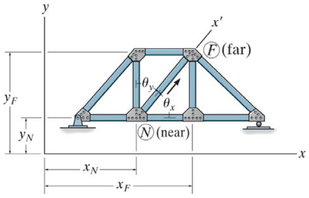
$$\lambda_y = \cos\theta_y = \frac{y_F - y_N}{L_{NF}}$$

$$= \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

Stiffness method of analysis: **trusses**

Displacement and Force Transformation Matrices

- The algebraic signs in these "generalized" equations will automatically account for members that are oriented in other quadrants of the *x*–*y* plane.



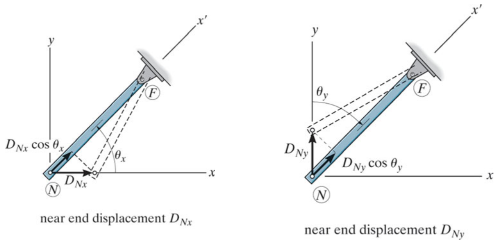
$$\lambda_x = \cos\theta_x = \frac{x_F - x_N}{L_{NF}} = \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

$$\lambda_y = \cos\theta_y = \frac{y_F - y_N}{L_{NF}} = \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

Stiffness method of analysis: **trusses**

Displacement Transformation Matrices

- In global coordinates, each end of the member can have two degrees of freedom or independent displacements; namely, joint *N* has D_{Nx} and D_{Ny}.



near end displacement D_{Nx}

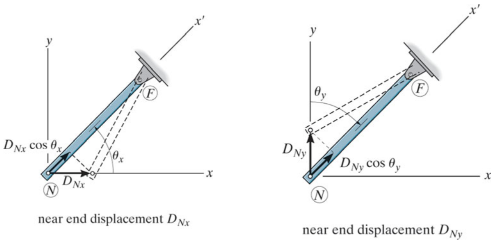
near end displacement D_{Ny}

Stiffness method of analysis: **trusses**

Displacement Transformation Matrices

- The effect of *both* global displacements causes the local displacement:

$$d_N = D_{Nx}\cos\theta_x + D_{Ny}\cos\theta_y$$



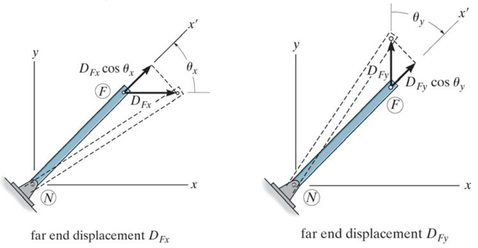
near end displacement D_{Nx}

near end displacement D_{Ny}

Stiffness method of analysis: **trusses**

Displacement Transformation Matrices

- In global coordinates, each end of the member can have two degrees of freedom or independent displacements; namely, joint *F* has D_{Fx} and D_{Fy}.



far end displacement D_{Fx}

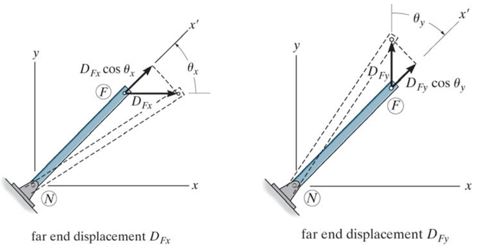
far end displacement D_{Fy}

Stiffness method of analysis: **trusses**

Displacement Transformation Matrices

- The effect of *both* global displacements causes the local displacement:

$$d_F = D_{Fx}\cos\theta_x + D_{Fy}\cos\theta_y$$



far end displacement D_{Fx}

far end displacement D_{Fy}

Stiffness method of analysis: **trusses**

Displacement Transformation Matrices

- Since: $\lambda_x = \cos\theta_x$ $\lambda_y = \cos\theta_y$

$$d_N = D_{Nx}\cos\theta_x + D_{Ny}\cos\theta_y = D_{Nx}\lambda_x + D_{Ny}\lambda_y$$

$$d_F = D_{Fx}\cos\theta_x + D_{Fy}\cos\theta_y = D_{Fx}\lambda_x + D_{Fy}\lambda_y$$

$$\underbrace{\begin{Bmatrix} d_N \\ d_F \end{Bmatrix}}_{\mathbf{d}} = \underbrace{\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix}}_{\mathbf{D}}$$

$$\mathbf{d} = \mathbf{TD}$$

$$\mathbf{T} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

- **T** transforms the four global *x*, *y* displacements **D** into the two local displacements **d**.

Stiffness method of analysis: **trusses**

Displacement Transformation Matrices

- Since: $\lambda_x = \cos\theta_x$ $\lambda_y = \cos\theta_y$

$$d_N = D_{Nx}\cos\theta_x + D_{Ny}\cos\theta_y = D_{Nx}\lambda_x + D_{Ny}\lambda_y$$

$$d_F = D_{Fx}\cos\theta_x + D_{Fy}\cos\theta_y = D_{Fx}\lambda_x + D_{Fy}\lambda_y$$

$$\overset{\mathbf{d}}{\begin{Bmatrix} d_N \\ d_F \end{Bmatrix}} = \overset{\mathbf{T}}{\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}} \overset{\mathbf{D}}{\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix}} \qquad \mathbf{d} = \mathbf{TD}$$

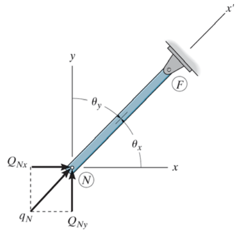
$$\mathbf{T} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

- Hence, **T** is referred to as the ***displacement transformation matrix***.

Stiffness method of analysis: **trusses**

Force Transformation Matrices

- Consider now application of the force q_N to the near end of the member, with the far end held pinned (fixed).



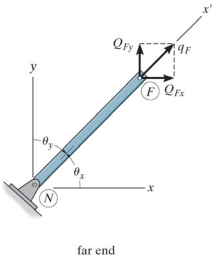
$$Q_{Nx} = q_N \cos\theta_x$$

$$Q_{Ny} = q_N \cos\theta_y$$

Stiffness method of analysis: **trusses**

Force Transformation Matrices

- Likewise, if the force q_F to the far end of the member, with the near end held pinned (fixed).



$$Q_{Fx} = q_F \cos\theta_x$$

$$Q_{Fy} = q_F \cos\theta_y$$

Stiffness method of analysis: **trusses**

Force Transformation Matrices

- Since: $\lambda_x = \cos\theta_x$ $\lambda_y = \cos\theta_y$

$$Q_{Nx} = q_N\lambda_x \qquad Q_{Ny} = q_N\lambda_y \qquad Q_{Fx} = q_F\lambda_x \qquad Q_{Fy} = q_F\lambda_y$$

$$\mathbf{T} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

$$\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx} \\ Q_{Fy} \end{Bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \begin{Bmatrix} q_N \\ q_F \end{Bmatrix} \qquad \mathbf{T}^T = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \qquad \mathbf{Q} = \mathbf{T}^T\mathbf{q}$$

- In this case this ***force transformation matrix*** transforms the two local (x') forces **q** acting at the ends of the member into the four global (*x*, *y*) force components **Q**.

Stiffness method of analysis: **trusses**

Force Transformation Matrices

- Since: $\lambda_x = \cos\theta_x$ $\lambda_y = \cos\theta_y$

$$Q_{Nx} = q_N\lambda_x \qquad Q_{Ny} = q_N\lambda_y \qquad Q_{Fx} = q_F\lambda_x \qquad Q_{Fy} = q_F\lambda_y$$

$$\mathbf{T} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

$$\begin{Bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Fx} \\ Q_{Fy} \end{Bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \begin{Bmatrix} q_N \\ q_F \end{Bmatrix} \qquad \mathbf{T}^T = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \qquad \mathbf{Q} = \mathbf{T}^T\mathbf{q}$$

- Notice that $\mathbf{T}^T$ is the transpose of the displacement transformation matrix **T**.

Stiffness method of analysis: **trusses**

Member Global Stiffness Matrix

- We will now combine the results of the preceding sections and thereby determine the stiffness matrix for a member in terms of its global force components **Q** and its global displacements **D**.

- To do this, we substitute **d** = **TD** into **q** = **k'd** so that the member's forces **q** are then expressed in terms of the global displacements **D**.

$$\mathbf{q} = \mathbf{k'd} \quad \Rightarrow \quad \mathbf{q} = \mathbf{k'TD}$$

- Substituting this equation into **Q** = $\mathbf{T}^T$**q** yields

$$\mathbf{Q} = \mathbf{T}^T\mathbf{q} \quad \Rightarrow \quad \mathbf{Q} = \mathbf{T}^T\mathbf{k'TD} \qquad \boxed{\mathbf{k} = \mathbf{T}^T\mathbf{k'T}}$$

Stiffness method of analysis: **trusses**

Member Global Stiffness Matrix

- The matrix **k** is the ***member stiffness matrix*** in global coordinates.
- Since $\mathbf{T}^T$, **T** and **k**′ are known, we have

$$\mathbf{k} = \overset{\mathbf{T}'}{\begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix}} \overset{\mathbf{k}'}{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \overset{\mathbf{T}}{\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}}$$

Stiffness method of analysis: **trusses**

Member Global Stiffness Matrix

- The matrix **k** is the ***member stiffness matrix*** in global coordinates.
- Since $\mathbf{T}^T$, **T** and **k**′ are known, we have

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c} \\ \left[\begin{array}{cccc} N_x & N_y & F_x & F_y \\ \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{array}\right] \end{array} \begin{array}{c} \\ N_x \\ N_y \\ F_x \\ F_y \end{array}$$

Member stiffness matrix in global coordinates

Stiffness method of analysis: **trusses**

Member Global Stiffness Matrix

- This is indicated by the code number notation along the rows and columns, that is, N_x, N_y, F_x, and F_y.

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c} \\ \left[\begin{array}{cccc} N_x & N_y & F_x & F_y \\ \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{array}\right] \end{array} \begin{array}{c} \\ N_x \\ N_y \\ F_x \\ F_y \end{array}$$

Member stiffness matrix in global coordinates

Stiffness method of analysis: **trusses**

Member Global Stiffness Matrix

- The *location* of each element in this symmetric matrix is referenced with each global degree of freedom associated with the near end *N*, followed by the far end *F*.

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c} \\ \left[\begin{array}{cccc} N_x & N_y & F_x & F_y \\ \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{array}\right] \end{array} \begin{array}{c} \\ N_x \\ N_y \\ F_x \\ F_y \end{array}$$

Member stiffness matrix in global coordinates

Stiffness method of analysis: **trusses**

Member Global Stiffness Matrix

- Here **k** represents the force-displacement relations for the member when the components of force and displacement at the ends of the member are in the global or *x*, *y* directions.

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c} \\ \left[\begin{array}{cccc} N_x & N_y & F_x & F_y \\ \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{array}\right] \end{array} \begin{array}{c} \\ N_x \\ N_y \\ F_x \\ F_y \end{array}$$

Member stiffness matrix in global coordinates

Stiffness method of analysis: **trusses**

Truss Stiffness Matrix

- Once all the member stiffness matrices are formed in global coordinates, it then becomes necessary to assemble them in the proper order so that the ***structure stiffness matrix*** **K** for the truss can be found.
- This process of combining the member stiffness matrices depends on careful identification of the row and column of each element in the member stiffness matrices as noted by the *four* code numbers N_x, N_y, F_x, and F_y.

Stiffness method of analysis: **trusses**

Truss Stiffness Matrix

- For all the truss members the structure stiffness matrix will then have an order that will be equal to the highest code number assigned to the truss, since this represents the total number of degrees of freedom for the truss.
- To form the structure stiffness matrix, **K**, it is therefore necessary to take each member's elements in **k** and place them in the *same* row and column designation in **K**.
- If some elements are assigned to the same location, then they must be added together algebraically.

Stiffness method of analysis: **trusses**

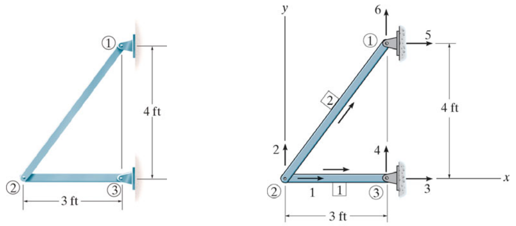
Truss Stiffness Matrix

- This method of assembling the member stiffness matrices to form the structure stiffness matrix will now be demonstrated by two numerical examples.
- Although this process is somewhat tedious when done by hand, it is rather easy to program on a computer.

Stiffness method of analysis: **trusses**

Example 14.1

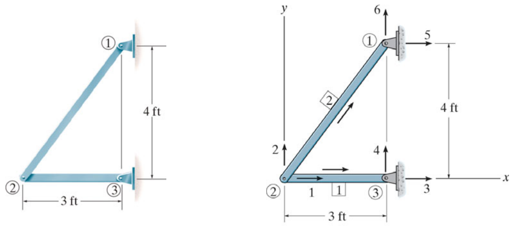
- Determine the structure stiffness matrix for the two-member truss shown below. *AE* is constant.



Stiffness method of analysis: **trusses**

Example 14.1

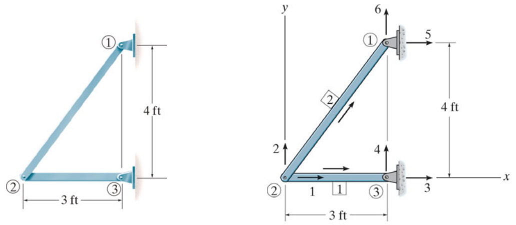
- By inspection, joint ② will have two unknown displacement components, whereas joints ① and ③ are constrained from displacement.



Stiffness method of analysis: **trusses**

Example 14.1

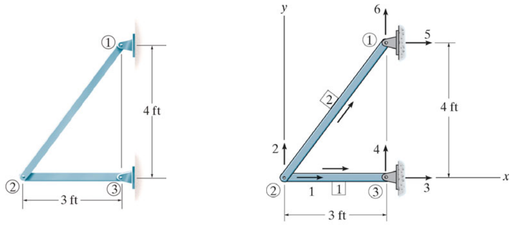
- Consequently, the displacements at joint ② are code numbered first, followed by those at joints ③ and, ①



Stiffness method of analysis: **trusses**

Example 14.1

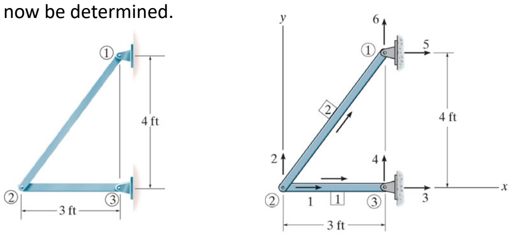
- The origin of the global coordinate system is at joint ②.



Stiffness method of analysis: **trusses**

Example 14.1

- The members are identified, and arrows are written along the two members to identify the near and far ends of each member.
- The direction cosines and the stiffness matrix for each member can now be determined.



Stiffness method of analysis: **trusses**

Example 14.1 – Member 1

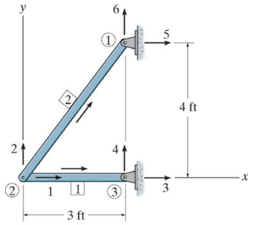
- Since ② is the near end and ③ is the far end, then:

$$\lambda_x = \frac{x_F - x_N}{L_{NF}} = \frac{3-0}{3} = 1$$

$$\lambda_y = \frac{y_F - y_N}{L_{NF}} = \frac{0-0}{3} = 0$$

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c c} \begin{array}{cccc} N_x & N_y & F_x & F_y \end{array} & \\ \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix} & \begin{array}{c} N_x \\ N_y \\ F_x \\ F_y \end{array} \end{array}$$

$$\mathbf{k}_1 = AE\begin{array}{c c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 0.333 & 0 & -0.333 & 0 \\ 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$



Stiffness method of analysis: **trusses**

Example 14.1 – Member 2

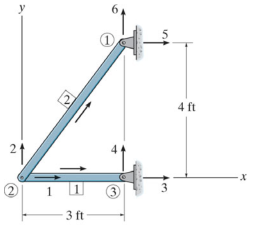
- Since ② is the near end and ① is the far end, we have:

$$\lambda_x = \frac{x_F - x_N}{L_{NF}} = \frac{3-0}{5} = 0.6$$

$$\lambda_y = \frac{y_F - y_N}{L_{NF}} = \frac{4-0}{5} = 0.8$$

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c c} \begin{array}{cccc} N_x & N_y & F_x & F_y \end{array} & \\ \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix} & \begin{array}{c} N_x \\ N_y \\ F_x \\ F_y \end{array} \end{array}$$

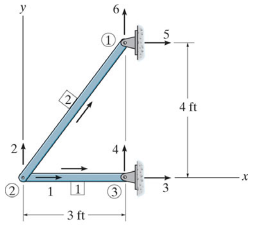
$$\mathbf{k}_2 = AE\begin{array}{c c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.072 & 0.096 & -0.072 & -0.096 \\ 0.096 & 0.128 & -0.096 & -0.128 \\ -0.072 & -0.096 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0.096 & 0.128 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array}$$



Stiffness method of analysis: **trusses**

Example 14.1 – Structure Stiffness Matrix

- This matrix has an order 6 × 6 of since there are six designated degrees of freedom for the truss.

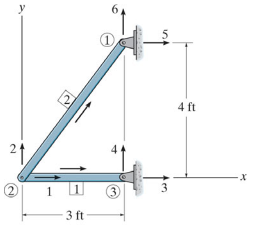


Stiffness method of analysis: **trusses**

Example 14.1 – Structure Stiffness Matrix

- Corresponding elements of the member stiffness matrices are now added algebraically to form the structure stiffness matrix.

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$$



Stiffness method of analysis: **trusses**

Example 14.1 – Structure Stiffness Matrix

- The assembly process is easier to see if the missing numerical columns and rows in $\mathbf{k}_1$ and $\mathbf{k}_2$ are filled with zeros to form two matrices.

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$$

$$\mathbf{K} = AE\begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.333 & 0 & -0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array} + AE\begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.072 & 0.096 & 0 & 0 & -0.072 & -0.096 \\ 0.096 & 0.128 & 0 & 0 & -0.096 & -0.128 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.072 & -0.096 & 0 & 0 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0 & 0 & 0.096 & 0.128 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$\mathbf{K} = AE\begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.405 & 0.096 & -0.333 & 0 & -0.072 & -0.096 \\ 0.096 & 0.128 & 0 & 0 & -0.096 & -0.128 \\ -0.333 & 0 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.072 & -0.096 & 0 & 0 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0 & 0 & 0.096 & 0.128 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Stiffness method of analysis: **trusses**

Application of the Stiffness Method for Truss Analysis

- Once the structure stiffness matrix is formed, the global force components **Q** acting on the truss can then be related to its global displacements **D** using: $\mathbf{Q} = \mathbf{KD}$
- This equation is referred to as the ***structure stiffness equation***.
- Since we have always assigned the lowest code numbers to identify the unconstrained degrees of freedom, this will allow us now to partition this equation in the following form:

$$\begin{Bmatrix} \mathbf{Q}_k \\ \mathbf{Q}_u \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_u \\ \mathbf{D}_k \end{Bmatrix}$$

$\mathbf{Q}_k$, $\mathbf{D}_k$ ***known*** loads and displacements

$\mathbf{Q}_u$, $\mathbf{D}_u$ ***unknown*** loads and displacements

Stiffness method of analysis: **trusses**

Application of the Stiffness Method for Truss Analysis

- Expanding the partitioned matrix yields:

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k \qquad \mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k$$

- Most often $\mathbf{D}_k$ = 0 since the supports are not displaced.

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u$$

- Solving for $\mathbf{D}_u$ gives: $\mathbf{D}_u = \mathbf{K}_{11}^{-1}\mathbf{Q}_k$
- Here we can obtain a direct solution for all the unknown joint displacements: $\mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u$

Stiffness method of analysis: **trusses**

Application of the Stiffness Method for Truss Analysis

- Member forces can be determined using: $q = k'\mathbf{TD}$

$$\begin{Bmatrix} q_N \\ q_F \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix}$$

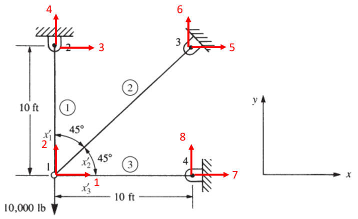
- Since $q_N = -q_F$ for equilibrium, only one of these forces has to be found.
- Here we will determine q_F, the one that exerts tension in the member:

$$\boxed{q_F = \frac{AE}{L}\begin{bmatrix} -\lambda_x & -\lambda_y & \lambda_x & \lambda_y \end{bmatrix}\begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix}}$$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

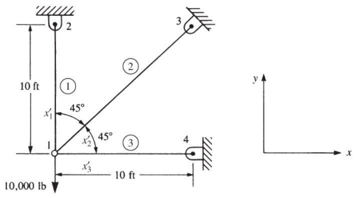
- The plane truss shown below is composed of three bars subjected to a downward force of 10 *kips* at node 1. Assume the cross-sectional area *A* = 2 in^2 and *E* is 30 x 10^6 *psi* for all elements.
- Determine the *x* and *y* displacement at node 1 and stresses in each element.



Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

Element	Node 1	Node 2	θ_x	θ_y	λ_x	λ_y
1	1	2	90°	0°	0	1
2	1	3	45°	45°	0.707	0.707
3	1	4	0°	90°	1	0



Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

$$\mathbf{k} = \frac{AE}{L}\begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix}$$

- The global elemental stiffness matrix are:

element 1: $\lambda_x = 0 \quad \lambda_y = 1$

$$\Rightarrow \mathbf{k}^{(1)} = \frac{(2\,in^2)(30\times10^6\,psi)}{120\,in}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\ {}^{lb}\!/_{in}$$

element 2: $\lambda_x = \frac{\sqrt{2}}{2} \quad \lambda_y = \frac{\sqrt{2}}{2}$

$$\Rightarrow \mathbf{k}^{(2)} = \frac{(2\,in^2)(30\times10^6\,psi)}{240\sqrt{2}\,in}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}\ {}^{lb}\!/_{in}$$

element 3: $\lambda_x = 1 \quad = \lambda_y = 0$

$$\Rightarrow \mathbf{k}^{(3)} = \frac{(2\,in^2)(30\times10^6\,psi)}{120\,in}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \end{array}\ {}^{lb}\!/_{in}$$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

- The total global stiffness matrix is:

$$\mathbf{K} = 5\times10^5 \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} {}^{lb}\!/_{in}$$

element 1

element 2

element 3

- The total global force-displacement equations are:

$$\begin{Bmatrix} 0 \\ -10,000 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = 5\times10^5 \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

- Applying the boundary conditions for the truss, the above equations reduce to:

$$\left\{\begin{array}{c} 0 \\ -10,000 \\ \hdashline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{array}\right\} = 5\times10^5 \left[\begin{array}{cc:cccccc} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} D_1 \\ D_2 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}$$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

- Applying the boundary conditions for the truss, the above equations reduce to:

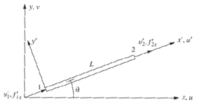
$$\begin{Bmatrix} 0 \\ -10,000 \end{Bmatrix} = 5\times10^5 \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}$$

- Solving the equations gives: $D_1 = 0.414\times10^{-2}\,in$

  $D_2 = -1.59\times10^{-2}\,in$

- Stress in an element is: $\sigma = \frac{E}{L}\left[-\lambda_x D_{Nx} - \lambda_y D_{Ny} + \lambda_x D_{Fx} + \lambda_y D_{Fy}\right]$

  where $\boxed{i}$ is the local node number



Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

Element	Node 1	Node 2	θ_x	θ_y	λ_x	λ_y
1	1	2	90°	0°	0	1
2	1	3	45°	45°	0.707	0.707
3	1	4	0°	90°	1	0

$$\sigma = \frac{E}{L}\left[-\lambda_x D_{Nx} - \lambda_y D_{Ny} + \lambda_x D_{Fx} + \lambda_y D_{Fy}\right]$$

element 1 $\sigma^{(1)} = \frac{30\times10^6}{120}\left[-D_2\right] = 3{,}965\ psi$

element 2 $\sigma^{(2)} = -\frac{30\times10^6}{120}\left[(0.707)D_1 + (0.707)D_2\right] = 1{,}471\ psi$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

Element	Node 1	Node 2	θ_x	θ_y	λ_x	λ_y
1	1	2	90°	0°	0	1
2	1	3	45°	45°	0.707	0.707
3	1	4	0°	90°	1	0

$$\sigma = \frac{E}{L}\left[-\lambda_x D_{Nx} - \lambda_y D_{Ny} + \lambda_x D_{Fx} + \lambda_y D_{Fy}\right]$$

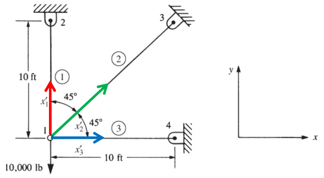
element 3 $\sigma^{(3)} = \frac{30\times10^6}{120}\left[-D_1\right] = -1{,}035\ psi$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

- Let's check equilibrium at node 1:

$F = \sigma A$



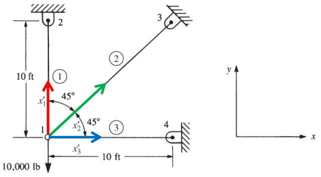
$$\sum F_x = \underline{F^{(2)}\cos(45°)} + \underline{F^{(3)}}$$

$$\sum F_y = \underline{F^{(2)}\sin(45°)} + \underline{F^{(1)}} - 10{,}000\,lb$$

Stiffness method of analysis: **trusses**

Example 14.2 – Plane Truss Problem

- Let's check equilibrium at node 1:



$$\sum F_x = (1{,}471\, psi)(2\, in^2)(0.707) - (1{,}035\; psi)(2\, in^2) = 0$$

$$\sum F_y = (3{,}965\; psi)(2\, in^2) + (1{,}471\, psi)(2\, in^2)(0.707) - 10{,}000 = 0$$

Stiffness method of analysis: **trusses**

Let's work some problems

Stiffness method of analysis: **trusses**

Any questions?