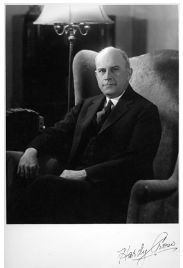
# Chapter 11

## Displacement Method of Analysis: **Moment Distribution**

- In 1930, Hardy Cross developed a method of analyzing beams and frames using ***moment distribution***.
- This method has been recognized as one of the most notable early advances in structural analysis during the 20th century.



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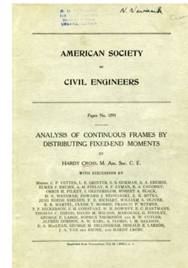
# Chapter 11

## Displacement Method of Analysis: **Moment Distribution**

- Hardy Cross was Professor of Structural Engineering in the Department of Civil Engineering at the University of Illinois from 1921 to 1937.
- During that period, his technical achievements significantly changed the field of structural analysis and the understanding of structural behavior.


AMERICAN SOCIETY OF CIVIL ENGINEERS

ANALYSIS OF CONTINUOUS FRAMES BY DISTRIBUTING FIXED-END MOMENTS

HARDY CROSS, M. AM. SOC. C. E.

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# Chapter 11

## Displacement Method of Analysis: **Moment Distribution**

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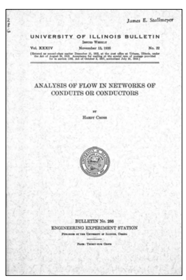
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BULLETIN No. 286

ENGINEERING EXPERIMENT STATION

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Displacement method of analysis: **moment distribution**

**Objectives**:

- To show how to apply the moment-distribution method to solve problems involving beams and frames.
- Moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy.

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Displacement method of analysis: **moment distribution**

- Essentially, the method begins by assuming each joint of a structure is fixed.
- Then, by unlocking and locking each joint in succession, the internal moments at the joints are “distributed” and balanced until the joints have rotated to their final or nearly final positions.
- It will be shown that this repetitive process of calculation is rather easy to learn to apply.

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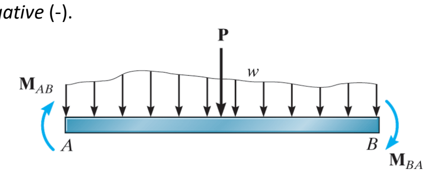
Displacement method of analysis: **moment distribution**

**Sign Convention**

- We will use the same sign convention as that established for the ***slope-deflection equations***:

  ***Clockwise moments*** that act *on the member* are considered *positive* (+), whereas ***counterclockwise moments*** are *negative* (-).

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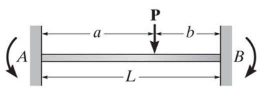
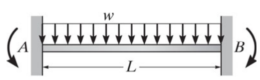
Displacement method of analysis: **moment distribution**

**Fixed-End Moments**

- The moments at the "walls" or fixed joints of a loaded member are called ***fixed-end moments (FEM)***.
- These moments can be determined from the table given on the inside back cover.

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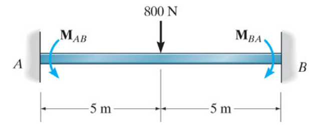
Displacement method of analysis: **moment distribution**

**Fixed-End Moments**

- For example, consider the beam loaded as shown below:

$$(FEM) = \frac{PL}{8}$$




$$(FEM) = \frac{PL}{8}$$

- Noting the action of these moments ***on the beam*** and applying our sign convention:

$$M_{AB} = (FEM)_{AB} = -\frac{PL}{8} = -\frac{800\,N(10\,m)}{8} = -1{,}000\,Nm$$

$$M_{BA} = (FEM)_{BA} = \frac{PL}{8} = \frac{800\,N(10\,m)}{8} = 1{,}000\,Nm$$

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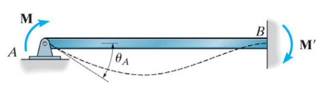
Displacement method of analysis: **moment distribution**

**Member Stiffness Factor**

- Consider the beam loaded as shown below which is pinned at one end and fixed at the other.




- Application of the moment **M** causes the end *A* to rotate through an angle $\theta_A$.
- In Chapter 10, we related **M** to $\theta_A$ using the conjugate-beam method.

$$M = \left(\frac{4EI}{L}\right)\theta_A$$

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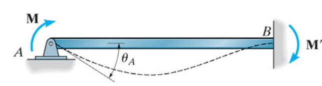
Displacement method of analysis: **moment distribution**

**Member Stiffness Factor**

- Consider the beam loaded as shown below which is pinned at one end and fixed at the other.




- The term in parentheses is the ***stiffness factor** K* at *A*,
- It is the amount of moment **M** required to rotate the end *A* of the beam $\theta_A = 1$ rad.

$$M = \left(\frac{4EI}{L}\right)\theta_A \qquad K = \frac{4EI}{L}$$

*Far end fixed*

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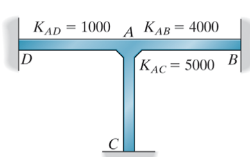
Displacement method of analysis: **moment distribution**

**Joint Stiffness Factor**

- If several members are fixed connected to a joint and each of their far ends is *fixed*, then by the principle of superposition, the ***total stiffness factor*** at the joint is the sum of the member stiffness factors at the joint: $K_T = \sum K$
- Consider joint *A* in this frame:
- The total stiffness factor of joint *A* is:




$$K_T = \sum 1{,}000 + 4{,}000 + 5{,}000$$

$$K_T = 10{,}000$$

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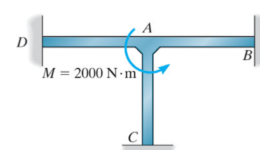
Displacement method of analysis: **moment distribution**

**Distribution Factor (DF)**

- If a moment **M** is applied to joint *A*, the three connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint.
- That fraction of the total resisting moment supplied by a member is called the ***distribution factor*** (**DF**).




- To obtain its value, imagine when **M** causes the joint to rotate an amount $\theta$, then all three members rotate by this same amount.

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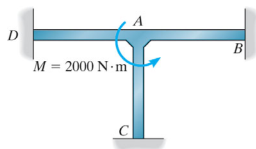
Displacement method of analysis: **moment distribution**

**Distribution Factor (DF)**

- For example, if $\mathbf{M}$ = 2,000 *Nm* it causes *A* to rotate $\theta_A$.
- Using the stiffness factor of $K_{AB}$, for member *AB*, then the moment contributed by this member is: $M_{AB} = K_{AB}\theta_A$




- Equilibrium requires:

$$M = M_{AB} + M_{AC} + M_{AD}$$

$$M = K_{AB}\theta_A + K_{AC}\theta_A + K_{AD}\theta_A \;= \theta_A \sum K$$

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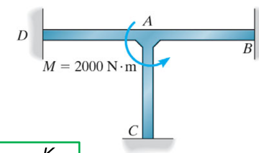
Displacement method of analysis: **moment distribution**

**Distribution Factor (DF)**

- For example, if $\mathbf{M}$ = 2,000 *Nm* it causes *A* to rotate $\theta_A$.
- Using the stiffness factor of $K_{AB}$, for member *AB*, then the moment contributed by this member is: $M_{AB} = K_{AB}\theta_A$




- The distribution factor for member *AB* is:

$$DF_{AB} = \frac{M_{AB}}{M} = \frac{K_{AB}\theta_A}{\theta_A \sum K} \qquad DF_{AB} = \frac{K_{AB}}{\sum K}$$

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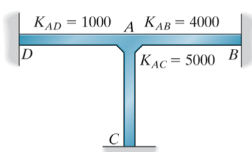
Displacement method of analysis: **moment distribution**

**Distribution Factor (DF)**

- Therefore, the distribution factors for members *AB*, *AC*, and *AD* at joint *A* are:




$$DF_{AB} = \frac{K_{AB}}{\sum K} = \frac{4{,}000}{10{,}000} = 0.4$$

$$DF_{AD} = \frac{K_{AD}}{\sum K} = \frac{1{,}000}{10{,}000} = 0.1$$

$$DF_{AC} = \frac{K_{AC}}{\sum K} = \frac{5{,}000}{10{,}000} = 0.5$$

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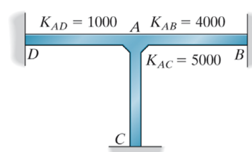
Displacement method of analysis: **moment distribution**

**Distribution Factor (DF)**

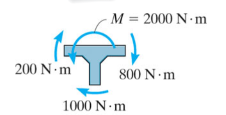
- If acts **M** = 2,000 *kNm* at joint *A*, the equilibrium moments exerted by the members on the joint become:




$$M_{AB} = 0.4(2{,}000\,Nm) = 800\,Nm$$

$$M_{AC} = 0.5(2{,}000\,Nm) = 1{,}000\,Nm$$

$$M_{AD} = 0.1(2{,}000\,Nm) = 200\,Nm$$

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Displacement method of analysis: **moment distribution**

**Member Relative-Stiffness Factor**

- Quite often a continuous beam or a frame will be made from the same material so its modulus of elasticity ***E*** will be the *same* for all the members.
- If this is the case, the common factor **4*E*** in the *Member Stiffness Factor* will *cancel* from the numerator and denominator of *Distribution Factor*.
- Hence, it is *easier* just to determine the member's ***relative-stiffness factor***

$$K_R = \frac{I}{L} \qquad \textit{Far end fixed}$$

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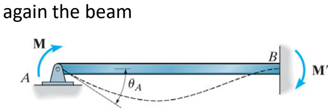
Displacement method of analysis: **moment distribution**

**Carry-Over Factor**

- Consider again the beam




- It was shown in Chapter 10 that: $M = \left(\frac{4EI}{L}\right)\theta_A \qquad M' = \left(\frac{2EI}{L}\right)\theta_A$
- Combining these equations, we get: $M' = \frac{1}{2}M$

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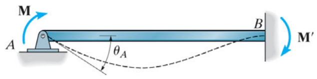
Displacement method of analysis: **moment distribution**

**Carry-Over Factor**

- Consider again the beam




- The ***carry-over factor*** represents the fraction of **M** that is "carried over" from the pin to the wall.
- Hence, in this case of a beam with *the far end fixed*, the carry-over factor is +½.
- The plus sign indicates that both moments act in the same direction.

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Displacement method of analysis: **moment distribution for beams**

- Moment distribution is based on the principle of successively locking and unlocking the joints of a structure to allow the moments at the joints to be distributed and balanced.
- The best way to explain the method is by examples.

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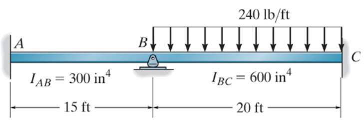
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Consider the beam with a constant modulus of elasticity *E* and having the dimensions and loading shown below:




- Before we begin, we must first determine the ***distribution factors*** at the two ends of each span.
- The stiffness factors on either side of *B* are:

$$K = \frac{4EI}{L} \qquad K_{BA} = \frac{4E\left(300\,in^4\right)}{15\,ft} = 4E\left(20\,{}^{in^4}\!/_{ft}\right)$$

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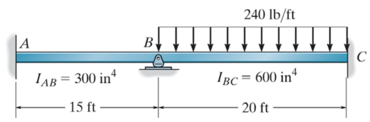
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Consider the beam with a constant modulus of elasticity *E* and having the dimensions and loading shown below:




- Before we begin, we must first determine the ***distribution factors*** at the two ends of each span.
- The stiffness factors on either side of *B* are:

$$K = \frac{4EI}{L} \qquad K_{BC} = \frac{4E\left(600\,in^4\right)}{20\,ft} = 4E\left(30\,{}^{in^4}\!/_{ft}\right)$$

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: For the ends connected to joint *B*:

$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{4E\left(20\,{}^{in^4}\!/_{ft}\right)}{4E\left(20\,{}^{in^4}\!/_{ft}\right) + 4E\left(30\,{}^{in^4}\!/_{ft}\right)} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{4E\left(30\,{}^{in^4}\!/_{ft}\right)}{4E\left(20\,{}^{in^4}\!/_{ft}\right) + 4E\left(30\,{}^{in^4}\!/_{ft}\right)} = 0.6$$

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: At the locked joints *A* and *C*, the ***distribution factor*** depends on the member stiffness factor and the "stiffness factor" of the joint.
- Since in theory it would take a moment of infinite magnitude to rotate a fixed joint one radian, the stiffness factor is infinite, and so for joints *A* and *C* we have:

$$DF_{AB} = \frac{K_{AB}}{\sum K} = \frac{4E\left(20\,{}^{in^4}\!/_{ft}\right)}{\infty + 4E\left(20\,{}^{in^4}\!/_{ft}\right)} = 0$$

$$DF_{CB} = \frac{K_{CB}}{\sum K} = \frac{4E\left(30\,{}^{in^4}\!/_{ft}\right)}{\infty + 4E\left(30\,{}^{in^4}\!/_{ft}\right)} = 0$$

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Note that the above results could also have been obtained if the relative-stiffness factor $K = I/L$ had been used for the calculations.
- Furthermore, if a ***consistent*** set of units is used for the stiffness factor, the ***DF*** will always be dimensionless, and at a joint, except where it is located at a fixed wall, the sum of the ***DFs*** will always equal 1.

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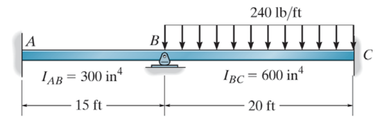
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Having calculated the *DF*s, we will now determine the FEMs.
- Only span *BC* is loaded, and using the table on the inside back cover for a uniform load, we can compute the FEMs:

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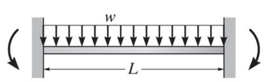
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Having calculated the *DF*s, we will now determine the FEMs.
- Only span *BC* is loaded, and using the table on the inside back cover for a uniform load, we can compute the FEMs:

$(FEM) = \frac{wL^2}{12}$




$(FEM) = \frac{wL^2}{12}$

$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{240\,lb/ft(20\,ft)^2}{12} = -8{,}000\,lbft$$

$$(FEM)_{CB} = \frac{wL^2}{12} = \frac{240\,lb/ft(20\,ft)^2}{12} = 8{,}000\,lbft$$

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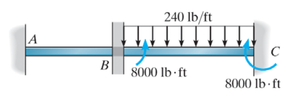
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: We begin by assuming joint *B* is fixed or locked.
- The fixed-end moment at *B* then holds span *BC* in this fixed or locked position as shown below:




- This, of course, does not represent the actual equilibrium situation at *B*, since the moments on ***each side*** of this joint must be equal but opposite.

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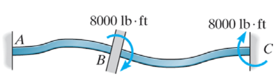
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: To correct this, we will apply an equal, but opposite moment of to the joint and *allow the joint to rotate freely.*




- As a result, portions of this moment are then distributed in spans *BC* and *BA* in accordance with the *DF*s (or stiffness) of these spans at the joint.
- Specifically, the moment in *BA* is 0.4(8,000 *lb·ft*) = 3,200 *lb·ft* and the moment in *BC* is 0.6(8,000 *lb·ft*) = 4,800 *lb·ft*.

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Displacement method of analysis: **moment distribution for beams**

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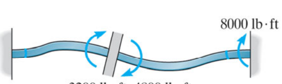


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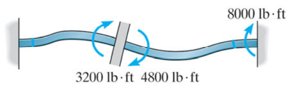
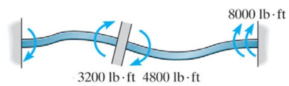
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Finally, due to the released rotation at *B*, these moments must be "carried over" since moments at *B* create reactions at the other ends.




- Using the carry-over factor of +½, the results are shown in:

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Displacement method of analysis: **moment distribution for beams**

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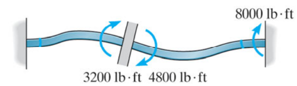


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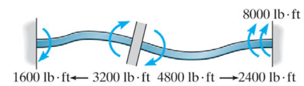


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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: This example indicates the basic steps necessary when distributing moments at a joint:

  1. Determine the unbalanced moment acting at the initially "locked" joint (8,000 *lb·ft*),
  2. Unlock the joint and apply an equal but opposite unbalanced moment to correct the equilibrium,
  3. **Distribute the moment** among the connecting spans (3,200 *lb·ft* and 4,800 *lb·ft*), and
  4. **Carry the moment** in each span over to its other end (1,600 *lb·ft* and 2,400 *lb·ft*).

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: These steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4 | 0.6 | 0 |
| FEM | | | | |
| Dist. | | | | |
| CO | | | | |
| Dist. | | | | |
| Σ | | | | |

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: These steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4 | 0.6 | 0 |
| FEM | | | -8,000 | 8,000 |
| Dist. | | 3,200 | 4,800 | |
| CO | 1,600 | | | 2,400 |
| Dist. | | | | |
| Σ | **1,600** | **3,200** | **-3,200** | **10,400** |

- Here the notation **Dist.** indicates a line where moments are distributed, then carried over.

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: These steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4 | 0.6 | 0 |
| FEM | | | -8,000 | 8,000 |
| Dist. | | 3,200 | 4,800 | |
| CO | 1,600 | | | 2,400 |
| Dist. | | | | |
| Σ | **1,600** | **3,200** | **-3,200** | **10,400** |

- In this case, only ***one cycle*** of moment distribution is necessary, since the wall supports at *A* and *C* "absorb" the moments.

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: These steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4 | 0.6 | 0 |
| FEM | | | -8,000 | 8,000 |
| Dist. | | 3,200 | 4,800 | |
| CO | 1,600 | | | 2,400 |
| Dist. | | | | |
| Σ | **1,600** | **3,200** | **-3,200** | **10,400** |

- No further joints must be balanced or unlocked to satisfy joint equilibrium.

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: These steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4 | 0.6 | 0 |
| FEM | | | -8,000 | 8,000 |
| Dist. | | 3,200 | 4,800 | |
| CO | 1,600 | | | 2,400 |
| Dist. | | | | |
| Σ | **1,600** | **3,200** | **-3,200** | **10,400** |

- Notice that joint *B* is now in equilibrium.

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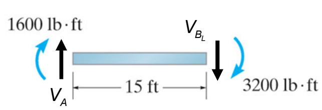
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Since $M_{BA}$ is positive, this moment is applied to span *BA* in a clockwise sense as shown on its free-body diagram:




$\circlearrowleft^+ \sum M_A = 0 = -1{,}600\,lbft - 3{,}200\,lbft - V_{B_L}(15\,ft)$ $\quad V_{B_L} = 320\,lb$

$+\uparrow \sum F_y = 0 = V_A - V_{B_L}$ $\quad V_A = -320\,lb$

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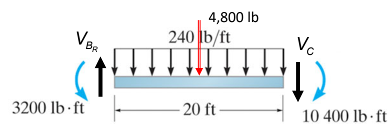
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: Since $M_{BC}$ is negative, this moment is applied to span *BC* in a counterclockwise sense as shown on its free-body diagram:




$\circlearrowleft^+ \sum M_B = 0 = -4{,}800\,lb(10\,ft) + 3{,}200\,lbft - 10{,}400\,lbft - V_C(20\,ft)$

$V_C = -2{,}760\,lb$

$+\uparrow \sum F_y = 0 = -V_C + V_{B_R} - 4{,}800\,lb$ $\quad V_{B_R} = 2{,}040\,lb$

$B_y = 2{,}360\,lb$

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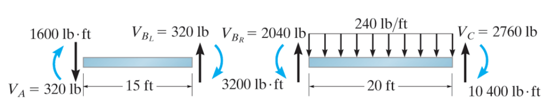
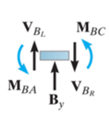
Displacement method of analysis: **moment distribution for beams**

- **Example 11-1**: The resulting support reaction can be determined from the free-body diagrams:




$+\uparrow \sum F_y = 0 = V_{B_L} + B_y - V_{B_R}$

$= 320\,lb + B_y - 2{,}040\,lb$

$B_y = 1{,}720\,lb$

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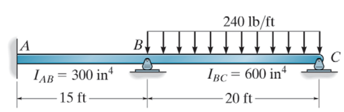
Displacement method of analysis: **moment distribution for beams**

- **Example 11-2**: Consider the beam with a constant modulus of elasticity *E* and a rocker support at *C*.




- In this case, only *one member* is at joint *C*, so the distribution factor for member *CB* at joint *C* is:

$$DF_{CB} = \frac{K_{BA}}{\sum K} = \frac{4E\left(20\,{in^4}/{ft}\right)}{4E\left(20\,{in^4}/{ft}\right)} = 1$$

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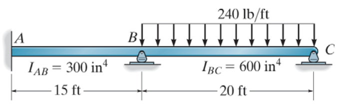
## Displacement method of analysis: **moment distribution for beams**

- **Example 11-2**: Consider the beam with a constant modulus of elasticity *E* and a rocker support at *C*.




- The other distribution factors are the same:

$DF_{AB} = 0$ $DF_{BA} = 0.4$ $DF_{BC} = 0.6$

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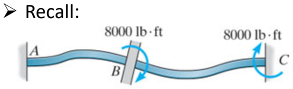
## Displacement method of analysis: **moment distribution for beams**

| Joint | A | B | | C | |
|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | | | | 3 |

- Recall:




- In line 3, we fix all joints, and the moments are distributed to *BA, BC* and *CB.*

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## Displacement method of analysis: **moment distribution for beams**

| Joint | A | B | | C | |
|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | 3200 | 4800 | −8000 | 3 |
| CO | 1600 | | | | 4 |
| Dist. | | | | | 5 |

- In line 4, we unlocked *B* and *C,* and the moments are carried over to the other end of each span.
- In line 5, the joints are locked, and the moments are balanced and distributed.
- *BA* is 0.4(4,000 *lb·ft*) = 1,600 *lb·ft*
*BC* is 0.6(4,000 *lb·ft*) = 2,400 *lb·ft*.

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## Displacement method of analysis: **moment distribution for beams**

| Joint | A | B | | C | |
|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | 3200 | 4800 | −8000 | 3 |
| CO | 1600 | | −4000 | 2400 | 4 |
| Dist. | | 1600 | 2400 | −2400 | 5 |
| CO | 800 | | | | 6 |
| Dist. | | | | | 7 |

- In line 6, we unlocked *B* and *C,* and the moments are carried over to the other end of each span.
- In line 7, the joints are locked, and the moments are balanced and distributed.
- *BA* is 0.4(1,200 *lb·ft*) = 480 *lb·ft*
*BC* is 0.6(1,200 *lb·ft*) = 720 *lb·ft*.

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## Displacement method of analysis: **moment distribution for beams**

| Joint | A | B | | C | |
|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | 3200 | 4800 | −8000 | 3 |
| CO | 1600 | | −4000 | 2400 | 4 |
| Dist. | | 1600 | 2400 | −2400 | 5 |
| CO | 800 | | −1200 | 1200 | 6 |
| Dist. | | 480 | 720 | −1200 | 7 |
| CO | 240 | | | | 8 |
| Dist. | | | | | 9 |

- In line 8, we unlocked *B* and *C,* and the moments are carried over to the other end of each span.
- In line 9, the joints are locked, and the moments are balanced and distributed.
- *BA* is 0.4(600 *lb·ft*) = 240 *lb·ft*
*BC* is 0.6(600 *lb·ft*) = 360 *lb·ft*.

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## Displacement method of analysis: **moment distribution for beams**

| Joint | A | B | | C | |
|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | 3200 | 4800 | −8000 | 3 |
| CO | 1600 | | −4000 | 2400 | 4 |
| Dist. | | 1600 | 2400 | −2400 | 5 |
| CO | 800 | | −1200 | 1200 | 6 |
| Dist. | | 480 | 720 | −1200 | 7 |
| CO | 240 | | −600 | 360 | 8 |
| Dist. | | 240 | 360 | −360 | 9 |
| CO | | | −180 | 180 | 10 |
| Dist. | 120 | 72 | 108 | −180 | 11 |
| CO | | | −90 | 54 | 12 |
| Dist. | 36 | 36 | 54 | −54 | 13 |
| CO | | | −27 | 27 | 14 |
| Dist. | 18 | 10.8 | 16.2 | −27 | 15 |
| CO | | | −13.5 | 8.1 | 16 |
| Dist. | 5.4 | 5.4 | 8.1 | −8.1 | 17 |
| CO | | | −4.05 | 4.05 | 18 |
| Dist. | 2.7 | 1.62 | 2.43 | −4.05 | 19 |
| CO | | | −2.02 | 1.22 | 20 |
| Dist. | 0.81 | 0.80 | 1.22 | −1.22 | 21 |
| CO | | | −0.61 | 0.61 | 22 |
| Dist. | 0.40 | 0.24 | 0.37 | −0.61 | 23 |
| $\Sigma M$ | 2823 | 5647 | −5647 | 0 | 24 |

- Repeat until the changes in moments are negligible.
- The final estimates of the moments are the summation of all the distribution and carry overs.

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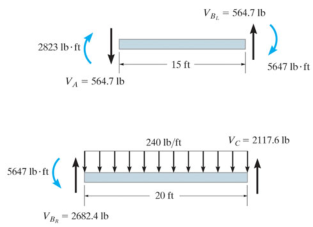
Displacement method of analysis: **moment distribution for beams**

| Joint | $A$ | $B$ | | $C$ | |
|---|---|---|---|---|---|
| Member | $AB$ | $BA$ | $BC$ | $CB$ | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | 3200 | 4800 | −8000 | 3 |
| CO | 1600 | | −4000 | 2400 | 4 |
| Dist. | | 1600 | 2400 | −2400 | 5 |
| CO | 800 | | −1200 | 1200 | 6 |
| Dist. | | 480 | 720 | −1200 | 7 |
| CO | 240 | | −600 | 360 | 8 |
| Dist. | | 240 | 360 | −360 | 9 |
| CO | | | −180 | 180 | 10 |
| Dist. | 120 | 72 | 108 | −180 | 11 |
| CO | | | −90 | 54 | 12 |
| Dist. | 36 | 36 | 54 | −54 | 13 |
| CO | | | −27 | 27 | 14 |
| Dist. | 18 | 10.8 | 16.2 | −27 | 15 |
| CO | | | −13.5 | 8.1 | 16 |
| Dist. | 5.4 | 5.4 | 8.1 | −8.1 | 17 |
| CO | | | −4.05 | 4.05 | 18 |
| Dist. | 2.7 | 1.62 | 2.43 | −4.05 | 19 |
| CO | | | −2.02 | 1.22 | 20 |
| Dist. | 0.81 | 0.80 | 1.22 | −1.22 | 21 |
| CO | | | −0.61 | 0.61 | 22 |
| Dist. | 0.40 | 0.24 | 0.37 | −0.61 | 23 |
| $\Sigma M$ | 2823 | 5647 | −5647 | 0 | 24 |

- Using statics, the final values for shear can be determined.

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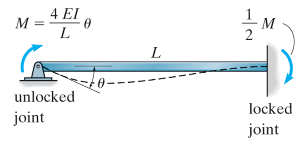
Displacement method of analysis: **moment distribution for beams**

**Stiffness-Factor Modifications**

- In the previous examples of moment distribution, we have considered each beam span to be constrained by a ***fixed support*** (locked joint) at its far end when distributing and carrying over the moments.
- For this reason, we have calculated the stiffness factors, distribution factors, and the carry-over factors based on the case shown below:

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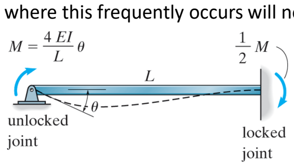
Displacement method of analysis: **moment distribution for beams**

**Stiffness-Factor Modifications**

- Here, of course, the stiffness factor is: $K = \dfrac{4EI}{L}$
- And the carry-over factor is: +½
- In some cases, it is possible to modify the stiffness factor of a span and thereby simplify the process of moment distribution.
- Three cases where this frequently occurs will now be considered.

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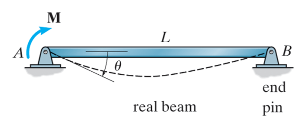
Displacement method of analysis: **moment distribution for beams**

**Member Pin Supported at Far End**

- Many beams are supported at their ends by a ***pin*** (or roller).




- We can determine the stiffness factor at joint $A$ of this beam by applying a moment **M** at the joint and relating it to the angle $\theta$.

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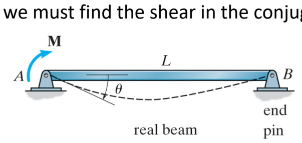
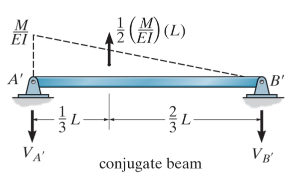
Displacement method of analysis: **moment distribution for beams**

**Member Pin Supported at Far End**

- To do this we must find the shear in the conjugate beam at $A'$.

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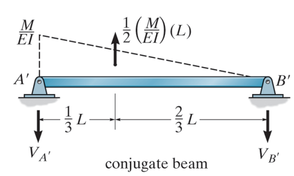
Displacement method of analysis: **moment distribution for beams**

**Member Pin Supported at Far End**

$$\circlearrowleft^{+}\sum M_{B'} = 0 = -\left[\frac{1}{2}\left(\frac{M}{EI}\right)(L)\right]\left(\frac{2}{3}L\right) + V_{A}(L) \qquad V_{A'} = \frac{ML}{3EI} \Rightarrow \theta_{A'}$$

$$M = \left(\frac{3EI}{L}\right)\theta \qquad \boxed{\Rightarrow K = \frac{3EI}{L}}$$

*Far end pinned or roller supported*

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Displacement method of analysis: **moment distribution for beams**

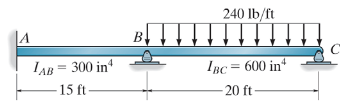
**Member Pin Supported at Far End**

- Also, note that *the **carry-over factor** is **zero*** since the pin at *B* does not support a moment.
- By comparison, if the far end were fixed supported, the stiffness factor $K = 4EI/L$ would have to be modified by ¾ to model the case of having the far end pin supported.
- If this modification is considered, the moment-distribution process is simplified since the end pin does *not* have to be locked–unlocked successively when distributing the moments.
- Also, since the end span is pinned, the fixed-end moments for the span are calculated using the formulas in the right column of the table on the inside back cover.



Displacement method of analysis: **moment distribution for beams**

- **Example 11-2a**: Consider the beam with a constant modulus of elasticity *E* and a rocker support at *C*.

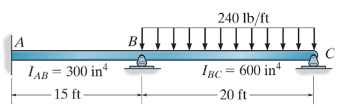



- The distribution factor for *CB* at joint *C* is: $DF_{CB} = 1$
- The distribution factor for *AB* at joint *A* is: $DF_{AB} = 0$



Displacement method of analysis: **moment distribution for beams**

- **Example 11-2a**: Consider the beam with a constant modulus of elasticity *E* and a rocker support at *C*.




- The stiffness factors on either side of *B* are:

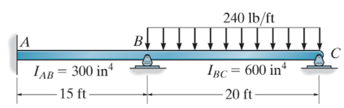
$$K_{BA} = \frac{4E(300\,in^4)}{15\,ft} = E\left(80\,in^4/ft\right)$$

$$K_{BC} = \frac{3E(600\,in^4)}{20\,ft} = E\left(90\,in^4/ft\right)$$



Displacement method of analysis: **moment distribution for beams**

- **Example 11-2a**: Consider the beam with a constant modulus of elasticity *E* and a rocker support at *C*.




- The distribution factors for *B* are:

$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{E\left(80\,in^4/ft\right)}{E\left(80\,in^4/ft\right) + E\left(90\,in^4/ft\right)} = 0.4706$$

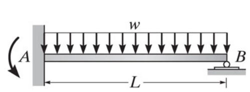
$$DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{E\left(80\,in^4/ft\right)}{E\left(80\,in^4/ft\right) + E\left(90\,in^4/ft\right)} = 0.5294$$



Displacement method of analysis: **moment distribution for beams**

- **Example 11-1a**: We will now determine the FEM for pinned end.
- Only span *BC* is loaded, and using the table on the inside back cover for a uniform load, we can compute the FEMs:

$$(FEM) = \frac{wL^2}{8}$$




$$(FEM)_{BC} = -\frac{wL^2}{8} = -\frac{240\,lb/ft\,(20\,ft)^2}{8} = -12{,}000\,lbft$$



Displacement method of analysis: **moment distribution for beams**

- **Example 11-1a**: The steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4706 | 0.5294 | 1 |
| FEM | | | | |
| Dist. | | | | |
| CO | | | | |
| Dist. | | | | |
| Σ | | | | |

Displacement method of analysis: **moment distribution for beams**

- **Example 11-1a**: The steps are usually presented in tabular form:

| Joint | A | B | | C |
|---|---|---|---|---|
| Members | AB | BA | BC | CB |
| DF | 0 | 0.4706 | 0.5294 | 1 |
| FEM | | | -12,000 | |
| Dist. | | 5,647 | 6,352 | |
| CO | 2,824 | | | |
| Dist. | | | | |
| Σ | **2,824** | **5,647** | **-5,647** | |



Displacement method of analysis: **moment distribution for beams**

Let's work some problems



Displacement method of analysis: **moment distribution for beams**

**Symmetric Beam and Loading**

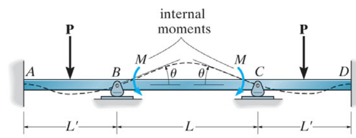
- If a beam is symmetric with respect to both its loading and geometry, the bending-moment diagram for the beam will also be symmetric.
- As a result, a modification of the stiffness factor for the center span can be made, so that moments in the beam only have to be distributed through a joint lying on either half of the beam.



Displacement method of analysis: **moment distribution for beams**

**Symmetric Beam and Loading**

- To develop the appropriate stiffness-factor modification, consider the beam shown below:



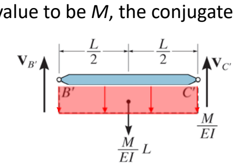

- Due to the symmetry, the internal moments at *B* and *C* are equal.



Displacement method of analysis: **moment distribution for beams**

**Symmetric Beam and Loading**

- Assuming this value to be *M*, the conjugate beam for span *BC* is:




conjugate beam

- The slope $\theta$ at each end is:

$$\circlearrowleft^{+} \sum M_{C'} = 0 = \left[\left(\frac{M}{EI}\right)(L)\right]\left(\frac{1}{2}L\right) - V_{B'}(L) \qquad V_{B'} = \frac{ML}{2EI}$$

$$M = \left(\frac{2EI}{L}\right)\theta \qquad \boxed{\Rightarrow K = \frac{2EI}{L}}$$

*Symmetric beam and loading*



Displacement method of analysis: **moment distribution for beams**

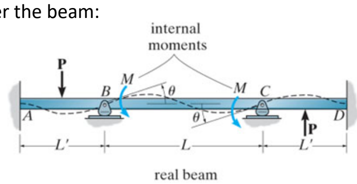
**Symmetric Beam with Antisymmetric Loading**

- If a symmetric beam is subjected to antisymmetric loading, the resulting moment diagram will be ***antisymmetric***.
- As in the previous case, we can modify the stiffness factor of the center span so that only one-half of the beam has to be considered for the moment-distribution analysis.

Displacement method of analysis: **moment distribution for beams**

**Symmetric Beam with Antisymmetric Loading**

- Consider the beam:




- Due to the antisymmetric loading, the internal moment at *B* is equal but opposite to that at *C*.

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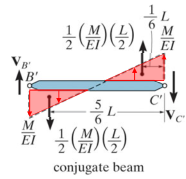
Displacement method of analysis: **moment distribution for beams**

**Symmetric Beam with Antisymmetric Loading**

- Assuming this value to be *M*, the conjugate beam for its center span *BC* is:




$$\circlearrowleft^+ \sum M_{C'} = 0 = -\left[\frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\right]\left(\frac{L}{6}\right) + \left[\frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\right]\left(\frac{5L}{6}\right) - V_{B'}(L)$$

$$V_{B'} = \frac{ML}{6EI} \qquad M = \left(\frac{6EI}{L}\right)\theta \qquad \Rightarrow K = \frac{6EI}{L}$$

*Symmetric beam with antisymmetric loading*

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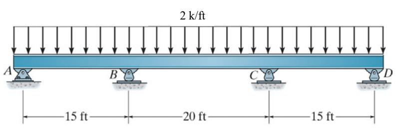
Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: Determine the internal moments at the supports for the beam shown below. Assume *EI* is constant.




- By inspection, the beam and loading are symmetrical.

- We will apply *K = 2EI/L* to calculate the stiffness factor of the center span *BC* and therefore use only the left half of the beam for the analysis.

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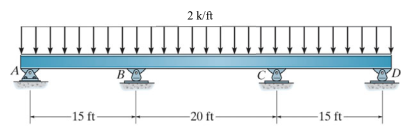
Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: Determine the internal moments at the supports for the beam shown below. Assume *EI* is constant.




- The analysis can be shortened even further by using *K = 3EI/L* for calculating the stiffness factor of segment *AB* since the far end *A* is pinned.

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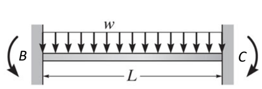
Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: For span *BC*, the FEM are:

$$(FEM) = \frac{wL^2}{12}$$




$$(FEM) = \frac{wL^2}{12}$$

$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{2\,k/ft(20\,ft)^2}{12} = -66.67\,kft$$

$$(FEM)_{CB} = \frac{wL^2}{12} = \frac{2\,k/ft(20\,ft)^2}{12} = 66.67\,kft$$

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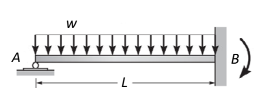
Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: For span *AB*, the FEM are:




$$(FEM) = \frac{wL^2}{8}$$

$$(FEM)_{BA} = \frac{wL^2}{8} = \frac{2\,k/ft(15\,ft)^2}{8} = 56.25\,kft$$

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: The stiffnesses at joint *B* are:

- The stiffness for *AB*: $K = \frac{3EI}{L}$ $K_{BA} = \frac{3EI}{15ft}$
- The stiffness for *BC*: $K = \frac{2EI}{L}$ $K_{BC} = \frac{2EI}{20ft}$

$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{3/15}{3/15 + 2/20} = \frac{2}{3}$$

$$DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{2/20}{3/15 + 2/20} = \frac{1}{3}$$

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: Putting the data into a table:

| Joint | A | B | |
|---|---|---|---|
| Members | AB | BA | BC |
| DF | 1 | 2/3 | 1/3 |
| FEM | 0 | 56.25 | -66.67 |
| Dist. | | | |
| Σ | | | |

-10.42

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Displacement method of analysis: **moment distribution for beams**

- **Example 11-3**: Putting the data into a table:

| Joint | A | B | |
|---|---|---|---|
| Members | AB | BA | BC |
| DF | 1 | 2/3 | 1/3 |
| FEM | 0 | 56.25 | -66.67 |
| Dist. | | 6.9467 | 3.4733 |
| Σ | | **63.1967** | **-63.1967** |

$M_{BC} = 63.1967\ kft$ $M_{CB} = -63.1967\ kft$

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Displacement method of analysis: **moment distribution for beams**

Let's work some problems

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Displacement method of analysis: **moment distribution method**

Any questions?

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