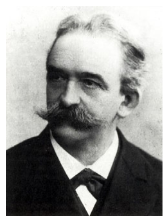
Chapter 10

Displacement Method of Analysis: **Slope-Deflection Equations**

The method was originally developed by Heinrich Manderla and Otto Mohr in 1880 for the purpose of studying secondary stresses in trusses.

Later, in 1915, George. A. Maney developed a refined version of this technique and applied it to analyzing indeterminate beams and framed structures.



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Chapter 10

Displacement Method of Analysis: **Slope-Deflection Equations**

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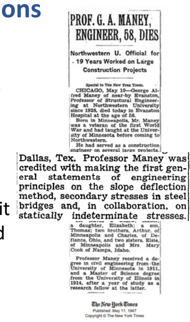
PROF. G. A. MANEY, ENGINEER, 58, DIES

Northwestern U. Official for 19 Years Worked on Large Construction Projects

Special to THE NEW YORK TIMES.

CHICAGO, May 10—George Alfred Maney of near-by Evanston, Professor of Structural Engineering at Northwestern University since 1928, died today in Evanston Hospital at the age of 58.

Born in Minneapolis, Mr. Maney was a veteran of the first World War and had taught at the University of Minnesota before coming to Northwestern.

He had served as a construction engineer on several large projects.

Dallas, Tex. Professor Maney was credited with making the first general statements of engineering principles on the slope deflection method, secondary stresses in steel bridges and, in collaboration, on statically indeterminate stresses.

a daughter, Elizabeth; a son, Thomas; two brothers, Arthur, of Minneapolis and Charles, of Defiance, Ohio, and two sisters, Elsie, of Minneapolis and Mrs. Mary Cook of Nampa, Idaho.

Professor Maney received a degree in civil engineering from the University of Minnesota in 1911, and a Master of Science degree from the University of Illinois in 1914, after a year of study as a research fellow at the latter.

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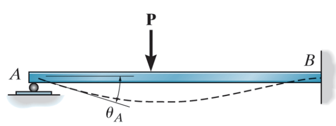
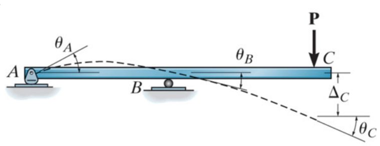
Displacement method of analysis: **slope-defection method**

Objectives:

To show how to apply the ***slope-deflection equations*** to analyze statically indeterminate ***beams***.





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Displacement method of analysis: **slope-defection method**

- All statically indeterminate structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements to ensure their safety.
- In the previous chapter, we used the ***force method*** to satisfy these requirements.
- In this chapter, we will use a ***displacement method***.
- The ***displacement method*** requires writing the unknown displacements in terms of the loads using the ***load-displacement relationships*** and then solving the equilibrium equation for these displacements.

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Displacement method of analysis: **slope-defection method**

- Once displacements are determined, the unknown loads are determined from the ***compatibility equations***.
- Every displacement method follows this general procedure.
- In this chapter, the procedure will be generalized to produce the ***slope-deflection equations***.

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Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- When a structure is loaded, specified points on it, called ***nodes***, will undergo unknown displacements.
- These displacements are referred to as the ***degrees of freedom*** for the structure.
- It is important to specify these ***degrees of freedom*** since they become the unknowns when the method is applied.
- The number of these unknowns is referred to as the degree in which the structure is ***kinematically indeterminate***.

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Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Nodes on a structure are usually located at ***joints, supports,*** at the ***ends*** of a member, or where the members have a sudden ***change in cross-section***.
- In three dimensions, each node on a frame or beam can have at most ***three linear displacements and three rotational displacements***.
- In two dimensions, each node can have at most ***two linear displacements and one rotational displacement***.

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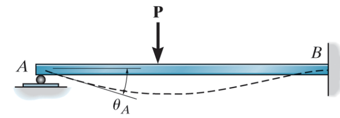
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Let's consider some examples, beginning with this beam.



- Here any load **P** applied to the beam will cause node *A* only to rotate (neglecting axial deformation), while node *B* is completely restricted from moving.

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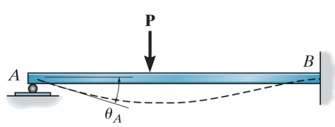
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Let's consider some examples, beginning with this beam.



- The beam has only ***one unknown degree of freedom*** θ_A and is therefore, kinematically indeterminate to the first degree.

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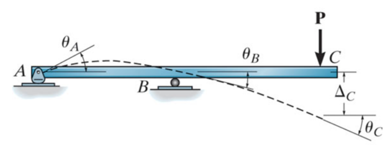
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Consider this beam.



- The beam has nodes at *A*, *B*, and *C*, and so has four degrees of freedom, designated by the unknown rotational displacements θ_A , θ_B , θ_C , and the vertical displacement Δ_C.

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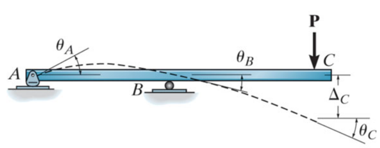
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Consider this beam.



- It is kinematically ***indeterminate to the 4th degree***.

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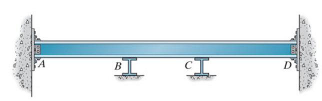
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

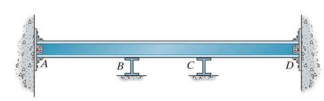
- Consider this beam.



rotational displacements θ_B and θ_C

indeterminate to the 2nd degree

Assume *A* and *D* are fixed, and *B* and *C* are rollers



rotational displacements θ_A, θ_B, θ_C, and θ_D

indeterminate to the 4th degree

Assume *A* and *D* are pins, and *B* and *C* are rollers

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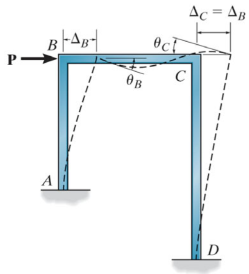
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Consider this frame.
- If we neglect the axial deformation of the members, the load **P** will cause nodes *B* and *C* to rotate, and these nodes will be displaced horizontally by an ***equal*** amount.



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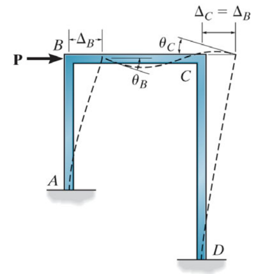
Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Consider this frame.
- The frame, therefore, has three degrees of θ_B , θ_C , and the horizontal displacement Δ_B; thus, it is kinematically ***indeterminate to the 3rd degree***.



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Displacement method of analysis: **slope-defection method**

Degrees of Freedom

- Specifying the kinematic indeterminacy or the number of unconstrained degrees of freedom for the structure is a necessary first step when applying a ***displacement method*** of analysis.
- It identifies the number of unknowns in the problem based on the assumptions made regarding the deformation behavior of the structure.
- Once these ***nodal displacements are determined***, then the deformation of the structural members will be completely specified, and the loadings within the members can be obtained.

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Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations

- The method of consistent displacements studied in Chapter 9 is called a ***force method of analysis***, because it requires writing equations that relate the unknown forces or moments in a structure.
- It's limited to structures which are ***not*** highly indeterminate.
- The work required to set up the compatibility equations, and furthermore each equation written involves ***all the unknowns.***
- It's difficult to solve the resulting set of equations unless a computer is available.

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Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations

- By comparison, the ***slope-deflection method*** is not as involved.
- It requires less work both to write the necessary equations for the solution of a problem and to solve these equations.
- Also, the ***slope-deflection method*** can be easily programmed on a computer and used to analyze a wide range of indeterminate structures.

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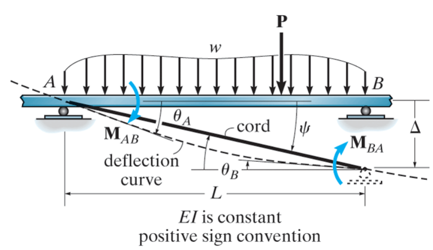
Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations – General Case

- To develop the general form of the ***slope-deflection equations***, we will consider the typical span *AB* of a continuous beam.



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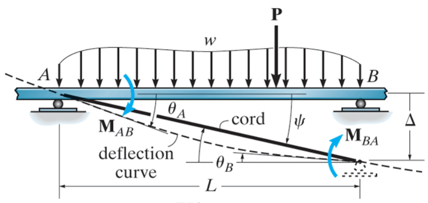
Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations – General Case

- We wish to relate the beam's internal end moments M_{AB} and M_{BA} in terms of its three degrees of freedom.



EI is constant
positive sign convention

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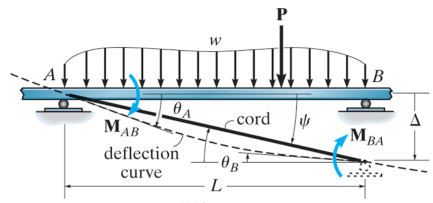
Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations – General Case

- Namely, its angular displacements θ_A and θ_B at the supports, and a linear displacement Δ, which could be caused by a relative settlement between the supports.



EI is constant
positive sign convention

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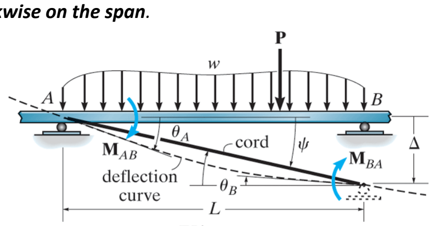
Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations – General Case

- Since we will be developing a "formula," ***moments*** and ***angular displacements*** will be considered ***positive*** when they act ***clockwise on the span***.



EI is constant
positive sign convention

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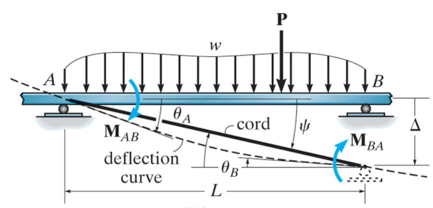
Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations – General Case

- Also, the ***linear displacement*** Δ is considered ***positive*** since this displacement causes the cord of the span and the span's cord ψ angle to rotate ***clockwise***.



EI is constant
positive sign convention

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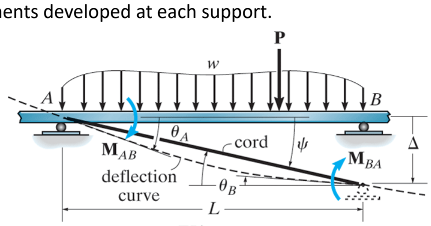
Displacement method of analysis: **slope-defection method**

Slope-Deflection Equations – General Case

- We will formulate the ***slope-deflection equations*** using the principle of superposition by considering ***separately*** the moments developed at each support.



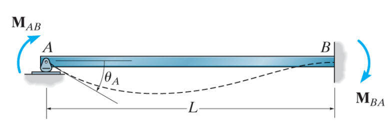
EI is constant
positive sign convention

23

Displacement method of analysis: **slope-defection method**

Angular Displacement at *A*, θ_A

- Consider node *A* of the member shown below to rotate θ_A while its far-end node *B* is *held fixed*.



- To determine the moment M_{AB} needed to cause this rotation, we will use the conjugate-beam method.

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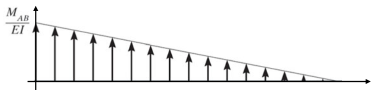
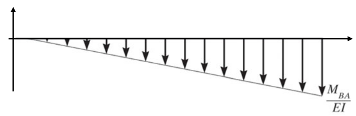
Displacement method of analysis: **slope-defection method**

Angular Displacement at *A*, θ_A

- Moment distribution due to M_{AB} is:



- Moment distribution due to M_{BA} is:



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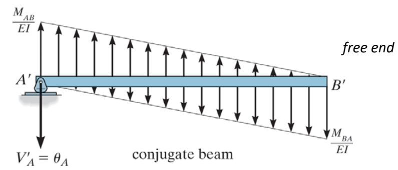
Displacement method of analysis: **slope-defection method**

Angular Displacement at *A*, θ_A

- The conjugate beam is:



- The end shear at *A′* acts downward on the beam, since θ_A is clockwise.

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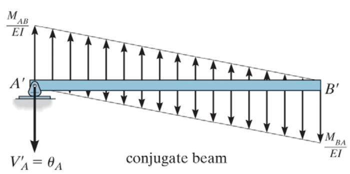
Displacement method of analysis: **slope-defection method**

Angular Displacement at *A*, θ_A

- The conjugate beam is:



- Since the deflection of the "real beam" is zero at *A* and *B*, then the corresponding sum of the ***moments*** at each end *A′* and *B′* of the conjugate beam must also be zero.

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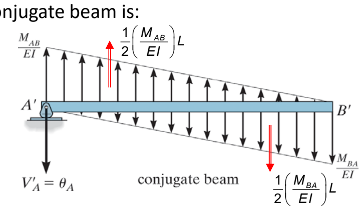
Displacement method of analysis: **slope-defection method**

Angular Displacement at *A*, θ_A

- The conjugate beam is:



$$\circlearrowleft^{+}\sum M_{A'}=0 \qquad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3}-\left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3}=0$$

$$\circlearrowleft^{+}\sum M_{B'}=0 \qquad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3}-\left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3}+\theta_A L=0$$

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Displacement method of analysis: **slope-defection method**

Angular Displacement at *A*, θ_A

- The solving the equations for M_{AB} and M_{BA} give:

$$\left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3}-\left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3}=0 \quad \Rightarrow M_{AB}=2M_{BA}$$

$$\left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3}-\left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3}+\theta_A L=0$$

$$\boxed{M_{BA}=\frac{2EI}{L}\theta_A} \qquad \boxed{M_{AB}=\frac{4EI}{L}\theta_A}$$

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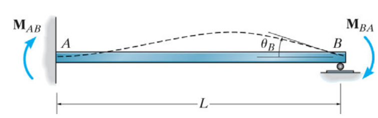
Displacement method of analysis: **slope-defection method**

Angular Displacement at *B*, θ_B

- Consider node *B* of the member shown below to rotate θ_B while its far-end node *A* is *held fixed*.



- We can relate the rotation θ_B to the applied moment M_{BA} and the reaction moment M_{AB} at the wall.

$$\boxed{M_{BA}=\frac{4EI}{L}\theta_B} \qquad \boxed{M_{AB}=\frac{2EI}{L}\theta_B}$$

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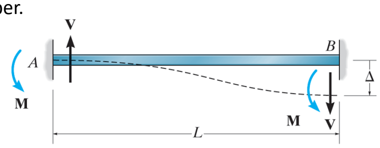
Displacement method of analysis: **slope-defection method**

Relative Linear Displacement, Δ

- If the far node *B* of the member is displaced relative to *A*, so that the cord of the member rotates ***clockwise*** (positive displacement) and both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member.



- As before, the moment **M** can be related to the displacement Δ using the conjugate-beam method.

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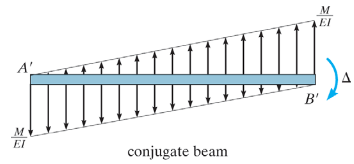
Displacement method of analysis: **slope-defection method**

Relative Linear Displacement, Δ

- Since the real beam is fixed at both ends, the conjugate beam is free at both ends. Summing moments about *B′* we have:



- However, due to the ***displacement*** of the real beam at *B*, the ***moment*** at the end *B′* of the conjugate beam must have a magnitude of Δ.

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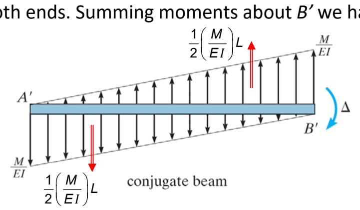
Displacement method of analysis: **slope-defection method**

Relative Linear Displacement, Δ

- Since the real beam is fixed at both ends, the conjugate beam is free at both ends. Summing moments about *B′* we have:



$$\circlearrowleft^{+} \sum M_{B'} = 0$$

$$= \left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right]\frac{2L}{3} - \left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right]\frac{L}{3} - \Delta = 0$$

$$\boxed{M_{AB} = M_{BA} = M = -\frac{6EI}{L^2}\Delta}$$

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Displacement method of analysis: **slope-defection method**

Fixed-End Moments

- In the previous cases we have considered relationships between the displacements θ_A, θ_B, and Δ the necessary moments M_{AB} and M_{BA} acting at nodes *A* and *B*.

- However, the loads acting over the span of the beam will also produce moments M_{AB} and M_{BA} and at the nodes.

- For this case ***both*** *A* and *B* are held fixed, and the moments at the supports are then referred to as ***fixed-end moments.***

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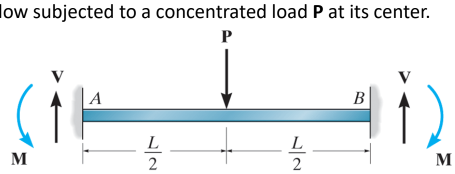
Displacement method of analysis: **slope-defection method**

Fixed-End Moments

- For example, consider the fixed-supported member shown below subjected to a concentrated load **P** at its center.



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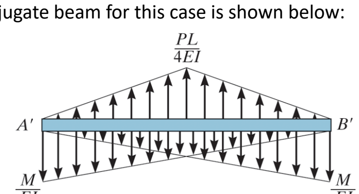
Displacement method of analysis: **slope-defection method**

Fixed-End Moments

- The conjugate beam for this case is shown below:



- Since we require the slope at each end to be zero on the real beam, then the end shears must be zero on the conjugate beam.

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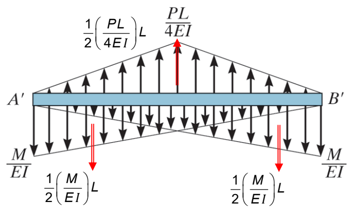
Displacement method of analysis: **slope-defection method**

Fixed-End Moments

- The conjugate beam for this case is shown below:



$$^{+}\uparrow \sum F_y = 0 = \frac{1}{2}\left(\frac{PL}{4EI}\right)L - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] \qquad \boxed{M = \frac{PL}{8}}$$

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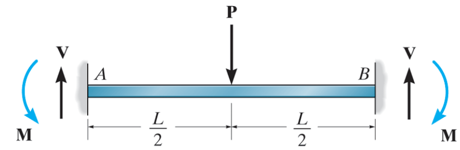
Displacement method of analysis: **slope-defection method**

Fixed-End Moments

- According to our sign convention, this moment is ***negative*** at node *A* (counterclockwise) and ***positive*** at node *B* (clockwise).



- For convenience in solving problems, fixed-end moments have been calculated for other loadings and are tabulated on the inside back cover.

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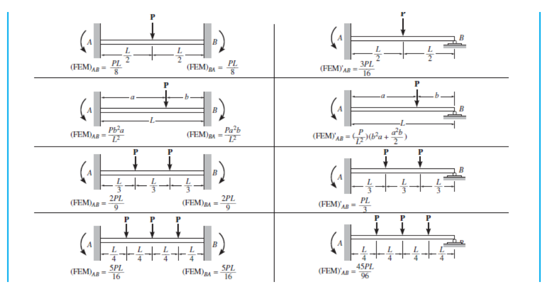
Displacement method of analysis: **slope-defection method**

Fixed-End Moments



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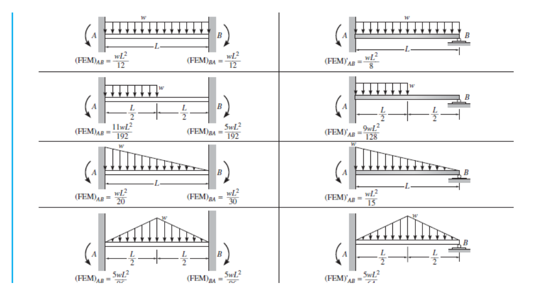
Displacement method of analysis: **slope-defection method**

Fixed-End Moments



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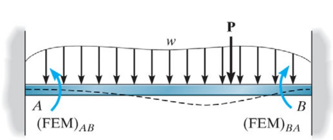
Displacement method of analysis: **slope-defection method**

Fixed-End Moments

- Assuming these FEMs have been determined for a specific problem:



$$\boxed{M_{AB} = (FEM)_{AB}} \qquad \boxed{M_{BA} = (FEM)_{BA}}$$

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Displacement method of analysis: **slope-defection method**

Slope-Deflection Equation

- Recall the end moments for the displacements θ_A, θ_B, and Δ, and those due to the loading:

$$\boxed{M_{AB} = \frac{4EI}{L}\theta_A}$$

$$\boxed{M_{AB} = \frac{2EI}{L}\theta_B}$$

$$\boxed{M_{AB} = -\frac{6EI}{L^2}\Delta}$$

$$\boxed{M_{AB} = (FEM)_{AB}}$$

$$\boxed{M_{AB} = \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (FEM)_{AB}}$$

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Displacement method of analysis: **slope-defection method**

Slope-Deflection Equation

- The resultant moments at each end of the beam can be written as:

$$M_{AB} = \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (FEM)_{AB}$$

$$M_{BA} = \frac{2EI}{L}\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (FEM)_{BA}$$

- Since these two equations are similar, the result can be expressed as a single equation.

43

Displacement method of analysis: **slope-defection method**

Slope-Deflection Equation

- Referring to one end of the span as the ***near end*** (N) and the other end as the ***far end*** (F), and letting the *member or span stiffness* be represented as $k = I/L$ and the *span's cord rotation* as $\psi = \Delta/L$ we can write:

$$M_N = 2Ek\left(2\theta_N + \theta_F - 3\psi\right) + (FEM)_N$$

- This equation considers both ***compatibility*** and ***load-displacement relationship*** due to bending and neglecting axial and shear deformations.

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Displacement method of analysis: **slope-defection method**

Slope-Deflection Equation

- Referring to one end of the span as the ***near end*** (N) and the other end as the ***far end*** (F), and letting the *member or span stiffness* be represented as $k = I/L$ and the *span's cord rotation* as $\psi = \Delta/L$ we can write:

$$M_N = 2Ek\left(2\theta_N + \theta_F - 3\psi\right) + (FEM)_N$$

- It is referred to as the general ***slope-deflection equation***.

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Displacement method of analysis: **slope-defection method**

Slope-Deflection Equation

- Referring to one end of the span as the ***near end*** (N) and the other end as the ***far end*** (F), and letting the *member or span stiffness* be represented as $k = I/L$ and the *span's cord rotation* as $\psi = \Delta/L$ we can write:

$$M_N = 2Ek\left(2\theta_N + \theta_F - 3\psi\right) + (FEM)_N$$

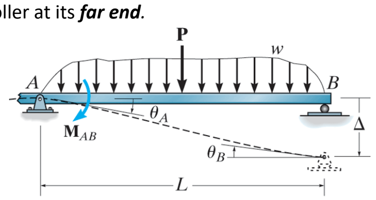
- When used for the solution of problems, this equation is applied ***twice*** for each member span (AB); that is, application is from A to B and from B to A for span AB.

46

Displacement method of analysis: **slope-defection method**

Pin-Supported End Span

- Occasionally an end span of a beam or frame is supported by a pin or roller at its ***far end***.



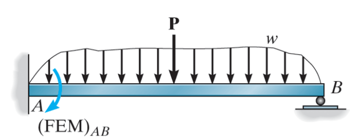
- The moment at the roller or pin must be zero; and if the angular displacement θ_F ($=\theta_B$) at this support does not have to be determined, the general slope-deflection equation is applied ***only once*** to the span rather than twice.

47

Displacement method of analysis: **slope-defection method**

Pin-Supported End Span

- Occasionally an end span of a beam or frame is supported by a pin or roller at its ***far end***.



$$M_N = 2Ek\left(2\theta_N + \theta_F - 3\psi\right) + (FEM)_N$$

$$M_F = 2Ek\left(2\theta_F + \theta_N - 3\psi\right) + 0 = 0$$

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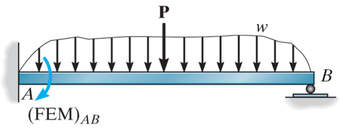
Displacement method of analysis: **slope-defection method**

Pin-Supported End Span

- Multiplying the first equation by 2 and subtracting the second equation from it ***eliminates*** the unknown θ_F and yields:



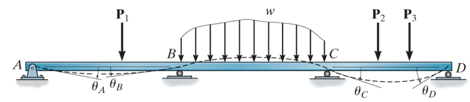
$$\boxed{M_N = 3Ek\left(\theta_N - \psi\right) + \left(FEM\right)_N}$$

49

Displacement method of analysis: **slope-defection method**

Pin-Supported End Span

- To summarize application of these two ***slope-deflection equations***, consider the continuous beam which has four degrees of freedom.



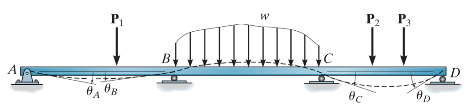
- Here the fixed-end equation can be applied ***twice*** to each of the three spans, i.e., from *A* to *B*, *B* to *A*, *B* to *C*, *C* to *B*, *C* to *D*, and *D* to *C*.

50

Displacement method of analysis: **slope-defection method**

Pin-Supported End Span

- These equations would involve the ***four unknown*** rotations, θ_A θ_B , θ_C , and θ_D .



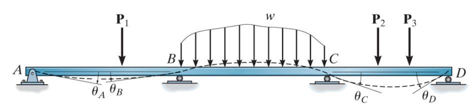
- However, since the end moments at *A* and *D* are zero, it is not necessary to determine θ_A and θ_D.

51

Displacement method of analysis: **slope-defection method**

Pin-Supported End Span

- A shorter solution occurs if we apply pin-end equation from *B* to *A* and *C* to *D* and then fixed-end from *B* to *C* and *C* to *B*.



- These ***four equations*** will involve only the unknown rotations θ_B and θ_C .

- Once the rotations are obtained, the internal moments at *B* and *C* can be found from the equilibrium equations applied at these supports.

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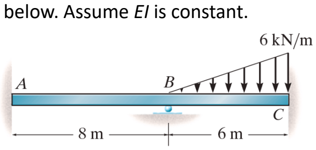
Displacement method of analysis: **slope-defection method**

- **Example 10-1**: Draw the shear and moment diagrams for the beam shown below. Assume *EI* is constant.



- Span *AB* and *BC* must be considered in this problem.

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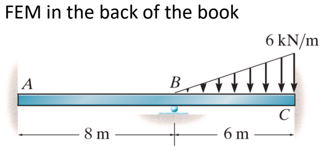
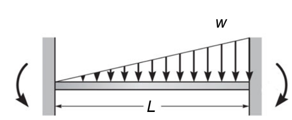
Displacement method of analysis: **slope-defection method**

- **Example 10-1**: Since the far ends are fixed, we can look up the formulas for FEM in the back of the book



$$\left(FEM\right) = -\frac{wL^2}{30}$$



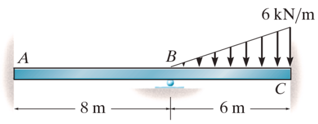
$$\left(FEM\right) = \frac{wL^2}{20}$$

- Here $(FEM)_{BC}$ is ***negative*** since it acts counterclockwise on the beam at B.

54

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: Since the far ends are fixed, we can look up the formulas for FEM in the back of the book



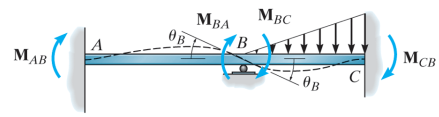
$$(FEM)_{BC} = -\frac{wL^2}{30} = -\frac{(6\,kN/m)(6\,m)^2}{30} = -7.2\,kNm$$

$$(FEM)_{CB} = \frac{wL^2}{20} = \frac{(6\,kN/m)(6\,m)^2}{20} = 10.8\,kNm$$

55

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: The elastic curve for the beam is:



- There are ***four unknown*** moments and an unknown slope at *B*.
- Since the supports do not settle, $\psi_{AB} = \psi_{BC} = 0$.

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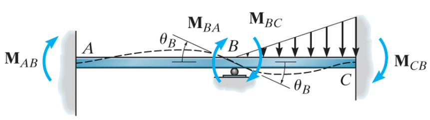
Displacement method of analysis: **slope-defection method**

- **Example 10-1**: For span *AB*, consider *A* to be near and *B* to be far



$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$

$$M_{AB} = 2\frac{EI}{L}(2\cancelto{0}{\theta_A} + \theta_B - 3\cancelto{0}{\psi}) + \cancelto{0}{(FEM)_{AB}}$$

$$M_{AB} = 2\frac{EI}{8}[2(0) + \theta_B - 3(0)] + 0 \boxed{= \frac{EI}{4}\theta_B}$$ 1

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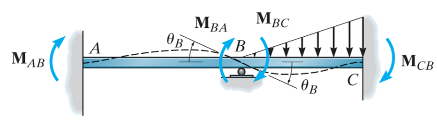
Displacement method of analysis: **slope-defection method**

- **Example 10-1**: For span *AB*, consider *B* to be near and *A* to be far



$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$

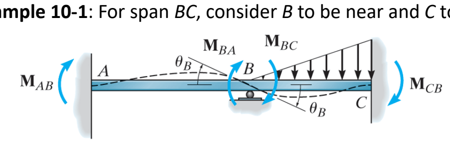
$$M_{BA} = 2\frac{EI}{L}(2\theta_B + \cancelto{0}{\theta_A} - 3\cancelto{0}{\psi}) + \cancelto{0}{(FEM)_{BA}}$$

$$M_{BA} = 2\frac{EI}{8}[2\theta_B + (0) - 3(0)] + 0 \boxed{= \frac{EI}{2}\theta_B}$$ 2

58

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: For span *BC*, consider *B* to be near and *C* to be far



$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$

$$M_{BC} = 2\frac{EI}{L}(2\theta_B + \cancelto{0}{\theta_C} - 3\cancelto{0}{\psi}) + (FEM)_{BC}$$

$$M_{BC} = 2\frac{EI}{6}[2\theta_B + (0) - 3(0)] - 7.2\,kNm \boxed{= \frac{2EI}{3}\theta_B - 7.2\,kNm}$$ 3

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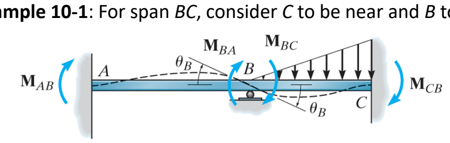
Displacement method of analysis: **slope-defection method**

- **Example 10-1**: For span *BC*, consider *C* to be near and *B* to be far



$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$

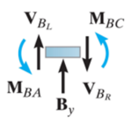
$$M_{CB} = 2\frac{EI}{L}(2\cancelto{0}{\theta_C} + \theta_B - 3\cancelto{0}{\psi}) + (FEM)_{CB}$$

$$M_{CB} = 2\frac{EI}{6}[2(0) + \theta_B - 3(0)] + 10.8\,kNm \boxed{= \frac{EI}{3}\theta_B + 10.8\,kNm}$$ 4

60

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: These four equations contain five unknowns: θ_B, M_{AB}, M_{BA}, M_{BC}, and M_{CB}.
- The necessary fifth equation comes from the condition of moment equilibrium at support *B*.
- The free-body diagram of a segment of the beam at *B* is:



$$\circlearrowleft^{+} \sum M_B = 0 = M_{BC} + M_{BA}$$

$$\boxed{M_{BC} + M_{BA} = 0} \quad 5$$

- Here and the moments act in the ***positive*** direction to be consistent with the slope-deflection equations.

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Displacement method of analysis: **slope-defection method**

- **Example 10-1**: To solve, substitute Equations (2) and (3), into Equation (5):

$$M_{BC} + M_{BA} = 0 \Rightarrow \underbrace{\left(\frac{2EI}{3}\theta_B - 7.2\,kNm\right)}_{M_{BC}} + \underbrace{\left(\frac{EI}{2}\theta_B\right)}_{M_{BA}} = 0 \qquad \boxed{\theta_B = \frac{6.17}{EI}}$$

- Substituting this result into the remaining equations gives:

$$M_{AB} = \frac{EI}{4}\theta_B = \boxed{1.543\,kNm} \qquad M_{BA} = \frac{EI}{2}\theta_B = \boxed{3.086\,kNm}$$

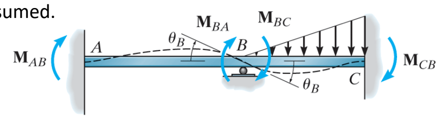
$$M_{BC} = \frac{2EI}{3}\theta_B - 7.2\,kNm = \boxed{-3.086\,kNm}$$

$$M_{CB} = \frac{EI}{3}\theta_B + 10.8\,kNm = \boxed{12.857\,kNm}$$

62

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: The negative value for M_{BC} indicates that this moment acts ***counterclockwise*** on the beam, not clockwise as assumed.



$$M_{AB} = \frac{EI}{4}\theta_B = \boxed{1.543\,kNm} \qquad M_{BA} = \frac{EI}{2}\theta_B = \boxed{3.086\,kNm}$$

$$M_{BC} = \frac{2EI}{3}\theta_B - 7.2\,kNm = \boxed{-3.086\,kNm}$$

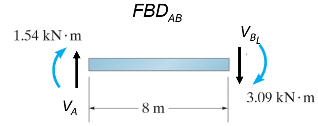
$$M_{CB} = \frac{EI}{3}\theta_B + 10.8\,kNm = \boxed{12.857\,kNm}$$

63

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: Using these results, the shears at the end spans are determined from the equilibrium equations.

FBD_{AB}



$$\circlearrowleft^{+} \sum M_B = 0 \quad = -3.086\,kNm - 1.543\,kNm - A_y(8\,m) \qquad \boxed{V_A = -0.579\,kN}$$

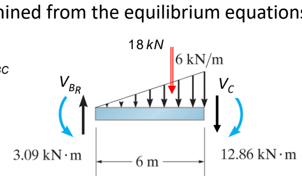
$$^{+}\uparrow \sum F_y = 0 = V_A - V_{B_L} \qquad \boxed{V_{B_L} = -0.579\,kN}$$

64

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: Using these results, the shears at the end spans are determined from the equilibrium equations.

FBD_{BC}



$$\circlearrowleft^{+} \sum M_C = 0 \quad = 3.086\,kNm - 12.857\,kNm - V_{B_R}(6\,m) + 18\,kN(2\,m)$$

$$\boxed{V_{B_R} = 4.372\,kN}$$

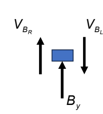
$$^{+}\uparrow \sum F_y = 0 = -V_C + V_{B_R} - 18\,kN \qquad \boxed{V_C = -13.627\,kN}$$

65

Displacement method of analysis: **slope-defection method**

- **Example 10-1**: Using these results, the shears at the end spans are determined from the equilibrium equations.

FBD_B



$$^{+}\uparrow \sum F_y = 0 = V_{B_R} - V_{B_L} + B_y$$

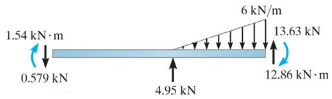
$$B_y = 4.372\,kN - (-0.579\,kN)$$

$$\boxed{B_y = 4.95\,kN}$$

66

Displacement method of analysis: **slope-defection method**

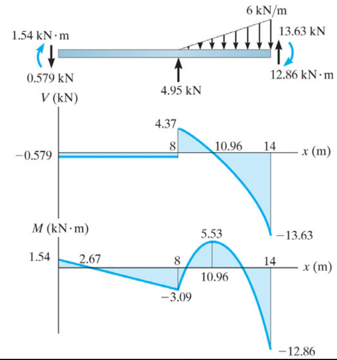
- **Example 10-1**: The shear and moment diagrams are:



67

Displacement method of analysis: **slope-defection method**

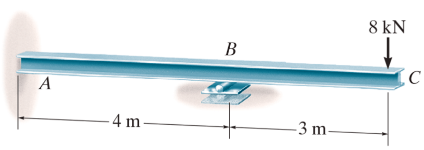
- **Example 10-1**: The shear and moment diagrams are:



68

Displacement method of analysis: **slope-defection method**

- **Example 10-2**: Determine the moments at *A* and *B* for the beam shown below. The support at *B* is displaced (settles) 80 mm.

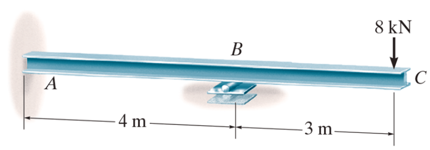


- Use these values: E = 200 GPa and $I = 5(10^{-6})$ m^4
- Only span *AB* must be considered since the M_{BC} can be computed from statics.

69

Displacement method of analysis: **slope-defection method**

- **Example 10-2**: Determine the moments at *A* and *B* for the beam shown below. The support at *B* is displaced (settles) 80 mm.

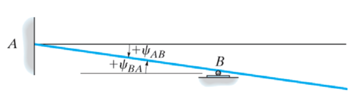


- Since there is no load on *AB,* the FEMs are zero.

70

Displacement method of analysis: **slope-defection method**

- **Example 10-2**: The downward displacement (settlement) of *B* causes the cord for span *AB* to rotate clockwise.



$$\psi_{AB} = \psi_{BA} = \frac{0.08\,m}{4\,m} = 0.02\,rad$$

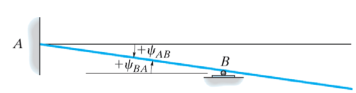
- Next, apply the slope-displacement equations to span *AB*

$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi) + (FEM)_N \qquad (FEM)_N = (FEM)_F = 0$$

71

Displacement method of analysis: **slope-defection method**

- **Example 10-2**: The downward displacement (settlement) of *B* causes the cord for span *AB* to rotate clockwise.



$$\psi_{AB} = \psi_{BA} = \frac{0.08\,m}{4\,m} = 0.02\,rad$$

- Next, apply the slope-displacement equations to span *AB*

$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi) + (FEM)_N \qquad (FEM)_N = (FEM)_F = 0$$

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Displacement method of analysis: **slope-defection method**

- **Example 10-2**: The slope-displacement equations to span *AB* are:

$$M_N = 2\frac{EI}{L}(2\theta_N + \theta_F - 3\psi)$$

$$M_{AB} = 2\frac{EI}{L}(2\cancel{\theta_A}^{\,0} + \theta_B - 3\psi) \quad = 2\frac{EI}{4}\left[2(0) + \theta_B - 3(0.02)\right] \quad (1)$$

$$M_{BA} = 2\frac{EI}{L}(2\theta_B + \cancel{\theta_A}^{\,0} - 3\psi) \quad = 2\frac{EI}{4}\left[2\theta_B + (0) - 3(0.02)\right] \quad (2)$$

$$2\frac{EI}{4} = \frac{1}{2m}\left[200(10^9)\,{}^{N}\!/_{m^2}\right]\left[5(10^{-6})m^4\right] = 500(10^3)Nm$$

- Two equations in three unknowns: θ_B , M_{AB}, and M_{BA}

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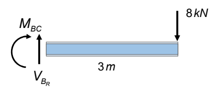
Displacement method of analysis: **slope-defection method**

- **Example 10-2**: The necessary third equation comes from the condition of moment equilibrium at support *B*.
- The free-body diagram of a segment of the beam at *BC* is:



$$\circlearrowleft^{+} \sum M_B = 0 = -M_{BC} - 8\,kN(3\,m)$$

$$\boxed{M_{BC} = -24\,kNm}$$

$$^{+}\!\uparrow \sum F_y = 0 = V_{B_L} - 8\,kN$$

$$V_{B_L} = 8\,kN$$

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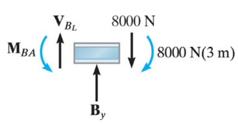
Displacement method of analysis: **slope-defection method**

- **Example 10-2**: The necessary third equation comes from the condition of moment equilibrium at support *B*.
- The free-body diagram of a segment of the beam at *B* is:



$$\circlearrowleft^{+} \sum M_B = 0 = M_{BA} - 24\,kNm \quad (3)$$

$$\boxed{M_{BA} = 24\,kNm}$$

75

Displacement method of analysis: **slope-defection method**

- **Example 10-2**: Substituting Equation (3) into Equation (2) gives:

$$M_{BA} = 2\frac{EI}{4}\left[2\theta_B + (0) - 3(0.02)\right]$$

$$24\,kNm = 500\,kNm\left[2\theta_B - 0.06\right] \qquad \boxed{\theta_B = 0.054\,rad}$$

- From Equation (1), the final moment is:

$$M_{AB} = 2\frac{EI}{4}\left[\theta_B - 3(0.02)\right] = 500\,kNm\left[0.054 - 0.06\right]$$

$$\boxed{M_{AB} = -3\,kNm}$$

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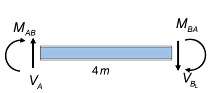
Displacement method of analysis: **slope-defection method**

- **Example 10-2**: With the moments, the reactions at A and B can be found.



$$\circlearrowleft^{+} \sum M_B = 0 = -M_{AB} - M_{BA} - V_A(4\,m)$$

$$= -(-3kNm) - 24kNm - V_A(4\,m)$$

$$\boxed{V_A = -5.25\,kN}$$

$$^{+}\!\uparrow \sum F_y = 0 = V_A - V_{B_L}$$

$$\boxed{V_{B_L} = -5.25\,kN}$$

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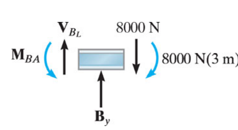
Displacement method of analysis: **slope-defection method**

- **Example 10-2**: With the moments, the reactions at A and B can be found.
- The free-body diagram of a segment of the beam at *B* is:



$$^{+}\!\uparrow \sum F_y = 0 = B_y + V_{B_L} - 8\,kN$$

$$= B_y - 5.25kN - 8\,kN$$

$$\boxed{B_y = 13.25\,kN}$$

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Displacement method of analysis: **slope-defection method**

Let’s work some problems

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Displacement method of analysis: **slope-defection method**

Any questions?

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