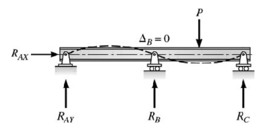
Chapter 9

Analysis of statically indeterminate **beams** by the force method

1

Analysis of statically indeterminate structures by the **force method**

- Selection of the ***redundant*** restraint

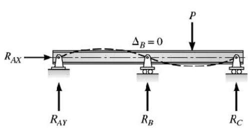


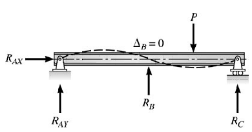
- The reaction at *B* is essential for the stability of the structure, it is termed ***redundant***

2

Analysis of statically indeterminate structures by the **force method**

- Selection of the ***redundant*** restraint



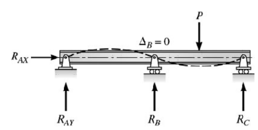


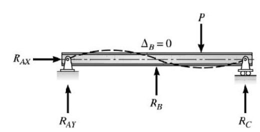
- To satisfy the original restraint at point *B*, we determine the ***deflection at B as a function of the redundant force R_B*** and set that equation equal to zero.

3

Analysis of statically indeterminate structures by the **force method**

- Selection of the ***redundant*** restraint





- The additional ***compatibility*** equation combined with the equations of ***equilibrium*** will account for all the indeterminate reactions.

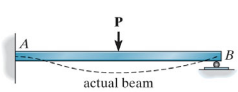
$\overset{+}{\rightarrow}\sum F_x = 0$ (1) $\quad \circlearrowleft^{+}\sum M_A = 0$ (3) $\quad \Delta_B = 0$ (4)

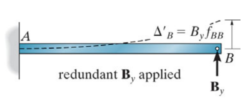
$+\uparrow\sum F_y = 0$ (2)

4

Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam



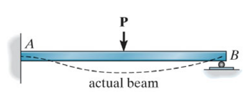


- One choice for the redundant in this problem is the reaction at point *B*.
- We know that the deflection at *B* is zero.

5

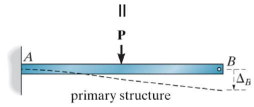
Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam



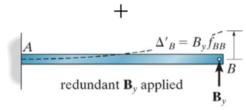
$$\Delta_B = 0$$

=



We know how to compute the deflection of at *B* due to **P**

+

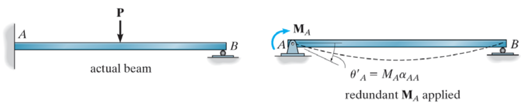


$$\Delta_B + \Delta'_B = 0$$

6

Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam

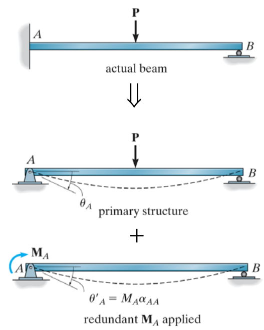


- In this case, the redundant is the slope at *A*.
- We know that the slope at *A* is zero.

7

Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam



$$\theta_A = 0$$

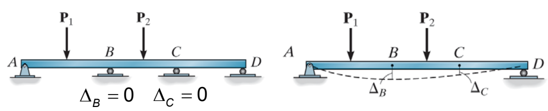
We know how to compute the slope of at *A* due to **P**

$$\theta_A + \theta'_A = 0$$

8

Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam



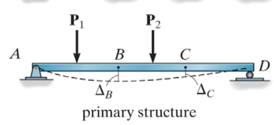
- In this case, the redundant is the reaction at *B* and *C*.
- We know that the displacement at *B* and *C* are zero.

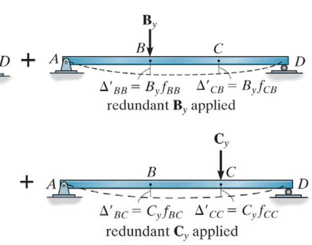
9

Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam

$$\Delta_B = 0 \qquad \Delta_C = 0$$





$$\Delta_B + \Delta'_{BB} + \Delta'_{BC} = 0$$

$$\Delta_C + \Delta'_{CB} + \Delta'_{CC} = 0$$

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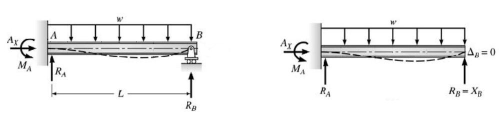
Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam

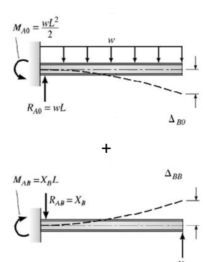


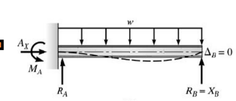
- One choice for the redundant in this problem is the reaction at point *B*.
- We know that the deflection at *B* is zero

11

Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam





$$\Delta_B = 0$$

$$\Delta_{BO} + \Delta_{BB} = 0$$

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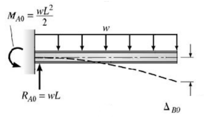
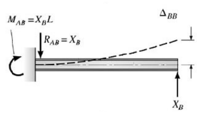
Analysis of statically indeterminate structures by the **force method**

- Consider the following indeterminate beam





- The deflections at *B* can be evaluations with virtual work
- Another method would be to used tables for deflections of structures and the method of superposition.

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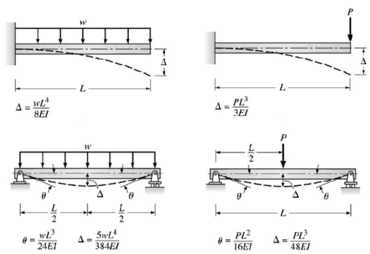
Analysis of statically indeterminate structures by the **force method**

- A few tabulated values for deflections are given below:



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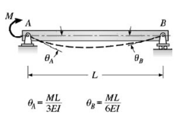
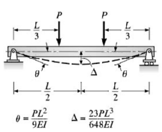
Analysis of statically indeterminate structures by the **force method**

- A few tabulated values for deflections are given below:





- As a sign convention, we will assume that displacements are positive (+) when they are in the positive y-direction.
- However, you are free to choose the direction of the redundant.

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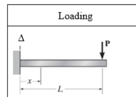
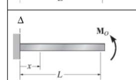
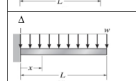
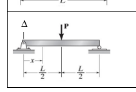
Analysis of statically indeterminate structures by the **force method**

- A few tabulated values for deflections are given below:

Loading	$\Delta^+ \uparrow$	θ^+ ↻	Equation
	$\Delta_{max} = -\frac{PL^3}{3EI}$ at $x = L$	$\theta_{max} = -\frac{PL^2}{2EI}$ at $x = L$	$\Delta = \frac{P}{6EI}(x^3 - 3Lx^2)$
	$\Delta_{max} = \frac{M_o L^2}{2EI}$ at $x = L$	$\theta_{max} = \frac{M_o L}{EI}$ at $x = L$	$\Delta = \frac{M_o x^2}{2EI}$
	$\Delta_{max} = -\frac{wL^4}{8EI}$ at $x = L$	$\theta_{max} = -\frac{wL^3}{6EI}$ at $x = L$	$\Delta = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
	$\Delta_{max} = -\frac{PL^3}{48EI}$ at $x = L/2$	$\theta_{max} = \pm\frac{PL^2}{16EI}$ at $x = 0$ or $x = L$	$\Delta = \frac{P}{48EI}(4x^3 - 3L^2x)$ $0 \le x \le L/2$

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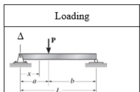
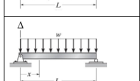
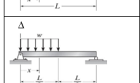
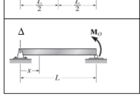
Analysis of statically indeterminate structures by the **force method**

- A few tabulated values for deflections are given below:

Loading	$\Delta^+ \uparrow$	θ^+ ↻	Equation
		$\theta_L = -\frac{Pab(L+b)}{6LEI}$ $\theta_R = \frac{Pab(L+a)}{6LEI}$	$\Delta = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\Delta_{max} = -\frac{5wL^4}{384EI}$ at $x = L/2$	$\theta_{max} = \pm\frac{wL^3}{24EI}$	$\Delta = -\frac{wx}{24EI}(x^3 - 2Lx^2 + L^3)$
		$\theta_L = -\frac{3wL^3}{128EI}$ $\theta_R = \frac{7wL^3}{384EI}$	$\Delta = -\frac{wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $\Delta = -\frac{wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x \le L$
	$\Delta_{max} = -\frac{M_o L^2}{9\sqrt{3}EI}$	$\theta_L = -\frac{M_o L}{6EI}$ $\theta_R = \frac{M_o L}{3EI}$	$\Delta = -\frac{M_o x}{6LEI}(L^2 - x^2)$

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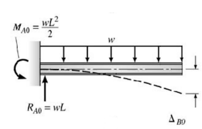
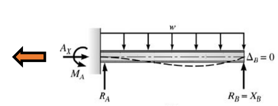
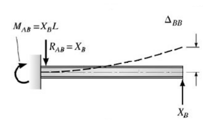
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.1 - Draw the shear and moment diagrams for the beam. *EI* is constant.





\+



$$\Delta_{BO} + \Delta_{BB} = 0$$

$$-\frac{wL^4}{8EI} + \frac{X_B L^3}{3EI} = 0$$

$$X_B = \frac{3wL}{8}$$

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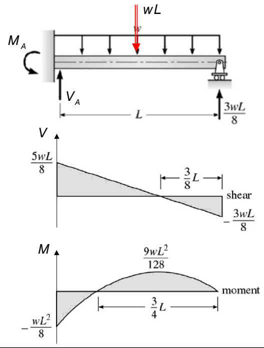
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.1 - Draw the shear and moment diagrams for the beam. *EI* is constant.



$$+\uparrow \sum F_y = 0 = V_A - wL + \frac{3wL}{8}$$

$$V_A = \frac{5wL}{8}$$

$$\circlearrowleft^+ \sum M_A = 0 = M_A - wL\left(\frac{L}{2}\right) + \frac{3wL}{8}(L)$$

$$M_A = -\frac{wL^2}{8}$$

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Analysis of statically indeterminate structures by the **force method**

Let's work some problems

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Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- When Maxwell developed the force method of analysis, he also published a theorem that relates the flexibility coefficients of any two points on an elastic structure—be it a truss, a beam, or a frame.
- This theorem is referred to as the ***theorem of reciprocal displacements*** and may be stated as follows:

The displacement of point B on a structure due to a unit load acting at point A is equal to the displacement of point A when the unit load is acting at point B.

$$f_{AB} = f_{BA}$$

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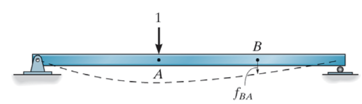
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- Proof of this theorem is easily demonstrated using the principle of virtual work.
- For example, consider the following beam



- When a ***real*** unit load acts at *A*, assume that the internal moments in the beam are represented by m_A.

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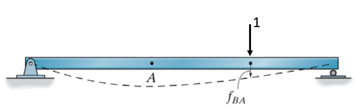
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- To determine the flexibility coefficient at *B*, a ***virtual*** unit load is placed at *B*, and the internal moments m_B are calculated.



$$f_{BA} = \int \frac{m_B m_A}{EI} dx$$

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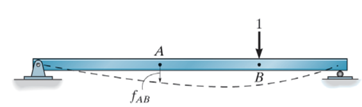
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- Likewise, if the flexibility coefficient f_{AB} is to be determined when a ***real*** unit load acts at B, then m_B represents the internal moments in the beam due to a real unit load



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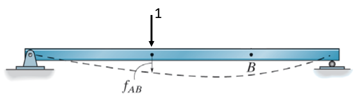
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- To determine the flexibility coefficient at *A*, a ***virtual*** unit load is placed at *A*, and the internal moments m_A are calculated.



$$f_{AB} = \int \frac{m_A m_B}{EI} dx \qquad f_{AB} = f_{BA}$$

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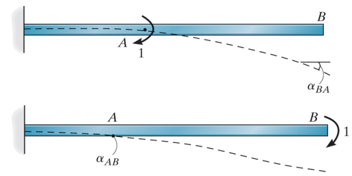
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- The theorem also applies to reciprocal rotations and may be stated as follows:

The rotation at point B on a structure, due to a unit moment acting at point A, is equal to the rotation at point A when the unit moment is acting at point B.



$$\alpha_{AB} = \alpha_{BA}$$

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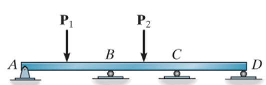
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- Because of this theorem, some work can be saved when applying the force method to problems that are statically indeterminate to the second degree or higher.



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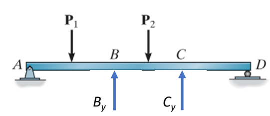
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

- Because of this theorem, some work can be saved when applying the force method to problems that are statically indeterminate to the second degree or higher.



$$\Delta_B = 0$$

$$\Delta_C = 0$$

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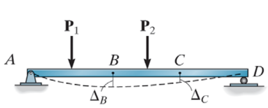
Analysis of statically indeterminate structures by the **force method**

Maxwell's Theorem of Reciprocal Displacements

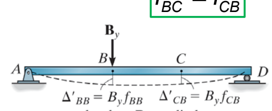
- Because of this theorem, some work can be saved when applying the force method to problems that are statically indeterminate to the second degree or higher.

$$f_{BC} = f_{CB}$$



primary structure

+

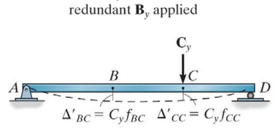


redundant B_y applied

$$\Delta_B + B_y f_{BB} + C_y f_{BC} = 0$$

$$\Delta_C B +_y f_{CB} + C_y f_{CC} = 0$$

+



redundant C_y applied

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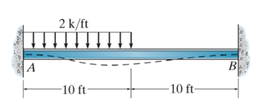
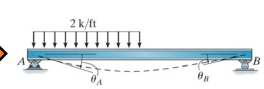
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



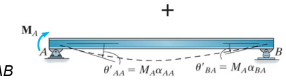


primary structure

$$\theta_A = 0$$

$$= \theta_A + M_A \alpha_{AA} + M_B \alpha_{AB}$$

+

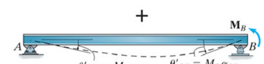


redundant moment M_A applied

$$\theta_B = 0$$

$$= \theta_B + M_A \alpha_{BA} + M_B \alpha_{BB}$$

+



redundant moment M_B applied

$$\alpha_{AB} = \alpha_{BA}$$

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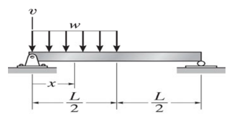
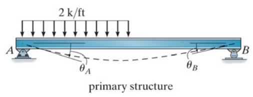
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.





$$\theta_L = \frac{3wL^3}{128EI} \qquad \theta_A = \frac{3(2\,k/ft)(20\,ft)^3}{128EI} = \frac{375\,kft^2}{EI}$$

$$\theta_R = \frac{7wL^3}{384EI} \qquad \theta_B = \frac{7(2\,k/ft)(20\,ft)^3}{384EI} = \frac{291.67\,kft^2}{EI}$$

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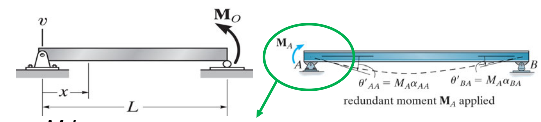
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



$$\theta_L = \frac{M_0 L}{6EI} \qquad \theta_R = \frac{M_0 L}{3EI}$$

$$\theta'_{AA} = M_A\alpha_{AA} = M_A\left(\frac{L}{3EI}\right) = M_A\frac{20\,ft}{3EI} = M_A\frac{6.67\,ft}{EI}$$

$$\theta'_{BA} = M_A\alpha_{BA} = M_A\left(\frac{L}{6EI}\right) = M_A\frac{20\,ft}{6EI} = M_A\frac{3.33\,ft}{EI}$$

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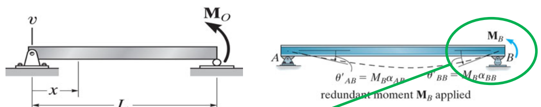
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



$$\theta_L = \frac{M_0 L}{6EI} \qquad \theta_R = \frac{M_0 L}{3EI}$$

$$\theta'_{AB} = M_B\alpha_{AB} = M_B\left(\frac{L}{6EI}\right) = M_B\frac{20\,ft}{6EI} = M_B\frac{3.33\,ft}{EI}$$

$$\theta'_{BB} = M_B\alpha_{BB} = M_B\left(\frac{L}{3EI}\right) = M_B\frac{20\,ft}{3EI} = M_B\frac{6.67\,ft}{EI}$$

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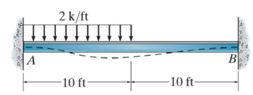
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



$$\theta_A + M_A\alpha_{AA} + M_B\alpha_{AB} = 0$$

$$\frac{375\,kft^2}{EI} + M_A\frac{6.67\,ft}{EI} + M_B\frac{3.33\,ft}{EI} = 0 \qquad (1)$$

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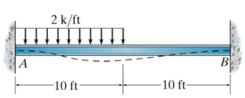
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



$$\theta_B + M_A\alpha_{BA} + M_B\alpha_{BB} = 0$$

$$\frac{291.67\,kft^2}{EI} + M_A\frac{3.33\,ft}{EI} + M_B\frac{6.67\,ft}{EI} = 0 \qquad (2)$$

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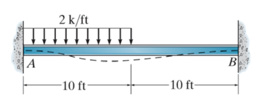
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



Solve these equations simultaneously.

$$\frac{375\,kft^2}{EI} + M_A\frac{6.67\,ft}{EI} + M_B\frac{3.33\,ft}{EI} = 0 \qquad (1)$$

$$\frac{291.67\,kft^2}{EI} + M_A\frac{3.33\,ft}{EI} + M_B\frac{6.67\,ft}{EI} = 0 \qquad (2)$$

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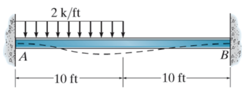
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



Solve these equations simultaneously.

$$\frac{375\,kft^2}{EI} + M_A\frac{6.67\,ft}{EI} + M_B\frac{3.33\,ft}{EI} = 0 \quad (1)$$

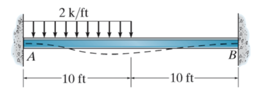
$$M_A = -M_B 0.5 - 56.25\,kft \quad (1)$$

$$\frac{291.67\,kft^2}{EI} + M_A\frac{3.33\,ft}{EI} + M_B\frac{6.67\,ft}{EI} = 0 \quad (2)$$

37

Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



Solve these equations simultaneously.

$$\frac{291.67\,kft^2}{EI} + \left(-M_B 0.5 - 56.25\,kft\right)\frac{3.33\,ft}{EI} + M_B\frac{6.67\,ft}{EI} = 0 \quad (2)$$

$$\frac{291.67\,kft^2}{EI} + M_A\frac{3.33\,ft}{EI} + M_B\frac{6.67\,ft}{EI} = 0 \quad (2)$$

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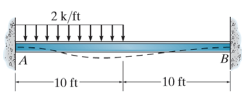
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



Solve these equations simultaneously.

$$\frac{291.67\,kft^2}{EI} + \left(-M_B 0.5 - 56.25\,kft\right)\frac{3.33\,ft}{EI} + M_B\frac{6.67\,ft}{EI} = 0 \quad (2)$$

Solving for M_B $\quad M_B = -20.82\,kft$

$$M_A = -M_B 0.5 - 56.25\,kft \quad (1) \qquad M_A = -45.84\,kft$$

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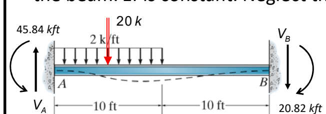
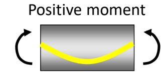
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



Positive moment



$$\circlearrowleft^{+}\sum M_B = 0 = -20.82\,kft + 45.84\,kft + 20k(15ft) - V_A(20ft) \qquad V_A = 16.25\,k$$

$$^{+}\uparrow \sum F_y = 0 = V_A - V_B - 20\,k \qquad V_B = -3.75\,k$$

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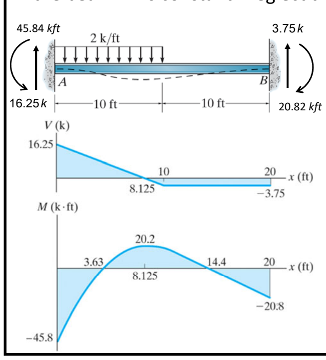
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.2 - Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



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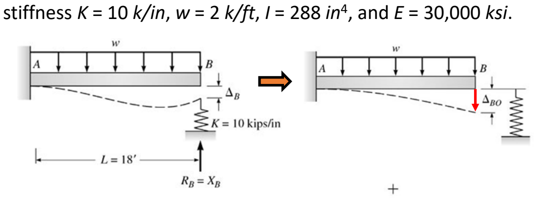
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.3 – Determine the deflection of point *B*. The stiffness *K* = 10 *k/in*, *w* = 2 *k/ft*, *I* = 288 *in*^{4}, and *E* = 30,000 *ksi*.



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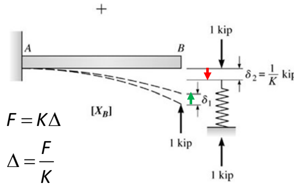

$$\Delta_{B,beam} = \Delta_{B,spring}$$

$$\delta_1 = \frac{X_B}{K} \qquad F = K\Delta$$

$$\delta_2 = -\frac{X_B}{K} \qquad \Delta = \frac{F}{K}$$

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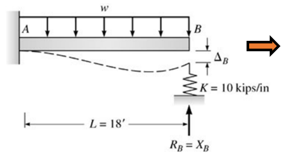
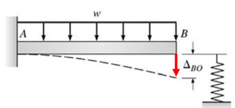
Analysis of statically indeterminate structures by the **force method**

Example Problem 9.3 – Determine the deflection of point *B*. The stiffness *K* = 10 *k/in*, *w* = 2 *k/ft*, *I* = 288 *in*⁴, and *E* = 30,000 *ksi*.





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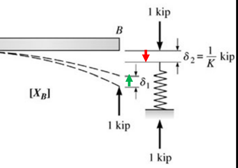

$$\Delta_{B,beam} = \Delta_{B,spring}$$

$$\Delta_{B,beam} = -\Delta_{B0} + \delta_1 X_B = -\delta_2 X_B$$

$$-\Delta_{B0} + \delta_1 X_B = -\frac{X_B}{K}$$

$$F = K\Delta$$

$$\Delta = \frac{F}{K}$$

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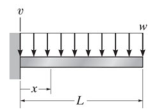
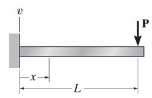
Analysis of statically indeterminate structures by the **force method**

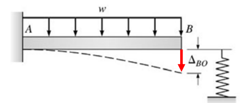
Example Problem 9.3 – Determine the deflection of point *B*. The stiffness *K* = 10 *k/in*, *w* = 2 *k/ft*, *I* = 288 *in*⁴, and *E* = 30,000 *ksi*.



$$\Delta = -\frac{wL^4}{8EI}$$



$$\Delta = -\frac{PL^3}{3EI}$$



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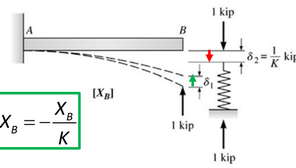

$$-\Delta_{B0} + \delta_1 X_B = -\frac{X_B}{K} \Rightarrow \frac{wL^4}{8EI} + \frac{L^3}{3EI} X_B = -\frac{X_B}{K}$$

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Analysis of statically indeterminate structures by the **force method**

Example Problem 9.3 – Determine the deflection of point *B*. The stiffness *K* = 10 *k/in*, *w* = 2 *k/ft*, *I* = 288 *in*⁴, and *E* = 30,000 *ksi*.

$$\frac{wL^4}{8EI} + \frac{L^3}{3EI} X_B = -\frac{X_B}{K}$$

$$\frac{\left(2\,{}^{k}\!/_{ft}\right)\left(18\,ft\right)^4 \left(1{,}728\,{}^{in^3}\!/_{ft^3}\right)}{8\left(30{,}000\,{}^{k}\!/_{in^2}\right)\left(288\,in^4\right)} + \frac{\left(18\,ft\right)^3 \left(1{,}728\,{}^{in^3}\!/_{ft^3}\right)}{3\left(30{,}000\,{}^{k}\!/_{in^2}\right)\left(288\,in^4\right)} X_B = -\frac{X_B}{\left(10\,{}^{k}\!/_{in}\right)}$$

$$X_B = 10.74\,k$$

With X_B, we can compute the reactions at the fixed end

$$\Delta_{B,spring} = \frac{X_B}{K} = \frac{10.74\,k}{10\,{}^{k}\!/_{in}} = 1.074\,in$$

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Analysis of statically indeterminate structures by the **force method**

Let's work some problems

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Analysis of statically indeterminate structures by the **force method**

Any questions?

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