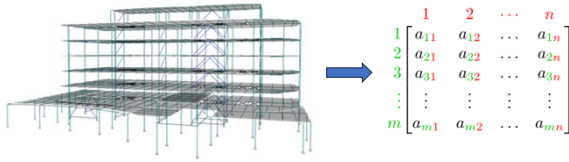


Appendix A: Matrix Algebra

Matrix Algebra for Structural Analysis



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Appendix A: Matrix Algebra

Basic Definitions and Types of Matrices

- With the accessibility of computers, the use of **matrix algebra** for the analysis of structures has become widespread.
- **Matrix algebra** provides an appropriate tool for this analysis.
- It is relatively easy to formulate the solution in a concise form and then perform the actual manipulations using a computer.
- For this reason, the structural engineer must be somewhat familiar with the fundamental operations of this branch of mathematics.

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Appendix A: Matrix Algebra

What is a matrix

- A **matrix** is a rectangular arrangement of numbers having ***m*** rows and ***n*** columns.
- The numbers, which are called **elements**, are assembled within brackets.
- For example, the ***A*** matrix is written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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Appendix A: Matrix Algebra

What is a matrix

- Such a matrix is said to have an order ***m* x *n*** (***m*** by ***n***).
- The first subscript for an element in the matrix denotes its **row position** and the second subscript is the **column position**.
- In general, ***a_{ij}*** is the element located in the *i*-th row and *j*-th column.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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Appendix A: Matrix Algebra

What is a row matrix

- If the matrix consists only of elements in a **single row**, it is called a **row matrix**.
- For example, a 1 x ***n*** **row matrix** is written as

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

- Here, only a single subscript is used to denote an element, since the row subscript is always understood to be equal to 1, that is, ***a₁*** = ***a₁₁***, ***a₂*** = ***a₁₂***, and so on.

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Appendix A: Matrix Algebra

What is a column matrix

- If the matrix consists only of elements in a **single column**, it is called a **column matrix**.
- For example, a ***m*** x 1 **column matrix** is written as

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

- Here the subscript notation symbolizes ***a₁*** = ***a₁₁***, ***a₂*** = ***a₂₁***, and so on.

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Appendix A: Matrix Algebra

What is a square matrix

- When the number of rows in a matrix equals the number of columns, the matrix is referred to as a **square matrix**.
- An $n \times n$ square matrix would be

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

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Appendix A: Matrix Algebra

What is a unit or identity matrix

- The **unit or identity matrix** is a diagonal matrix with all the diagonal elements equal to unity.
- For example,

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Appendix A: Matrix Algebra

What is a symmetric matrix

- A square matrix is **symmetric** provided $a_{ij} = a_{ji}$.
- For example,

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -1 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

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Appendix A: Matrix Operations

Equality of matrices

- Matrices **A** and **B** are said to be equal if they are of the same order and each of their corresponding elements are equal, that is, $a_{ij} = b_{ij}$.
- For example, if

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -1 & 4 \\ -2 & 4 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -1 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

- Then, $A = B$

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Appendix A: Matrix Operations

Addition and Subtraction of Matrices

- Two matrices can be **added** together or **subtracted** from one another if they are of the same order.
- The result is obtained by adding or subtracting the corresponding elements.

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 6 \\ 7 & -1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -1 & 9 \\ 9 & 8 \end{bmatrix} \quad A - B = \begin{bmatrix} 9 & -3 \\ -5 & 10 \end{bmatrix}$$

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Appendix A: Matrix Operations

Multiplication by a scalar

- When a matrix is multiplied by a **scalar**, each element of the matrix is multiplied by the scalar.
- For example, if

$$A = \begin{bmatrix} -4 & 9 \\ 3 & 12 \end{bmatrix} \quad k = -3 \quad kA = \begin{bmatrix} 12 & -27 \\ -9 & -36 \end{bmatrix}$$

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Appendix A: Matrix Operations

Matrix multiplication

- Two matrices **A** and **B** can be multiplied together only if they are **conformable**.
- This condition is satisfied if the number of **columns** in **A** equals the number of **rows** in **B**.
- For example,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

- Then **AB** can be determined since **A** has two columns and **B** has two rows.

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Appendix A: Matrix Operations

Matrix multiplication

- Two matrices **A** and **B** can be multiplied together only if they are **conformable**.
- This condition is satisfied if the number of **columns** in **A** equals the number of **rows** in **B**.
- For example,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

- Notice, however, that **BA** is not possible. Why?

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Appendix A: Matrix Operations

Matrix multiplication

- If matrix **A** having an order of **m × n** is multiplied by matrix **B** having an order of **n × q** it will yield matrix **C** having an order of **m × q**, that is,

$$AB = C$$

$$(m \times n)(n \times q) \quad (m \times q)$$

- The elements of matrix **C** are found by multiplying the elements **a_{ij}** of **A** and **b_{ij}** of **B** in the following manner:

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

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Appendix A: Matrix Operations

Matrix multiplication

- The methodology of this formula can be explained by a few simple examples. Consider

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

- By inspection, the product **AB** is possible since the matrices are **conformable**.

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Appendix A: Matrix Operations

Matrix multiplication

- The methodology of this formula can be explained by a few simple examples. Consider

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

- Multiply the elements in the first row of **A** by the corresponding elements in the column of **B** and add the results; that is,

$$c_{11} = \sum_{k=1}^3 a_{1k} b_{k1} = 3(2) + 5(-2) - 2(6) = -16$$

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Appendix A: Matrix Operations

Matrix multiplication

- The methodology of this formula can be explained by a few simple examples. Consider

$$A = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} \quad C = AB = \begin{bmatrix} -16 \\ 36 \end{bmatrix}$$

- Multiply the elements in the first row of **A** by the corresponding elements in the column of **B** and add the results; that is,

$$c_{21} = \sum_{k=1}^3 a_{2k} b_{k2} = 5(2) - 1(-2) + 4(6) = 36$$

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Appendix A: Matrix Operations

Matrix multiplication

- The following rules apply to matrix multiplication.
- In general, the product of two matrices is not commutative:

$$AB \neq BA$$
- The distributive law is valid: $A(B+C) = AB+AC$
- The associative law is valid: $A(BC) = (AB)C$

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Appendix A: Matrix Operations

Transposed matrix

- A matrix may be **transposed** by interchanging its rows and columns. For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad B^T = [b_1 \quad b_2 \quad b_3]$$

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Appendix A: Matrix Operations

Transposed matrix

- The following rules apply to transposed matrices.

$$(A+B)^T = A^T + B^T$$

$$(kA)^T = kA^T$$

$$(AB)^T = B^T A^T$$

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Appendix A: Matrix Operations

Inverse of a Matrix

- Consider the following set of three linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

$$\Rightarrow Ax = C$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{3 \times 1}$$

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Appendix A: Matrix Operations

Inverse of a Matrix

- One would think that a solution for x could be determined by dividing C by A .
- However, **division is not possible** in matrix algebra.
- Instead, one multiplies by the **inverse of the matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{3 \times 1}$$

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Appendix A: Matrix Operations

Inverse of a Matrix

- The **inverse of the matrix A** is another matrix of the same order and is symbolically written as A^{-1} .
- It has the following property, $AA^{-1} = A^{-1}A = I$

$$A^{-1}Ax = Ix = x \Rightarrow x = A^{-1}C$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{3 \times 1}$$

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Appendix A: Matrix Operations

Inverse of a Matrix

➤ The **inverse of the matrix A** is another matrix of the same order and is symbolically written as A^{-1} .

➤ It has the following property, $AA^{-1} = A^{-1}A = I$

$$A^{-1}Ax = Ix = x \quad \Rightarrow \quad x = A^{-1}C$$

➤ If A^{-1} can be obtained, a solution for x is possible.

➤ For hand calculations, A^{-1} can be developed using Cramer's rule (or any number of other methods).

➤ We will use the matrix operations in **Excel**.

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Appendix A: Matrix Operations

Any questions?



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