CHAPTER 6 – Rate of Return

Lecture 20
Rate of Return Analysis

Throughout the semester, we've been throwing around terms like ***rate of return***. Now we're going to find out what it really is (and isn't).

Let's begin with the somewhat technical definition provided by the textbook:

> *Rate of return is the rate paid on the unpaid balance of borrowed money or the rate earned on the unrecovered balance of an investment so that the final payment or receipt brings the balance to exactly zero with interest considered.*

This will be easier to understand if we break it into two parts. The first part pertains to borrowed money:

> *Rate of return is the rate paid on the unpaid balance of borrowed money … so that the final payment … brings the balance to exactly zero …*

Recall back in Chapter 2 when we looked at Mortgage Example 1:

Mortgage Example 1 (REPRISE)

Mary made a $150,000 offer on a new home. Her bank requires a 10% down-payment and will finance the rest over 30 years at a fixed rate of 0.5% per month. What are her monthly principal and interest (P&I) payments?

The initial loan ***balance*** was the $135,000 she ***borrowed*** from the bank and her ***payment*** amount was determined to be $809.39 per month.

We set up a spreadsheet to track the unpaid loan balance over time:

End of Month	Interest Owed	Loan Payment	Principal Repaid	Unpaid Balance
0				$135,000.00
1	$675.00	$809.39	$134.39	$134,865.61
2	$674.33	$809.39	$135.07	$134,730.54
⋮	⋮	⋮	⋮	⋮
359	$8.03	$809.39	$801.36	$805.37
360	$4.03	$809.39	$805.37	$0.00

Every month there's a new ***unpaid balance***. With the bank charging 0.5% per month on that ***unpaid balance***, Mary's 360 monthly payments of $809.39 bring the ***balance*** to exactly zero after the last payment. Therefore, the bank's rate of return is 0.5% per month.

In this problem, the bank's rate of return was known in advance and we had to calculate the payment amount needed to produce that rate of return. To do that, we set the present value of the loan amount equal to the present value of the 360 payments and solved for the payment amount:

$$A\ (P|A,0.5\%,360) = \$135{,}000$$

$$A = \frac{\$135{,}000}{(P\mid A, 0.5\%, 360)} = \frac{\$135{,}000}{166.7916} = \$809.39$$

So 0.5% per month is the interest rate that makes 360 monthly payments of $809.39 ***economically equivalent*** to the initial loan balance of $135,000.

That brings us to a rate-of-return definition that is a bit more user-friendly:

> *Rate of return is the interest rate that makes the cash outflows and cash inflows of a project economically equivalent to each other.*

To see how this works for an investment, let's take another look at the Pigs-R-Us example.

Pigs-R-Us Example 1 (REPRISE)

Pigs-R-Us is trying to decide whether or not to automate their meat packing process. The machine costs $200,000, has a useful life of 10 years and a salvage value of $10,000. The machine costs $9000 per year to operate and maintain but will save the company $50,000 per year in labor costs. Pigs-R-Us has asked you to evaluate the economics of this purchase. Assume their MARR is 14% per year.

We already know that the automated meat packing equipment is economically feasible because we calculated the present worth at our 14% MARR and it was greater than zero:

$$PW = -\$200{,}000 + \$41{,}000\ (P|A,14\%,10) + \$10{,}000\ (P|F,14\%,10) = \$16{,}558$$

So we know that the rate of return is <u>more</u> than 14%, but we don't know <u>how much</u> more.

To calculate the rate of return, we'll set the present value of the cash outflows equal to the present value of the cash inflows and solve for the interest rate (i*) that satisfies the equality:

$$-\$200{,}000 = \$41{,}000\ (P|A,i^*,10) + \$10{,}000\ (P|F,i^*,10)$$

Rearranging to get everything on the left-hand side of the equal sign:

$$-\$200{,}000 + \$41{,}000\ (P|A,i^*,10) + \$10{,}000\ (P|F,i^*,10) = 0$$

This is simply the project's ***present worth*** calculated at an unknown i* instead of at the MARR. This leads to another—even more user friendly—definition of rate of return:

> *Rate of return is the interest rate that makes the present worth (or annual worth or future worth) of a project exactly equal to zero.*

If you think about it, this makes perfect sense. To calculate the present worth, we set the interest rate equal to the MARR and if the present worth is exactly equal to zero, our return is exactly equal to the MARR. Well, if we set the interest rate equal to i* instead of the MARR and the present worth is equal to zero then our return is exactly i* (the rate of return).

Unfortunately, there's no way to solve the equation above directly for i*. No matter how hard we try, we can't isolate i* on the left-hand side of the equal sign. Instead, we have to use an equation solver (if we have one) or solve by ***trial-and-error***.

If we are going to solve this by trial-and-error, how do we decide where to start?

For this problem, we've already started! We know that the present worth equation evaluates to \$16,558 if the interest rate is 14% (the MARR). Since PW > 0, we know that the interest rate must be ***higher*** than the MARR, so let's try i* = 15% since that's the next table in the back of the textbook:

$$-\$200,000+\$41,000\overset{5.0188}{\left(P\mid A,15\%,10\right)}+\$10,000\overset{0.2472}{\left(P\mid F,15\%,10\right)}=\$8243$$

This cut the PW in half, but we're still too high. If a 1% change in interest rate cut the PW in half, another 1% change should get us pretty close to zero. So let's try i* = 16% next:

$$-\$200,000+\$41,000\overset{4.8332}{\left(P\mid A,16\%,10\right)}+\$10,000\overset{0.2267}{\left(P\mid F,16\%,10\right)}=\$428$$

This is awfully close to zero (relative to numbers like \$200,000) but the total is still a bit too high. The next table in the back of the book is for 18%, so let's try that next:

$$-\$200,000+\$41,000\overset{4.4941}{\left(P\mid A,18\%,10\right)}+\$10,000\overset{0.1911}{\left(P\mid F,18\%,10\right)}=-\$13,831$$

We have now bracketed the answer between 16% (which gives a PW>0) and 18% (which gives a PW<0). At this point, we can simply use linear interpolation to arrive at a reasonable ***approximation*** of the solution. If you don't remember how to do linear interpolation, here's the way I do it. Start by creating a table of i and PW:

i	PW (\$)
16%	428
i*	0
18%	−13,831

We'll start by choosing (18%,−13,831) as our ***anchor***. If the relationship between i and PW is linear (which is our assumption) then the ratio of (i*−18%) to (16%−18%) will be equal to the ratio of [0−(−13,831)] to [428−(−13,831)], so we can write

$$\frac{i^*-18\%}{16\%-18\%}=\frac{0-(-13831)}{428-(-13831)}=0.9670$$

This means that i* is 96.70% of the way from i = 18% (our anchor) down to i = 16%, just as PW = 0 is 96.70% of the way from PW = –$13,831 (our anchor) up to PW = $428.

Now we can rearrange this equation to solve for i*:

$$i^* - 18\% = (16\% - 18\%)(0.9670)$$

$$i^* = 18\% + (-2\%)(0.9670) = 16.06\%$$

We could just as easily chosen (16%,428) as our ***anchor***:

$$\frac{i^* - 16\%}{18\% - 16\%} = \frac{0 - 428}{-13831 - 428} = 0.0300$$

$$i^* - 16\% = (18\% - 16\%)(0.0300)$$

$$i^* - 16\% = (2\%)(0.0300) = 16.06\%$$

The important thing is to make sure your anchor values (whichever pair you choose) appear on the right side of every minus sign in the equation. Otherwise, you will not get the correct answer.

If we use an equation solver, we find that the answer really is 16.06%, so linear interpolation worked extremely well in this case. If you can bracket the answer with two adjacent interest rate tables in the back of the textbook, you'll usually get a good approximation.

What could we have done if we didn't know the MARR?

If we didn't know the MARR, we could just guess a reasonable value (let's say 12% per year or 1% per month) then see what happens. If the resulting PW is greater than zero, our next guess should be higher. If the resulting PW is less than zero, our next guess should be lower. It shouldn't take too many iterations to find the two interest rates that bracket the solution.

Now let's look at the second half of the rate-of-return definition, which deals with investments:

> *Rate of return is the rate … earned on the unrecovered balance of an investment so that the final … receipt brings the balance to exactly zero …*

Think of a revenue project in terms of a loan. When a company invests capital in a project it is, in effect, lending money to that project. As the project produces revenue, that revenue pays the company back with interest. The interest rate that brings the final "loan" balance to exactly zero is the company's rate of return for that project.

To illustrate this, let's make a spreadsheet for the Pigs-R-Us investment like the one we made for Mary's mortgage. At the end of each year, the "interest" owed will be 16.06% (the rate of return we solved for earlier) of the unrecovered loan balance (the amount that the project hasn't yet paid back to the company through revenue or cost savings):

End of Month	Interest Owed	Revenue Received	Balance Reduction	Unrecovered Balance
0				($200,000.00)
1	$32,113.54	$41,000.00	$8,886.46	($191,113.54)
2	$30,686.66	$41,000.00	$10,313.34	($180,800.20)
3	$29,030.67	$41,000.00	$11,969.33	($168,830.87)
⋮	⋮	⋮	⋮	⋮
8	$15,798.74	$41,000.00	$25,201.26	($73,191.78)
9	$11,752.23	$41,000.00	$29,247.77	($43,944.01)
10	$7,055.99	$51,000.00	$43,944.01	($0.00)

Sure enough, at an interest rate of 16.06% per year, the final $51,000 cash flow (which includes the $10,000 salvage value) brings the unrecovered balance to exactly zero. So at the end of 10 years, the automated meat packing equipment has paid back Pigs-R-Us for the $200,000 "loan" plus 16.06% per year interest (just as Mary paid back the bank its $135,000 loan plus 0.5% per month interest).

Let's look at another example:

Lockheed-Martin Example 1

Lockheed Martin has secured a satellite launch contract from a European communications company that will pay them €3.9M per year for the next 8 years. To land that contract, they invested €13M in a satellite tracking system. Of that amount, €8M was paid up front and the remaining €5M was paid during the first year of operation. The annual operating costs for the satellite tracking system are estimated at €0.9M per year (which includes the cost of personnel, electricity, maintenance, etc.). At the end of the contract, it is estimated that the equipment will have a salvage value of €0.5M. What is Lockheed Martin's internal rate of return (IRR) on this project?

We'll start by guessing a rate of return of 15% per year and see how close we are:

$$PW = -8^M - 5^M \overset{0.8696}{(P\,|\,F,15\%,1)} + 3.0^M \overset{4.4873}{(P\,|\,A,15\%,8)} + 0.5^M \overset{0.3269}{(P\,|\,F,15\%,8)} = 1.277^M$$

Since the PW is greater than zero, we know that Lockheed-Martin earns <u>more</u> than 15% and since $\$1.3^M$ is significantly larger than zero (relative to numbers like $\$8^M$ at Time 0), they probably earn significantly more than 15%, so let's try 20% next:

$$PW = -8^M - 5^M \overset{0.8333}{(P\,|\,F,20\%,1)} + 3.0^M \overset{3.8372}{(P\,|\,A,20\%,8)} + 0.5^M \overset{0.2326}{(P\,|\,F,20\%,8)} = -0.5386^M$$

This is now less than zero, so we've bracketed the solution. If we ***linearly*** interpolate between 15% and 20%, we might not get a very accurate answer because the underlying relationship between PW and i* is ***nonlinear***. In general, you want to bracket the solution between two adjacent tables in the textbook to minimize the error.

Since the next table back is for 18%, let's try that next:

$$PW = -8^M - 5^M \overset{0.8475}{(P\,|\,F,18\%,1)} + 3.0^M \overset{4.0776}{(P\,|\,A,18\%,8)} + 0.5^M \overset{0.2660}{(P\,|\,F,18\%,8)} = 0.1283^M$$

Now we've bracketed the answer between 18% and 20% and we can use linear interpolation to approximate the correct solution:

$$i^* = 18\% + (20\% - 18\%)\left(\frac{0 - 0.1283}{-0.5386 - 0.1283}\right) = 18.38\%$$

If we use an equation solver, we find that the real answer is 18.37%, which is very close.

Note that the ***present worth*** tells you ***how much*** wealth will be generated while the ***rate of return*** tells you ***how efficiently*** it is generated. So the two methods actually complement each other because they tell you different things about the project.