Chapter 2

Factors: How Time and Interest Affect Money

The McGraw-Hill Companies, Inc./John Flournoy

In Chapter 1 we learned the basic concepts of engineering economy and their role in decision making. The cash flow is fundamental to every economic study. Cash flows occur in many configurations and amounts—isolated single values, series that are uniform, and series that increase or decrease by constant amounts or constant percentages. This chapter develops the commonly used engineering economy factors that consider the time value of money for cash flows.

The application of factors is illustrated using their mathematical forms and a standard notation format. Spreadsheet functions are illustrated.

Objectives

Purpose: Use engineering economy factors to account for the time value of money.

1. Use the compound amount factor and present worth factor for single payments.	**F/P and P/F factors**
2. Use the uniform series factors.	**P/A, A/P, F/A and A/F factors**
3. Use the arithmetic gradient factors and the geometric gradient formula.	**Gradients**
4. Use uniform series and gradient factors when cash flows are shifted.	**Shifted cash flows**
5. Use a spreadsheet to make equivalency calculations.	**Spreadsheets**

2.1 SINGLE-PAYMENT FORMULAS (F/P AND P/F)

The most fundamental equation in engineering economy is the one that determines the amount of money F accumulated after n years (or periods) from a *single* present worth P, with interest compounded one time per year (or period). Recall that compound interest refers to interest paid on top of interest. Therefore, if an amount P is invested at time $t = 0$, the amount F_1 accumulated 1 year hence at an interest rate of i percent per year will be

$$\begin{aligned} F_1 &= P + Pi \\ &= P(1 + i) \end{aligned}$$

where the interest rate is expressed in decimal form. At the end of the second year, the amount accumulated F_2 is the amount after year 1 plus the interest from the end of year 1 to the end of year 2 on the entire F_1.

$$\begin{aligned} F_2 &= F_1 + F_1 i \\ &= P(1 + i) + P(1 + i)i \end{aligned}$$

The amount F_2 can be expressed as

$$\begin{aligned} F_2 &= P(1 + i + i + i^2) \\ &= P(1 + 2i + i^2) \\ &= P(1 + i)^2 \end{aligned}$$

Similarly, the amount of money accumulated at the end of year 3 will be

$$F_3 = P(1 + i)^3$$

By mathematical induction, the formula can be generalized for n years to

$$\mathbf{F = P(1 + i)^n} \qquad \textbf{[2.1]}$$

The term $(1 + i)^n$ is called a factor and is known as the *single-payment compound amount factor* (SPCAF), but it is usually referred to as the F/P *factor.* This is the conversion factor that yields the future amount F of an initial amount P after n years at interest rate i. The cash flow diagram is seen in Figure 2.1*a*.

Reverse the situation to determine the P value for a stated amount F. Simply solve Equation [2.1] for P.

$$\mathbf{P = F\left[\frac{1}{(1 + i)^n}\right]} \qquad \textbf{[2.2]}$$

The expression in brackets is known as the *single-payment present worth factor* (SPPWF), or the P/F *factor.* This expression determines the present worth P of a given future amount F after n years at interest rate i. The cash flow diagram is shown in Figure 2.1*b*.

Note that the two factors derived here are for *single payments;* that is, they are used to find the present or future amount when only one payment or receipt is involved.

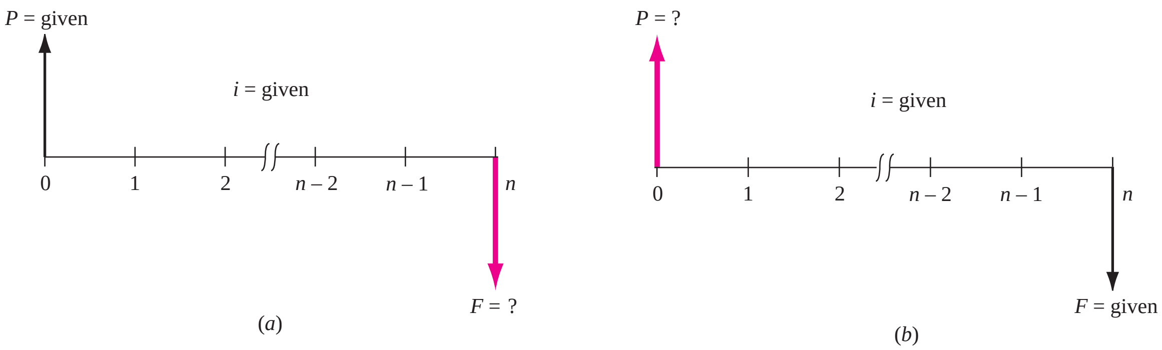


FIGURE 2.1 Cash flow diagrams for single-payment factors: (*a*) find *F* and (*b*) find *P*.

A standard notation has been adopted for all factors. It is always in the general form $(X/Y,i,n)$. The letter X represents what is sought, while the letter Y represents what is given. For example, F/P means *find F when given P.* The i is the interest rate in percent, and n represents the number of periods involved. Thus, $(F/P,6\%,20)$ represents the factor that is used to calculate the future amount F accumulated in 20 periods if the interest rate is 6% per period. The P is given. The standard notation, simpler to use than formulas and factor names, will be used hereafter. Table 2.1 summarizes the standard notation and equations for the F/P and P/F factors.

To simplify routine engineering economy calculations, tables of factor values have been prepared for a wide range of interest rates and time periods from 1 to large n values, depending on the i value. These tables are found at the end of this book following the Reference Materials.

For a given factor, interest rate, and time, the correct factor value is found at the intersection of the factor name and n. For example, the value of the factor $(P/F,5\%,10)$ is found in the P/F column of Table 10 at period 10 as 0.6139.

When it is necessary to locate a factor value for an i or n that is not in the interest tables, the desired value can be obtained in one of two ways: (1) by using the formulas derived in Sections 2.1 to 2.3 or (2) by linearly interpolating between the tabulated values.

For spreadsheet use, the F value is calculated by the FV function, and P is determined using the PV function. The formats are included in Table 2.1. Refer to Appendix A for more information on the FV and PV functions.

TABLE 2.1 ***F/P*** **and** ***P/F*** **Factors: Notation and Equation and Spreadsheet Function**

Factor					
Notation	**Name**	**Find/Given**	**Standard Notation Equation**	**Equation with Factor Formula**	**Excel Functions**
$(F/P,i,n)$	Single-payment compound amount	F/P	$F = P(F/P,i,n)$	$F = P(1+i)^n$	= FV(i%,n,,P)
$(P/F,i,n)$	Single-payment present worth	P/F	$P = F(P/F,i,n)$	$P = F[1/(1+i)^n]$	= PV(i%,n,,F)

EXAMPLE 2.1

An engineer received a bonus of $12,000 that he will invest now. He wants to calculate the equivalent value after 24 years, when he plans to use all the resulting money as the down payment on an island vacation home. Assume a rate of return of 8% per year for each of the 24 years. Find the amount he can pay down, using the tabulated factor, the factor formula, and a spreadsheet function.

Solution

The symbols and their values are

$$P = \$12{,}000 \qquad F = ? \qquad i = 8\% \text{ per year} \qquad n = 24 \text{ years}$$

The cash flow diagram is the same as that in Figure 2.1*a*.

Tabulated: Determine F, using the F/P factor for 8% and 24 years. Table 13 provides the factor value.

$$\begin{aligned} F &= P(F/P,i,n) = 12{,}000(F/P,8\%,24) \\ &= 12{,}000(6.3412) \\ &= \$76{,}094.40 \end{aligned}$$

Formula: Apply Equation [2.1] to calculate the future worth F.

$$\begin{aligned} F &= P(1 + i)^n = 12{,}000(1 + 0.08)^{24} \\ &= 12{,}000(6.341181) \\ &= \$76{,}094.17 \end{aligned}$$

Spreadsheet: Use the function = FV(i%,n,A,P). The cell entry is = FV(8%, 24,,12000). The F value displayed is ($76,094.17) in red or −$76,094.17 in black to indicate a cash outflow.

The slight difference in answers is due to round-off error. An equivalence interpretation of this result is that $12,000 today is worth $76,094 after 24 years of growth at 8% per year compounded annually.

EXAMPLE 2.2

Hewlett-Packard has completed a study indicating that $50,000 in reduced maintenance this year (i.e., year zero) on one processing line resulted from improved wireless monitoring technology.

a. If Hewlett-Packard considers these types of savings worth 20% per year, find the equivalent value of this result after 5 years.
b. If the $50,000 maintenance savings occurs now, find its equivalent value 3 years earlier with interest at 20% per year.

Solution

a. The cash flow diagram appears as in Figure 2.1*a*. The symbols and their values are

$$P = \$50{,}000 \qquad F = ? \qquad i = 20\% \text{ per year} \qquad n = 5 \text{ years}$$

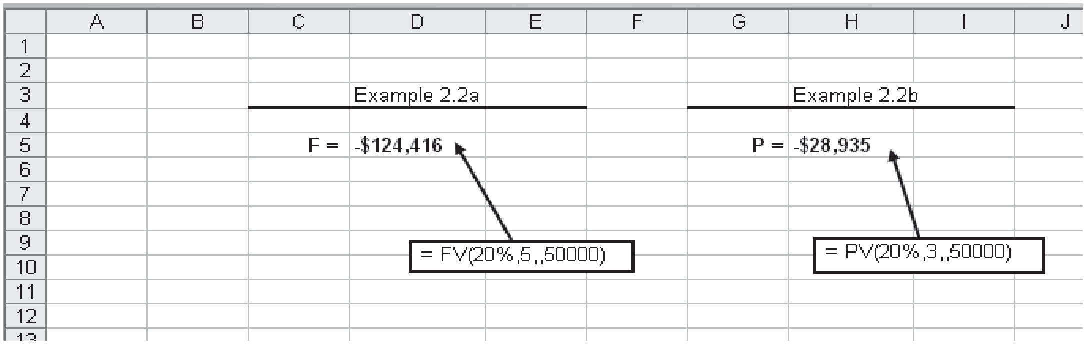


FIGURE 2.2 Use of single-cell spreadsheet functions to find F and P values, Example 2.2.

Use the F/P factor to determine F after 5 years.

$$
\begin{aligned}
F &= P(F/P,i,n) = \$50{,}000(F/P,20\%,5) \\
&= 50{,}000(2.4883) \\
&= \$124{,}415.00
\end{aligned}
$$

The function = FV(20%,5,,50000) also provides the answer. See Figure 2.2.

b. The cash flow diagram appears as in Figure 2.1*b* with F placed at time $t = 0$ and the P value placed 3 years earlier at $t = -3$. The symbols and their values are

$P = ?$ $\quad F = \$50{,}000$ $\quad i = 20\%$ per year $\quad n = 3$ years

Use the P/F factor to determine P three years earlier.

$$
\begin{aligned}
P &= F(P/F,i,n) = \$50{,}000\,(P/F,20\%,3) \\
&= 50{,}000(0.5787) = \$28{,}935.00
\end{aligned}
$$

Use the PV function and omit the A value. Figure 2.2 shows the result of entering = PV (20%,3,,50000) to be the same as using the P/F factor.

EXAMPLE 2.3

Jamie has become more conscientious about paying off his credit card bill promptly to reduce the amount of interest paid. He was surprised to learn that he paid $400 in interest in 2007 and the amounts shown in Figure 2.3 over the previous several years. If he made his payments to avoid interest charges, he would have these funds plus earned interest available in the future. What is the equivalent amount 5 years from now that Jamie could have available had he not paid the interest penalties? Let $i = 5\%$ per year.

Year	2002	2003	2004	2005	2006	2007
Interest paid, $	600	0	300	0	0	400

FIGURE 2.3 Credit card interest paid over the last 6 years, Example 2.3.

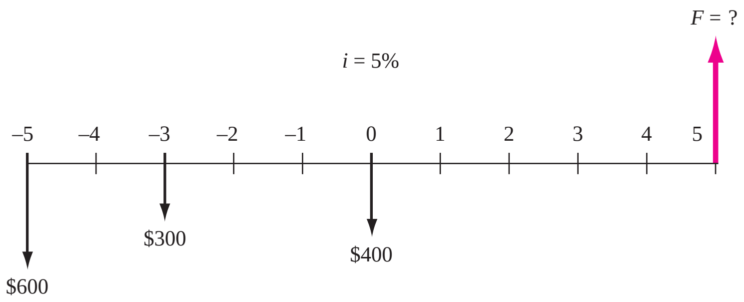


FIGURE 2.4 Cash flow diagram, Example 2.3.

Solution

Draw the cash flow diagram for the values $600, $300, and $400 from Jamie's perspective (Figure 2.4). Use F/P factors to find F in the year labeled 5, which is 10 years after the first cash flow.

$$\begin{aligned} F &= 600(F/P,5\%,10) + 300(F/P,5\%,8) + 400(F/P,5\%,5) \\ &= 600(1.6289) + 300(1.4775) + 400(1.2763) \\ &= \$1931.11 \end{aligned}$$

The problem could also be solved by finding the present worth in year -5 of the $300 and $400 costs using the P/F factors and then finding the future worth of the total in 10 years.

$$\begin{aligned} P &= 600 + 300(P/F,5\%,2) + 400(P/F,5\%,5) \\ &= 600 + 300(0.9070) + 400(0.7835) \\ &= \$1185.50 \\ F &= 1185.50(F/P,5\%,10) = 1185.50(1.6289) \\ &= \$1931.06 \end{aligned}$$

Comment: It should be obvious that there are a number of ways the problem could be worked, since any year could be used to find the equivalent total of the costs before finding the future value in year 5. As an exercise, work the problem using year 0 for the equivalent total and then determine the final amount in year 5. All answers should be the same except for round-off error.

2.2 UNIFORM SERIES FORMULAS (*P/A*, *A/P*, *A/F*, *F/A*)

There are four *uniform series* formulas that involve A, where A means that:

1. The cash flow occurs in *consecutive interest periods*, and
2. The cash flow *amount* is the *same* in each period.

The formulas relate a present worth P or a future worth F to a uniform series amount A. The two equations that relate P and A are as follows. (See Figure 2.5 for cash flow diagrams.)

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

In standard factor notation, the equations are $P = A(P/A, i, n)$ and $A = P(A/P, i, n)$, respectively. It is important to remember that in these equations, the P and the first A value are separated by one interest period. That is, the present worth P is always located one interest period prior to the first A value. It is also important to remember that the n is always equal to the number of A values.

The factors and their use to find P and A are summarized in Table 2.2. The spreadsheet functions shown in Table 2.2 are capable of determining both

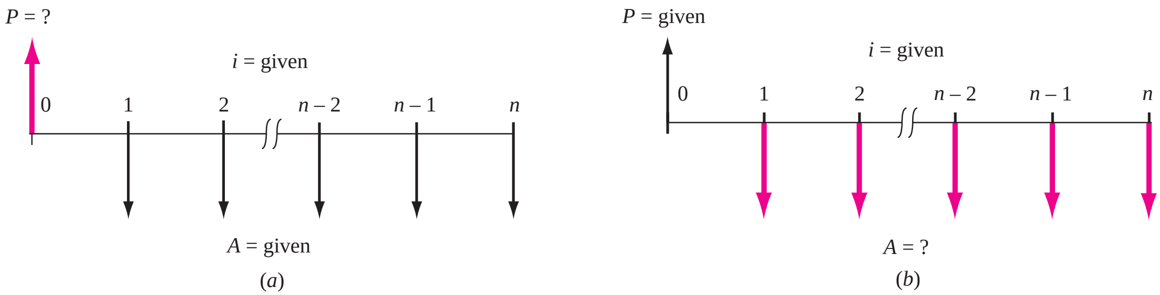


FIGURE 2.5 Cash flow diagrams used to determine (*a*) *P* of a uniform series and (*b*) *A* for a present worth.

TABLE 2.2 ***P/A* and *A/P* Factors: Notation, Equation and Spreadsheet Function**

Factor					
Notation	**Name**	**Find/Given**	**Factor Formula**	**Standard Notation Equation**	**Excel Function**
$(P/A,i,n)$	Uniform-series present worth	P/A	$\frac{(1+i)^n - 1}{i(1+i)^n}$	$P = A(P/A,i,n)$	$= \mathrm{PV}(i\%,n,A,F)$
$(A/P,i,n)$	Capital recovery	A/P	$\frac{i(1+i)^n}{(1+i)^n - 1}$	$A = P(A/P,i,n)$	$= \mathrm{PMT}(i\%,n,P,F)$

P and *A* values in lieu of applying the *P*/*A* and *A*/*P* factors. The PV function calculates the *P* value for a given *A* over *n* years, and a separate *F* value in year *n*, if present. The format is

= PV(*i*%,*n*,*A*,*F*)

Similarly, the *A* value is determined using the PMT function for a given *P* value in year 0 and a separate *F*, if present. The format is

= PMT(*i*%,*n*,*P*,*F*)

EXAMPLE 2.4

How much money should you be willing to pay now for a guaranteed $600 per year for 9 years starting next year, at a rate of return of 16% per year?

Solution

The cash flow diagram (Figure 2.6) fits the *P*/*A* factor. The present worth is:

$$P = 600(P/A,16\%,9) = 600(4.6065) = \$2763.90$$

The PV function = PV(16%,9,600) entered into a single spreadsheet cell will display the answer P = $2763.93.

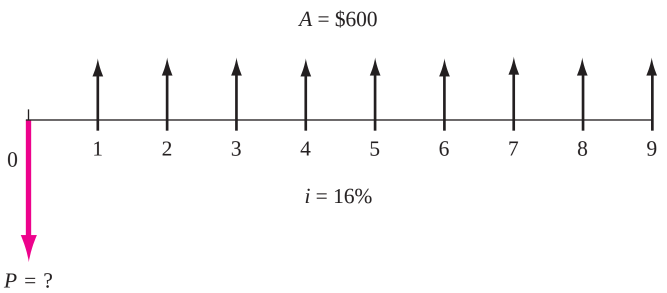


FIGURE 2.6 Diagram to find *P* using the *P*/*A* factor, Example 2.4.

The uniform series formulas that relate *A* and *F* follow. See Figure 2.7 for cash flow diagrams.

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

It is important to remember that these equations are derived such that the last *A* value occurs in the *same* time period as the future worth *F*, and *n* is always equal to the number of *A* values.

Standard notation follows the same form as that of other factors. They are (*F*/*A*,*i*,*n*) and (*A*/*F*,*i*,*n*). Table 2.3 summarizes the notations and equations.

If *P* is not present for the PMT function, the comma must be entered to indicate that the last entry is an *F* value.

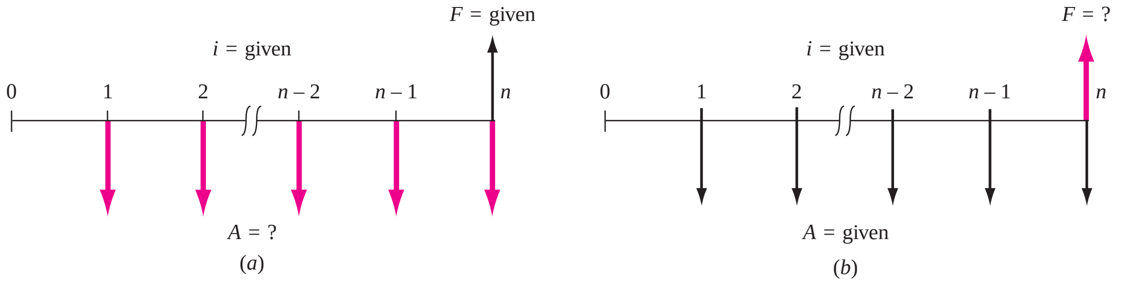


FIGURE 2.7 Cash flow diagrams to (*a*) find *A*, given *F*, and (*b*) find *F*, given *A*.

TABLE 2.3 ***F/A* and *A/F* Factors: Notation, Equation and Spreadsheet Function**

Factor					
Notation	**Name**	**Find/Given**	**Factor Formula**	**Standard Notation Equation**	**Excel Function**
$(F/A,i,n)$	Uniform-series compound amount	F/A	$\frac{(1+i)^n-1}{i}$	$F = A(F/A,i,n)$	$= \text{FV}(i\%,n,A,P)$
$(A/F,i,n)$	Sinking fund	A/F	$\frac{i}{(1+i)^n-1}$	$A = F(A/F,i,n)$	$= \text{PMT}(i\%,n,P,F)$

EXAMPLE 2.5

Formasa Plastics has major fabrication plants in Texas and Hong Kong. The president wants to know the equivalent future worth of $1 million capital investments each year for 8 years, starting 1 year from now. Formasa capital earns at a rate of 14% per year.

Solution

The cash flow diagram (Figure 2.8) shows the annual payments starting at the end of year 1 and ending in the year the future worth is desired. Cash flows are indicated in $1000 units. The F value in 8 years is

$$F = 1000(F/A,14\%,8) = 1000(13.2328) = \$13{,}232.80$$

The actual future worth is $13,232,800.

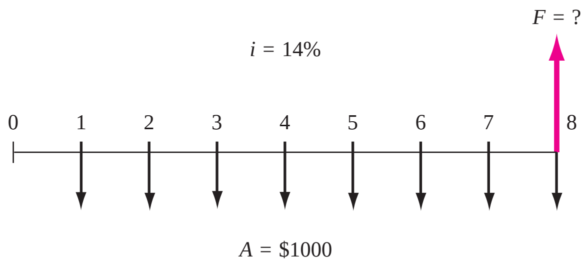


FIGURE 2.8 Diagram to find *F* for a uniform series, Example 2.5.

EXAMPLE 2.6

How much money must an electrical contractor deposit every year in her savings account starting 1 year from now at 5½% per year in order to accumulate $6000 seven years from now?

Solution

The cash flow diagram (Figure 2.9) fits the *A*/*F* factor.

$$A = \$6000(A/F,5.5\%,7) = 6000(0.12096) = \$725.76 \text{ per year}$$

The *A*/*F* factor value of 0.12096 was computed using the factor formula. Alternatively, use the spreadsheet function = PMT(5.5%,7,,6000) to obtain A = $725.79 per year.

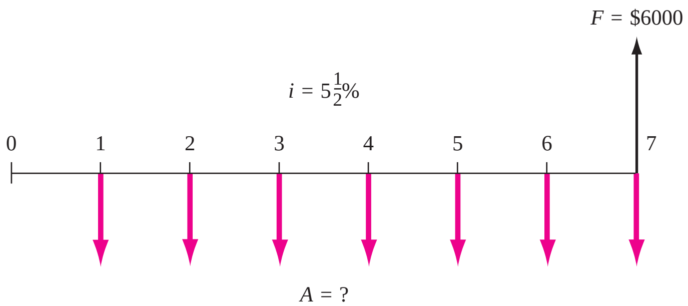


FIGURE 2.9 Cash flow diagram, Example 2.6.

When a problem involves finding i or n (instead of P, F, or A), the solution may require trial and error. Spreadsheet functions can be used to find i or n in most cases.

2.3 GRADIENT FORMULAS

The previous four equations involved cash flows of the *same magnitude A* in each interest period. Sometimes the cash flows that occur in consecutive interest periods are not the same amount (not an A value), but they do change in a predictable way. These cash flows are known as *gradients*, and there are two general types: arithmetic and geometric.

An *arithmetic gradient* is one wherein the cash flow changes (increases or decreases) by the same amount in each period. For example, if the cash flow in period 1 is $800 and in period 2 it is $900, with amounts increasing by $100 in each subsequent interest period, this is an arithmetic gradient G, with a value of $100.

The equation that represents the present worth of an arithmetic gradient series is:

$$P = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right] \qquad [2.3]$$

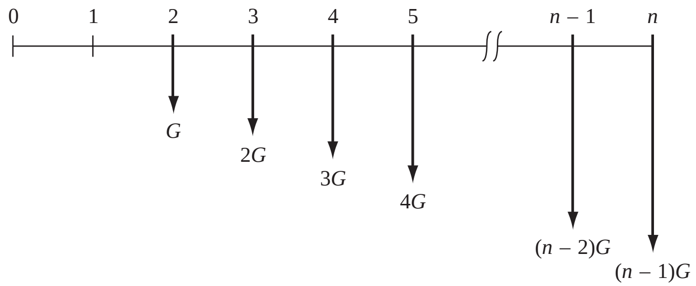


FIGURE 2.10 Conventional arithmetic gradient series without the base amount.

Equation [2.3] is derived from the cash flow diagram in Figure 2.10 by using the *P/F* factor to find the equivalent *P* in year 0 of each cash flow. Standard factor notation for the present worth of an arithmetic gradient is $P = G(P/G,i\%,n)$. This equation finds the present worth of the *gradient only* (the \$100 increases mentioned earlier). It does *not* include the base amount of money that the gradient was built upon (\$800 in the example). The base amount in time period 1 must be accounted for separately as a uniform cash flow series. Thus, the *general equation* to find the present worth of an arithmetic gradient cash flow series is

$$\begin{aligned} \boldsymbol{P} &= \textbf{Present worth of base amount} + \textbf{present worth of gradient amount} \\ &= \boldsymbol{A(P/A,i\%,n) + G(P/G,i\%,n)} \end{aligned} \quad \textbf{[2.4]}$$

where A = amount in *period* 1
G = amount of *change* in cash flow between periods 1 *and* 2
n = number of periods from 1 through n of gradient cash flow
i = interest rate per period

If the gradient cash flow *decreases* from one period to the next, the only change in the general equation is that the plus sign becomes a minus sign. Since the gradient G begins between years 1 and 2, this is called a *conventional gradient.*

EXAMPLE 2.7

The Highway Department expects the cost of maintenance for a piece of heavy construction equipment to be \$5000 in year 1, to be \$5500 in year 2, and to increase annually by \$500 through year 10. At an interest rate of 10% per year, determine the present worth of 10 years of maintenance costs.

Solution

The cash flow includes an increasing gradient with $G = \$500$ and a base amount of \$5000 starting in year 1. Apply Equation [2.4].

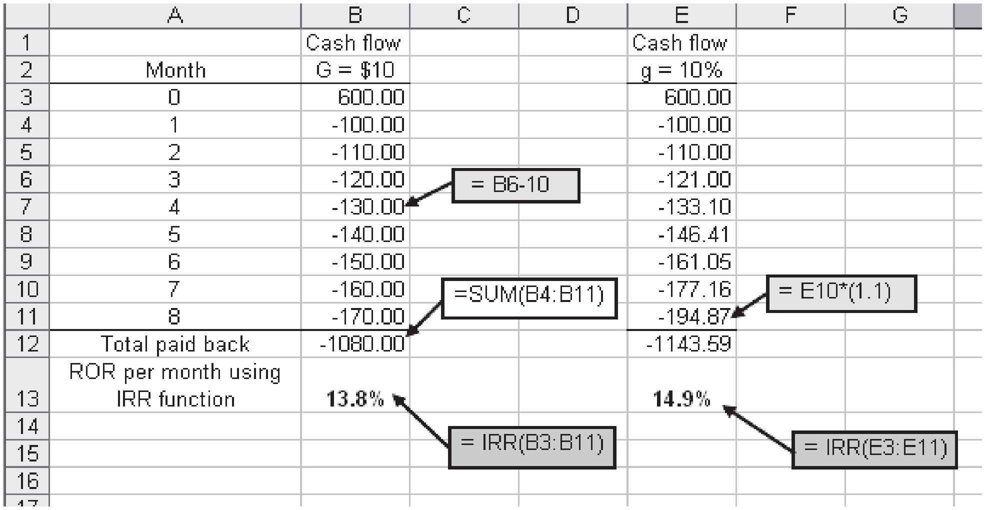
$$\begin{aligned} P &= 5000(P/A,10\%,10) + 500(P/G,10\%,10) \\ &= 5000(6.1446) + 500(22.8913) \\ &= \$42,169 \end{aligned}$$

EXAMPLE 2.13

Bobby was desperate. He borrowed \$600 from a pawn shop and understood he was to repay the loan starting next month with \$100, increasing by \$10 per month for a total of 8 months. Actually, he misunderstood. The repayments increased by 10% each month after starting next month at \$100. Use a spreadsheet to calculate the monthly interest rate that he thought he was to pay, and what he actually will pay.

Solution

Figure 2.20 lists the cash flows for the assumed arithmetic gradient $G = \$-10$ per month, and the actual percentage gradient $g = -10\%$ per month. Note the simple relations to construct the increasing cash flows for each type gradient. Apply the IRR function to each series using its format = IRR (first_cell:last_cell). Bobby is paying an exorbitant rate per month (and year) at 14.9% *per month*, which is higher than he expected it to be at 13.8% per month. (Interest rates are covered in detail in Chapter 3.)



	A	B	C	D	E	F	G
1		Cash flow			Cash flow		
2	Month	G = \$10			g = 10%		
3	0	600.00			600.00		
4	1	-100.00			-100.00		
5	2	-110.00			-110.00		
6	3	-120.00			-121.00		
7	4	-130.00			-133.10		
8	5	-140.00			-146.41		
9	6	-150.00			-161.05		
10	7	-160.00			-177.16		
11	8	-170.00			-194.87		
12	Total paid back	-1080.00			-1143.59		
13	ROR per month using IRR function	13.8%			14.9%		
14							
15							
16							

FIGURE 2.20 Use of a spreadsheet to generate arithmetic and percentage gradient cash flows and application of the IRR function, Example 2.13.

SUMMARY

In this chapter, we presented formulas that make it relatively easy to account for the time value of money. In order to use the formulas correctly, certain things must be remembered.

1. When using the P/A or A/P factors, the P and the first A value are separated by one interest period.
2. When using the F/A or A/F factors, the F and the last A value are in the *same* interest period.
3. The n in the uniform series formulas is equal to the number of A values involved.
4. Arithmetic gradients change by a uniform amount from one interest period to the next, and there are

This equation accounts for *all* of the cash flow, including the amount in period 1. For a decreasing geometric gradient, change the sign prior to both g values. When $g = i$, the P value is

$$P = A_1[n/(1 + i)] \qquad \textbf{[2.8]}$$

Geometric gradient factors are not tabulated; the equations are used. Spreadsheets are also an option.

EXAMPLE 2.9 A mechanical contractor has four employees whose combined salaries through the end of this year are \$250,000. If he expects to give an average raise of 5% each year, calculate the present worth of the employees' salaries over the next 5 years. Let $i = 12\%$ per year.

Solution

The cash flow at the end of year 1 is \$250,000, increasing by $g = 5\%$ per year (Figure 2.11). The present worth is found using Equation [2.7].

$$P = 250{,}000\left[\frac{1 - \left(\frac{1.05}{1.12}\right)^5}{0.12 - 0.05}\right]$$

$$= 250{,}000(3.94005)$$

$$= \$985{,}013$$

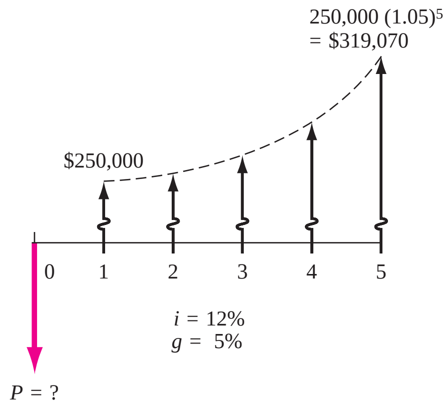


FIGURE 2.11 Cash flow with $g = 5\%$, Example 2.9

In summary, some basics for gradients are:

- Arithmetic gradients consist of two parts: a uniform series that has an A value equal to the amount of money in period 1, and a gradient that has a value equal to the change in cash flow between periods 1 and 2.
- For arithmetic gradients, the gradient factor is preceded by a plus sign for increasing gradients and a minus sign for decreasing gradients.

- Arithmetic and geometric cash flows start between periods 1 and 2, with the A value in each equation equal to the magnitude of the cash flow in period 1.
- Geometric gradients are handled with Equation [2.7] or [2.8], which yield the present worth of *all* the cash flows.

2.4 CALCULATIONS FOR CASH FLOWS THAT ARE SHIFTED

When a uniform series begins at a time other than at the end of period 1, it is called a *shifted series.* In this case several methods can be used to find the equivalent present worth P. For example, P of the uniform series shown in Figure 2.12 could be determined by any of the following methods:

- Use the P/F factor to find the present worth of each disbursement at year 0 and add them.
- Use the F/P factor to find the future worth of each disbursement in year 13, add them, and then find the present worth of the total using $P = F(P/F,i,13)$.
- Use the F/A factor to find the future amount $F = A(F/A,i,10)$, and then compute the present worth using $P = F(P/F,i,13)$.
- Use the P/A factor to compute the "present worth" (which will be located in year 3 not year 0), and then find the present worth in year 0 by using the $(P/F,i,3)$ factor. (Present worth is enclosed in quotation marks here only to represent the present worth as determined by the P/A factor in year 3, and to differentiate it from the present worth in year 0.)

Typically the last method is used. For Figure 2.12, the "present worth" obtained using the P/A factor is located in year 3. This is shown as P_3 in Figure 2.13.

Remember, the present worth is always located one period prior to the first uniform-series amount when using the P/A factor.

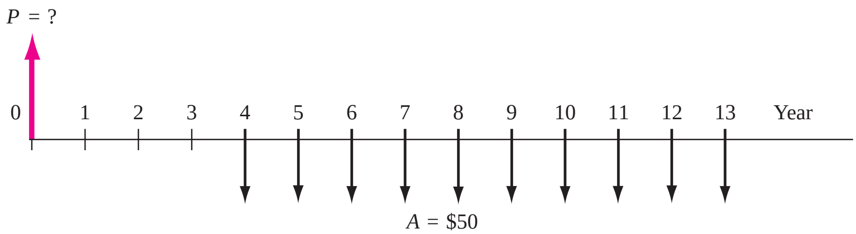


FIGURE 2.12
A uniform series that is shifted.

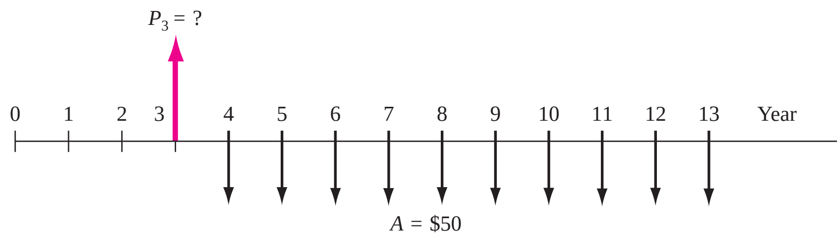
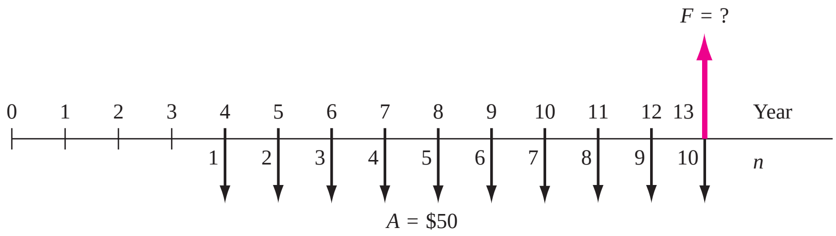


FIGURE 2.13
Location of present worth for the shifted uniform series in Figure 2.12.

FIGURE 2.14
Placement of F and renumbering for n for the shifted uniform series of Figure 2.12.



To determine a future worth or F value, recall that the F/A factor has the F located in the *same* period as the last uniform-series amount. Figure 2.14 shows the location of the future worth when F/A is used for Figure 2.12 cash flows.

Remember, the future worth is always located in the same period as the last uniform-series amount when using the F/A factor.

It is also important to remember that the number of periods n in the P/A or F/A factor is equal to the number of uniform-series values. It may be helpful to *renumber* the cash flow diagram to avoid errors in counting. Figure 2.14 shows Figure 2.12 renumbered to determine $n = 10$.

As stated above, there are several methods that can be used to solve problems containing a uniform series that is shifted. However, it is generally more convenient to use the uniform-series factors than the single-amount factors. There are specific steps that should be followed in order to avoid errors:

1. Draw a diagram of the positive and negative cash flows.
2. Locate the present worth or future worth of each series on the cash flow diagram.
3. Determine n for each series by renumbering the cash flow diagram.
4. Set up and solve the equations.

EXAMPLE 2.10

An engineering technology group just purchased new CAD software for $5000 now and annual payments of $500 per year for 6 years starting 3 years from now for annual upgrades. What is the present worth of the payments if the interest rate is 8% per year?

Solution

The cash flow diagram is shown in Figure 2.15. The symbol P_A is used throughout this chapter to represent the present worth of a uniform annual series A, and P'_A represents the present worth at a time other than period 0. Similarly, P_T represents the total present worth at time 0. The correct placement of P'_A and the diagram renumbering to obtain n are also indicated. Note that P'_A is located in actual year 2, not year 3. Also, $n = 6$, not 8, for the P/A factor. First find the value of P'_A of the shifted series.

$$P'_A = \$500(P/A,8\%,6)$$

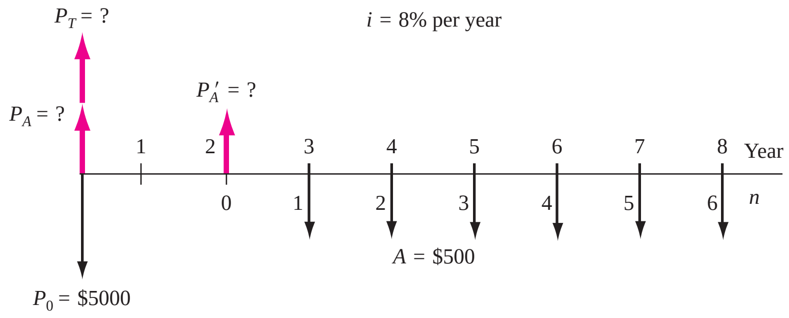


FIGURE 2.15

Cash flow diagram with placement of P values, Example 2.10.

Since P'_A is located in year 2, now find P_A in year 0.

$$P_A = P'_A(P/F,8\%,2)$$

The total present worth is determined by adding P_A and the initial payment P_0 in year 0.

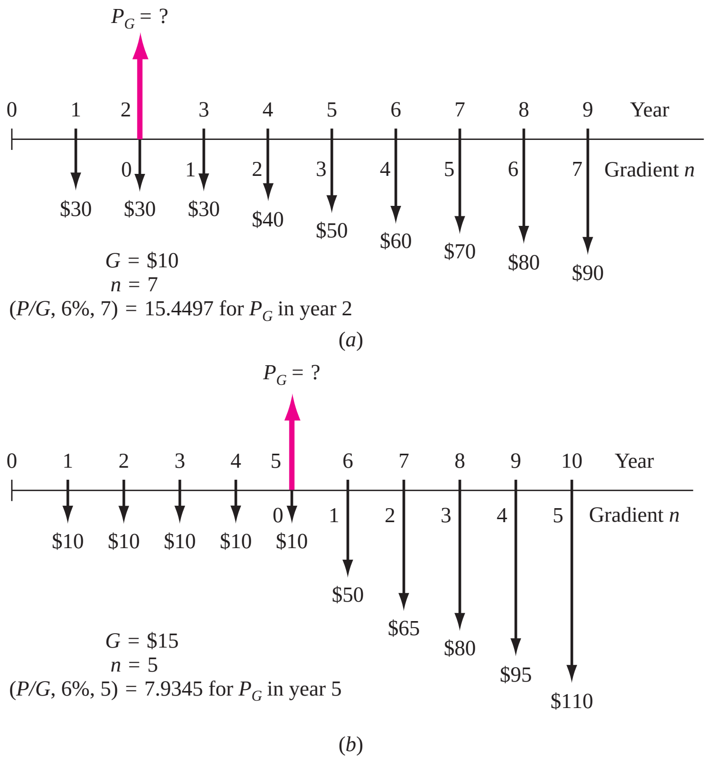
$$\begin{aligned} P_T &= P_0 + P_A \\ &= 5000 + 500(P/A,8\%,6)(P/F,8\%,2) \\ &= 5000 + 500(4.6229)(0.8573) \\ &= \$6981.60 \end{aligned}$$

To determine the present worth for a cash flow that includes both uniform series and single amounts at specific times, use the P/F factor for the single amounts and the P/A factor for the series. To calculate A for the cash flows, first convert everything to a P value in year 0, or an F value in the last year. Then obtain the A value using the A/P or A/F factor, where n is the total number of years over which the A is desired.

Many of the considerations that apply to shifted uniform series apply to gradient series as well. Recall that a conventional gradient series starts between periods 1 and 2 of the cash flow sequence. A gradient starting at any other time is called a *shifted gradient*. The n value in the P/G and A/G factors for the shifted gradient is determined by renumbering the time scale. The period in which the *gradient first appears is labeled period 2*. The n value for the factor is determined by the renumbered period where the last gradient increase occurs. The P/G factor values and placement of the gradient series present worth P_G for the shifted arithmetic gradients in Figure 2.16 are indicated.

It is important to note that the A/G factor *cannot* be used to find an equivalent A value in periods 1 through n for cash flows involving a shifted gradient. Consider the cash flow diagram of Figure 2.16*b*. To find the equivalent annual series in years 1 through 10 for the gradient series only, first find the present worth of the gradient in year 5, take this present worth back to year 0, and then annualize the present worth for 10 years with the A/P factor. If you apply the annual

FIGURE 2.16 Determination of G and n values used in factors for shifted gradients.



series gradient factor ($A/G,i,5$) directly, the gradient is converted into an equivalent annual series over years 6 through 10 only.

Remember, to find the equivalent A series of a shifted gradient through all the periods, first find the present worth of the gradient at actual time 0, then apply the ($A/P,i,n$) factor.

If the cash flow series involves a *geometric gradient* and the gradient starts at a time other than between periods 1 and 2, it is a shifted gradient. The P_g is located in a manner similar to that for P_G above, and Equation [2.7] is the factor formula.

EXAMPLE 2.11 Chemical engineers at a Coleman Industries plant in the Midwest have determined that a small amount of a newly available chemical additive will increase the water repellency of Coleman's tent fabric by 20%. The plant superintendent has arranged to purchase the additive through a 5-year contract at $7000 per year, starting 1 year from now. He expects the annual price to increase by 12% per year starting in the sixth year and thereafter through year 13. Additionally, an initial investment of $35,000 was made now to prepare a site suitable for the contractor to deliver the additive. Use i = 15% per year to determine the equivalent total present worth for all these cash flows.

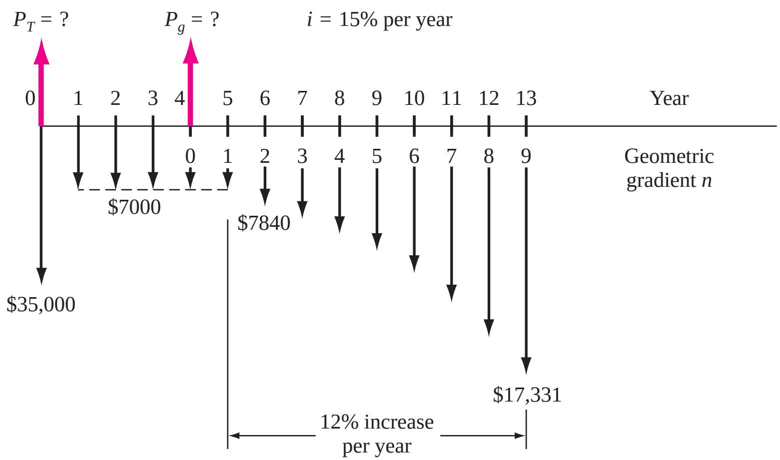


FIGURE 2.17 Cash flow diagram including a geometric gradient with $g = 12\%$, Example 2.11.

Solution

Figure 2.17 presents the cash flows. The total present worth P_T is found using $g = 0.12$ and $i = 0.15$. Equation [2.7] is used to determine the present worth P_g for the entire geometric series at actual year 4, which is moved to year 0 using $(P/F,15\%,4)$.

$$
\begin{aligned}
P_T &= 35{,}000 + A(P/A,15\%,4) + A_1(P/A,12\%,15\%,9)(P/F,15\%,4) \\
&= 35{,}000 + 7000(2.8550) + \left[7000\frac{1 - (1.12/1.15)^9}{0.15 - 0.12}\right](0.5718) \\
&= 35{,}000 + 19{,}985 + 28{,}247 \\
&= \$83{,}232
\end{aligned}
$$

Note that $n = 4$ in the $(P/A,15\%,4)$ factor because the \$7000 in year 5 is the initial amount A_1 in Equation [2.7].