Non-Prismatic Bars

“I do not pretend to understand the universe. It’s a great deal bigger than I am.”
– Tom Stoppard

Changing Cross Section

In our previous work, we have looked at the stress and strain in bars with constant cross sectional areas and with point loads applied axially.

Now we will consider what is happening in the bar if the cross section changes and/or if the load is an axially distributed load.

In the first case, we will vary the cross section of the material and see how the stress and deformation varies with the variation in the cross section.

If we take a cross-section at some distance x to the right from the support we can draw a free-body diagram of the right-hand section.
Changing Cross Section

The axial stress at a distance $x$ could then be calculated as

$$\sigma_x = \frac{P}{A_x}$$

We may develop a function that relates the cross sectional area as a function of the distance $x$

Now we need to utilize this changing stress through the bar to determine the overall change in length of the bar

This allows us to calculate the stress on any face at any depth in the bar
We can start by taking a differential section of the bar.

The stress on the differential area is

\[ \sigma_x = \frac{P}{A_x} \]

In terms of the strain, the stress on the differential area is

\[ \sigma_x = E \varepsilon_x = \frac{P}{A_x} \]

If we replace the strain by the deformation divided by the original length we have

\[ \sigma_x = E \frac{\delta_x}{dx} = \frac{P}{A_x} \]
Changing Cross Section

The deformation through this differential section represents a differential deformation within the bar.

\[ E \frac{d\delta}{dx} = \frac{P}{A_x} \]

Changing Cross Section

Isolating this differential deformation we have

\[ d\delta = \frac{Pdx}{EA_x} \]

Changing Cross Section

To calculate the deformation along the length of the bar we would sum up the differential deformations

\[ d\delta = \frac{Pdx}{EA_x} \]

Since we are looking at differential quantities, we make summations using the integral

\[ \int d\delta = \int \frac{Pdx}{EA_x} \]
Changing Cross Section

Since the axial load is constant along the length and the modulus of elasticity is constant we can bring those outside of the integral

\[ \int d\delta = \frac{P}{E} \int \frac{dx}{A_x} \]

And the integral of all the deformation along the length of the bar is equal to the total deformation we have

\[ \delta = \frac{P}{E} \int \frac{dx}{A_x} \]

NOTE: Even though we have been using deformation, with an axial load the deformation is an elongation or a shortening

Finally, the change in length of the bar is

\[ \delta = \frac{P}{E} \int_0^L \frac{dx}{A_x} \]
Changing Cross Section

- Since $A_x$ is a function of $x$, this is as far as we can go until we define the function of $A_x$.

\[
\delta = \frac{PL}{EA} \int_0^L \frac{dx}{A_x}
\]

Changing Cross Section

- Notice how the expression we just developed resembles the expression for a prismatic bar.

\[
\delta = \frac{P}{EA} \int_0^L \frac{dx}{A_x}
\]

Problem 3-4.1 and 3-4.2

- If we were to take small differential lengths along the axis they would be $dx$ lengths and the elongation of these lengths would still be

\[
\delta = \frac{PL}{EA}
\]

\[
\sigma_x = \frac{P}{A_x}
\]

- Problem 3-4.1.
  - The bar’s cross-sectional area is $A = (1 + 0.1x)$ in$^2$ and the modulus of elasticity of the material is $E = 12 \times 10^6$ psi. If the bar is subjected to tensile axial forces $P = 20$ kip at its ends, what is the normal stress at $x = 6$ in.?
Distributed Axial Loads

In some cases we may have a constant cross section but we may have an axial load that varies with along the bar.

3.4.2. What is the change in length of the bar in Problem 3.4.1?

For instance, we we have a bar loaded as shown:

If the weight of the bar is significant, we can look at a section of the bar and look at the forces on that section.
Distributed Axial Loads

In this case, the axial load at any distance up from the bottom of the beam is both the load and the weight of that section.

\[ \text{load} + \text{weight} \]

This means that the stress varies as we travel up the beam and therefore the strain at each point as we move up the beam.

\[ \text{stress} = \frac{\text{force}}{\text{area}} \]

If we label the loading axis as \( x \), the elongation expression again become a differential expression.

\[ d\delta = \frac{Pdx}{EA} \]

In this case, it is \( P \) that is a function of \( x \) rather than \( A \) so the total elongation of the beam would be.

\[ \delta = \frac{\int_{0}^{L} Pdx}{EA} \]
Changing Cross Section

In the most general case $P$, $E$, and $A$ would all be functions of $x$:

$$\delta = \int_0^L \frac{P(x)dx}{A(x)E(x)}$$

Distributed Axial Loads

3-4.9. The bar is fixed at the left end and subjected to a uniformly distributed axial force. It has cross-sectional area $A$ and modulus of elasticity $E$. (a) Determine the internal axial force $P$ in the bar as a function of $x$. (b) What is the bar’s change in length?

Homework

- Read section 3-6
- Problem 3-4.7
- Problem 3-4.15
- Problem 3-4.23 (Both the cross sectional area and the loading are functions)