Dimensions and Units

- You already know how important using the correct dimensions can be in the analysis of a problem in fluid mechanics
- If you don’t, I am not doing my job very well.
Dimensions and Units

- Dimensions are the what in a measurements
  - We measure length, time, mass, weight, etc.
- Units are just what system of measurement we are going to use to describe the dimensions
  - We can measure the length of something in feet
  - We can measure the elapsed time in minutes
  - We can measure the weight in stones
Dimensions and Units

- Based on our system of measurement, some dimensions are chosen as fundamental to the system.
- They are not defined in terms of any other dimension.
- Sort of the prime numbers of the system.
- The system will also give us the symbol and the common unit for this fundamental dimension.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Symbol*</th>
<th>SI Unit</th>
<th>English Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>m</td>
<td>kg (kilogram)</td>
<td>lbm (pound-mass)</td>
</tr>
<tr>
<td>Length</td>
<td>L</td>
<td>m (meter)</td>
<td>ft (foot)</td>
</tr>
<tr>
<td>Time$^1$</td>
<td>t</td>
<td>s (second)</td>
<td>s (second)</td>
</tr>
<tr>
<td>Temperature</td>
<td>T</td>
<td>K (kelvin)</td>
<td>R (rankine)</td>
</tr>
<tr>
<td>Electric current</td>
<td>I</td>
<td>A (ampere)</td>
<td>A (ampere)</td>
</tr>
<tr>
<td>Amount of light</td>
<td>C</td>
<td>cd (candela)</td>
<td>cd (candela)</td>
</tr>
<tr>
<td>Amount of matter</td>
<td>N</td>
<td>mol (mole)</td>
<td>mol (mole)</td>
</tr>
</tbody>
</table>
Dimensions and Units

- Every other quantity that we work with can be defined in terms of these fundamental dimensions.
- When we are describing the dimensions of something, we use the curly braces \{\} around both the thing we are describing and the dimensions we are using.

For example, if we take pressure down to fundamental dimensions we would have:

\[
\{P\} = \left\{ \frac{F}{A} \right\} \\
\{A\} = \{L^2\} \\
\{F\} = \left\{ \frac{mL}{t^2} \right\} \\
\{P\} = \left\{ \frac{mL}{t^2} \right\} = \left\{ \frac{m}{L \cdot t^2} \right\}
\]

Pressure has dimensions of force per area.
Area is dimensioned as the square of the fundamental dimension length.
Force is dimensioned as the quotient of the product of fundamental dimensions mass and length and the square of the fundamental dimension time.
So Pressure can be described as having fundamental dimension as shown.
Dimensions and Units

- So we can describe the units of pressure in fundamental units in each system

\[
\begin{align*}
\{P\} &= \left\{ \frac{F}{A} \right\} \\
\{A\} &= \left\{ L^2 \right\} \\
\{F\} &= \left\{ \frac{mL}{t^2} \right\} \\
\{P\} &= \left\{ \frac{mL}{t^2} \right\} = \left\{ \frac{m}{L^2t^2} \right\}
\end{align*}
\]

In SI, pressure would have fundamental units of kilograms per meter per second squared.

In USCS, pressure would have fundamental units of pound-mass per foot per second squared.

Dimensional Homogeneity

- One of the wonderful safety nets we have in engineering analysis is the ability to check out answers for the correct dimensions.

- While it is often very easy, especially with our sophisticated calculators, to just write down numbers and make the calculation assigning the units after the calculation is made.

- This can lead to some costly and embarrassing mistakes.
Dimensional Homogeneity

- If the dimensions aren't the same for each additive term in an expression or if the dimensions aren't the same on both sides of the equal sign, you made a mistake.

- Use them in your expressions, including all expressions leading up to your calculations and you will save yourself a lot of time and trouble.
- Yes, TIME.
- It is a lot easier to do it correctly the first time than to retrace your steps.
Angular momentum, also called moment of momentum \((H)\), is formed by the cross product of a moment arm \((r)\) and the linear momentum \((mv)\) of a fluid particle. What are the primary dimensions of angular momentum? List the units of angular momentum in primary SI units and in primary English units.

\[
\{ H \} = \{ rmv \} = \{ r \} \{ m \} \{ v \}
\]

\[
\{ r \} = \{ L \}
\]

\[
\{ m \} = \{ m \}
\]

\[
\{ v \} = \left\{ \frac{L}{t} \right\}
\]

\[
\{ H \} = \{ L \} \{ m \} \left\{ \frac{L}{t} \right\} = \left\{ \frac{mL^2}{t} \right\} = \left\{ mL^2t^{-1} \right\}
\]

Dimensional Homogeneity

- You may often see the inline form of dimensions given with both positive and negative powers for the exponents
- This is done to simplify the presentation of units in technical papers
Dimensional Homogeneity

- Dimensional homogeneity can also help you to remember the form of an expression.
- For example, if you wanted the Bernoulli equation in terms of head which is typically given in terms of length.
- You know that the dimension of the result is in units of length, \( L \) so all the other terms would also have to work down to units of \( L \).

For the Pressure term we know that the dimensions of pressure are \( mL^{-1}t^{-2} \) so we would have to find a factor that that had dimensions of \( m^{-1}L^2t^2 \) to bring the pressure term to dimensions of \( L \).

\[
\{ p \} = \left\{ \frac{m}{Lt^2} \right\} \\
\left\{ \frac{m}{Lt^2} \right\} \left\{ \frac{L^2 t^2}{m} \right\} = \{ L \}
\]
Dimensional Homogeneity

- From previous work with the Bernoulli equation we know that weight density, mass density, and the gravitational constant are somewhere in the expression so we can see which would work here.

\[
\{ \rho \} = \left\{ \frac{m}{L^3} \right\} \\
\{ \rho \} = \left\{ \frac{m}{L^3} \right\} \\
\{ g \} = \left\{ \frac{L}{t^2} \right\} \\
\{ \gamma \} = \left\{ \frac{m}{L^2} \right\} \left\{ \frac{L}{t^2} \right\} = \left\{ \frac{m}{L^2 t^2} \right\}
\]

From this dimensional analysis, you can develop that if you divide Pressure by weight density, you have the term in dimensions of length

\[
\{ \rho \} = \left\{ \frac{m}{L^3} \right\} \\
\{ \rho \} = \left\{ \frac{m}{L^3} \right\} \\
\{ g \} = \left\{ \frac{L}{t^2} \right\} \\
\{ \gamma \} = \left\{ \frac{m}{L^2} \right\} \left\{ \frac{L}{t^2} \right\} = \left\{ \frac{m}{L^2 t^2} \right\}
\]
Model Development

- We often have to evaluate systems and ideas in a lab scale setting because field scale is just too large and expensive to build and test.
- For example, if you consider the flow under piles on a bridge, we would have to build a lot of bridges to evaluate flow patterns around different types of supports and different conditions.

So we build models of the systems and test them at this much smaller scale but we have to be careful as to what characteristics we use when we make the models.

The rest of this chapter will be about first locating significant parameters to include in the model and then about just how to work with the parameters on the scale model to have them accurately represent the full scale system.
If we have a system where we are looking to develop some sort of predictive model we may want to first develop a relationship between dependent and independent parameters. We could just measure everything and use statistical methods to develop a relationship and that is done quite often. This is known as an empirical relationship (developed from data rather than first principles).
Rayleigh Method

- The Rayleigh method is a way to look at possible independent parameters and see how they might just relate to each other to describe the dependent variable.

You are presented a lab situation where the height $h$ a liquid attains inside a partly submerged capillary tube is a function of surface tension $\sigma$, tube radius $R$, gravity $g$, and liquid density $\rho$. How are the independent variables related in describing the dependent variable?
Rayleigh Method

The first step is to identify the fundamental units for both the dependent and independent parameters.

\[ \{h\} = \{L\} \]
\[ \{\sigma\} = \left\{ \frac{m}{t^2} \right\} \]
\[ \{R\} = \{L\} \]
\[ \{g\} = \left\{ \frac{L}{t^2} \right\} \]
\[ \{\rho\} = \left\{ \frac{m}{L^3} \right\} \]

The dimensions we are using here are shown in Table 4.1 in the text. We are using a M,L,T dimensional system.

Now we set up a relation between the dependent and independent variables

\[ h = C \sigma^{a_1} R^{a_2} g^{a_3} \rho^{a_4} \]
Rayleigh Method

- Each of the independent variables has been given an exponent
- C is an arbitrary constant

\[ h = C \sigma^{a_1} R^{a_2} g^{a_3} \rho^{a_4} \]

\[ \{h\} = \{L\} \]
\[ \{\sigma\} = \left\{ \frac{m}{L^2} \right\} \]
\[ \{R\} = \{L\} \]
\[ \{g\} = \left\{ \frac{L}{t^2} \right\} \]
\[ \{\rho\} = \left\{ \frac{m}{L^3} \right\} \]

Now we substitute the dimensions on both sides of the equals sign

\[ \{h\} = \{L\} \]
\[ \{\sigma\} = \left\{ \frac{m}{L^2} \right\} \]
\[ \{R\} = \{L\} \]
\[ \{g\} = \left\{ \frac{L}{t^2} \right\} \]
\[ \{\rho\} = \left\{ \frac{m}{L^3} \right\} \]
Rayleigh Method

Now we can group like fundamental units together.

\[
\{h\} = \{L\} \\
\{\sigma\} = \left( \frac{m}{L^2} \right) \\
\{R\} = \{L\} \\
\{g\} = \left( \frac{L}{t^2} \right) \\
\{\rho\} = \left( \frac{m}{L^3} \right)
\]

\[
h = C\sigma^{a_1}R^{a_2}g^{a_3}\rho^{a_4}
\]

\[
\{L\} = \left[ \frac{m}{L^2} \right]^{a_1} \left[ L \right]^{a_2} \left[ \frac{L}{t^2} \right]^{a_3} \left[ \frac{m}{L^3} \right]^{a_4}
\]

\[
0 = a_1 + a_4 \\
1 = a_2 + a_3 - 3a_4 \\
0 = -2a_1 - 2a_3
\]
Rayleigh Method

We now have three equations in four unknowns. We can’t solve for them exactly but we can try and reduce them to as few unknowns as possible.

\[
h = C \sigma a_1 R^2 g a_3 \rho a_4
\]

\[
0 = a_1 + a_4 \Rightarrow a_1 = -a_4
\]

\[
1 = a_2 + a_3 - 3a_4
\]

\[
0 = -2a_1 - 2a_3 \Rightarrow a_1 = -a_3
\]
Rayleigh Method

- So we are down to defining everything in terms of $a_1$.
- Returning to the original expression with the new expressions

$$h = C \sigma^{a_1} R^{k-2a_1} g^{-a_1} \rho^{-a_1}$$

$$a_1 = -a_4$$
$$a_2 = 1 - 2a_1$$
$$a_3 = -a_3$$

We can get it down to only one exponent because we started with four independent variables and we could generate three expressions

$$h = C \sigma^{a_1} R^{k-2a_1} g^{-a_1} \rho^{-a_1}$$

$$a_1 = -a_4$$
$$a_2 = 1 - 2a_1$$
$$a_3 = -a_3$$
Rayleigh Method

- Unless we have a very unique situation (or three or less independent variables) we cannot get the expression any better than three less than the number of independent variables

\[ h = C \sigma^{a_1} R^{1-2a_1} g^{-a_1} \rho^{-a_1} \]
\[ a_1 = -a_4 \]
\[ a_2 = 1 - 2a_1 \]
\[ a_3 = -a_3 \]

Rayleigh Method

- Now we can group variables with like exponents

\[ h = C \sigma^{a_1} R^{1-2a_1} g^{-a_1} \rho^{-a_1} \]
\[ \dot{h} = CR \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} \]
Dimensional Analysis

Rayleigh Method

- Normally, we write the expression with any terms on the right side that we know the exponent moved to the left side.

\[
h = C \sigma^{a_1} R^{1-2a_1} g^{-a_1} \rho^{-a_1} 
\]

\[
h = C R \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} 
\]

\[
\frac{h}{R} = C \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} 
\]

Rayleigh Method

- You may also see this written in function form

\[
h = C \sigma^{a_1} R^{1-2a_1} g^{-a_1} \rho^{-a_1} 
\]

\[
h = C R \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} 
\]

\[
\frac{h}{R} = C \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} 
\]

\[
\frac{h}{R} = f \left( \frac{\sigma}{R^2 g \rho} \right) 
\]
Dr. Janna does not use the f notation because we will have a parameter later that uses the f

\[ h = C \sigma^{-a_1} R^{1-2a_1} g^{-a_1} \rho^{-a_1} \]

\[ h = CR \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} \]

\[ \frac{h}{R} = C \left( \frac{\sigma}{R^2 g \rho} \right)^{a_1} \]

\[ \frac{h}{R} = f \left[ \frac{\sigma}{R^2 g \rho} \right] \]

An example from the text

4.21 Power input to a pump depends on flow rate \( Q \), pressure rise \( \Delta p \), liquid density \( \rho \), efficiency \( \eta \), and impeller diameter \( D \). Determine an expression for power by using dimensional analysis.
Rayleigh Method

- An example from the text

4.21 Power input to a pump depends on flow rate $Q$, pressure rise $\Delta p$, liquid density $\rho$, efficiency $\eta$, and impeller diameter $D$. Determine an expression for power by using dimensional analysis.

\[ P = C Q^{a_1} \Delta p^{a_2} \rho^{a_3} D^{a_4} \]

$\eta$ is excluded because it is dimensionless
Rayleigh Method

An example from the text

4.21 Power input to a pump depends on flow rate $Q$, pressure rise $\Delta p$, liquid density $\rho$, efficiency $\eta$, and impeller diameter $D$. Determine an expression for power by using dimensional analysis.

$$P = C Q^{a_1} \Delta p^{a_2} \rho^{a_3} D^{a_4}$$

$$\begin{align*}
\left\{ \frac{mL^2}{t^3} \right\} &= \left\{ \frac{L^3}{t} \right\}^{a_1} \left\{ \frac{m}{L^2} \right\}^{a_2} \left\{ \frac{m}{L^3} \right\}^{a_3} \left\{ L \right\}^{a_4} \\
\{ m \} &= \{ m \}^{a_2} \{ m \}^{a_3} \\
\{ L \}^2 &= \{ L \}^{3a_1} \{ L \}^{-a_2} \{ L \}^{-3a_3} \{ L \}^{a_4} \\
\{ t \}^{-3} &= \{ t \}^{-a_1} \{ t \}^{-2a_2}
\end{align*}$$
Rayleigh Method

- An example from the text

4.21 Power input to a pump depends on flow rate $Q$, pressure rise $\Delta p$, liquid density $\rho$, efficiency $\eta$, and impeller diameter $D$. Determine an expression for power by using dimensional analysis.

$$P = CQ^{a_1} \Delta p^{a_2} \rho^{a_3} D^{a_4}$$

1. $a_2 + a_3 \Leftrightarrow a_3 = 1 - a_2$
2. $2 = 3a_1 - a_2 - 3a_3 + a_4$
3. $-3 = -a_1 - 2a_2 \Leftrightarrow a_1 = 3 - 2a_2$

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Dimensional Analysis
Rayleigh Method

- An example from the text

4.21 Power input to a pump depends on flow rate $Q$, pressure rise $\Delta p$, liquid density $\rho$, efficiency $\eta$, and impeller diameter $D$. Determine an expression for power by using dimensional analysis.

$$P = C Q^a \Delta p^b \rho^c D^d$$
$$P = C Q^{3-2a} \Delta p^{a-2} \rho^{1-a} D^{-4+4a}$$
$$P = C Q^3 Q^{-2a} \Delta p^{a-2} \rho^{1-a} D^4 D^{4a}$$
$$P = C \rho^3 D^{-4} \left( Q^{-2a} \Delta p^{a-2} \rho^{-a} D^{4a} \right)$$
$$P = C \frac{Q^3 \rho^3}{D^4} \left( \frac{\Delta p D^4}{Q^2 \rho} \right)^{a^2}$$

Homework 16-1

4.10 In a falling-sphere viscometer, spheres are dropped through a liquid, and their terminal velocity is measured. The liquid viscosity is then determined. Perform a dimensional analysis for the viscometer assuming that viscosity $\mu$ is a function of sphere diameter $D$ and mass $m$, local acceleration due to gravity $g$, and liquid density $\rho$.

Use a4.
4.15 The friction factor $f$ is a dimensionless quantity used in pipe flow problems as an aid in calculating pressure drop. The friction factor is a function of fluid properties density $\rho$ and viscosity $\mu$, of average velocity $V$, of pipe diameter $D$, and of the roughness of the pipe wall. The wall roughness is traditionally represented by a single term $e$ having the dimension of length. Determine a relationship between these variables to predict friction factor.

Use a2 and a5.