Deflections

- The previously discussed geometric methods are very good and straightforward for simple loadings.
- However, these methods are very tedious for complex loadings.
- In cases like this, an ***energy method*** is the preferred technique.
- Energy methods are based on the principle of conservation of energy.

Deflections

- This principle states that the work done by all the external forces, U_e, acting on a structure is equal to the internal work or the strain energy, U_i, stored in the structure.

$$U_e = U_i$$

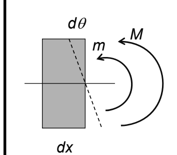
- Both shear and moment contribute to the deformation of beams.
- Typically, the effects of bending on deformation are much more significant than the effects of shear

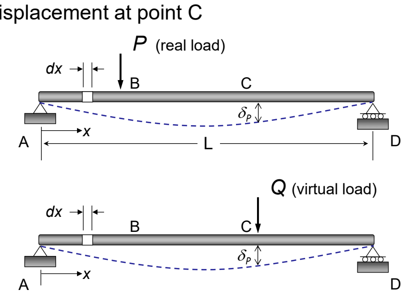
Deflections

- The procedure to compute a deflection component of a beam is like that for a truss
- Begin by applying a unit virtual load ***Q*** at the point where the deflection is to be computed
- Apply a unit couple at the point where the slope is to be computed

Deflections

Let's examine the following beam and use virtual work to compute the displacement at point C



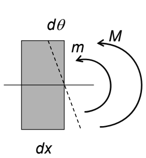


M – moment due to real loads P

m – moment due to virtual loads Q

Deflections

The change in slope due to the *P* system is: $d\theta = \frac{M}{EI}dx$



The external work done by the virtual force W_Q moving through the distance δ_P is:

$$W_Q = \sum Q\delta_P$$

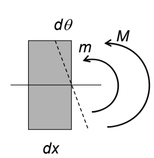
The virtual strain energy dU_Q stored in each element as the moment *m* moves through angle $d\theta$ is:

$$dU_Q = m\,d\theta$$

Deflections

To find the magnitude of dU_Q we must sum – or integrate – the energy:

$$U_Q = \int_{x=0}^{x=L} m\,d\theta$$



The principle of conservation of energy states the external virtual work W_Q equals the virtual strain energy U_Q:

$$\sum Q\delta_P = \int_{x=0}^{x=L} m\,d\theta$$

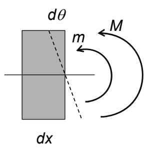
The $d\theta$ is known in terms of *M*, then the virtual work expression is:

$$\sum Q\delta_P = \int_{x=0}^{x=L} \frac{mM}{EI}dx$$

Deflections

Since the virtual load is a unit load the previous expression reduces to:

$$\delta_P = \int_{x=0}^{x=L} \frac{mM}{EI} dx$$



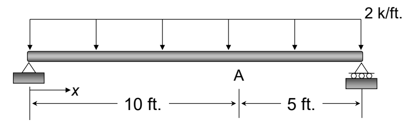
If the external work is done by a virtual moment M_Q moving through a slope θ_P, then the virtual work equation is:

$$\theta_P = \int_{x=0}^{x=L} \frac{m_\theta M}{EI} dx$$

Where virtual moment m_θ is caused by a virtual couple.

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 *in*^{4} and E = 29(10^3) *ksi*).



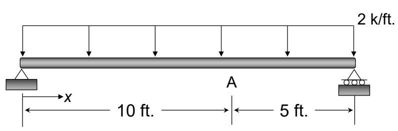
The first step is to find the M expression for the real forces. Integrating the loading function, we can get the shear force equation:

$$V(x) = \int w(x)dx = -\int 2\,dx = -2x + C_1 \qquad \begin{bmatrix} V(x=0) = 15k \\ V(x=15) = -15k \end{bmatrix}$$

$$V(x) = \left[15 - 2x\right]k$$

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 *in*^{4} and E = 29(10^3) *ksi*).



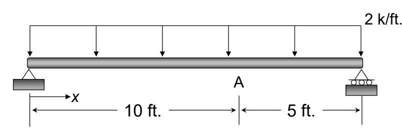
The next step is to find the M equation from the shear equation.

$$M(x) = \int V(x)dx = \int (15 - 2x)dx = 15x - x^2 + C_2 \qquad \begin{bmatrix} M(x=0) = 0 \\ M(x=15) = 0 \end{bmatrix}$$

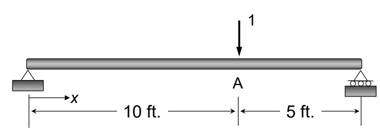
$$M(x) = \left[15x - x^2\right] \text{k ft.}$$

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 *in*^{4} and E = 29(10^3) *ksi*).



The next step is to find the virtual moment equation.



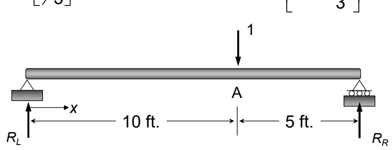
Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 *in*^{4} and E = 29(10^3) *ksi*).

$$\circlearrowleft^{+} \sum M_L = 0 = -1(10\text{ ft.}) + R_R(15\text{ ft.}) \Rightarrow R_R = \tfrac{2}{3} \Rightarrow R_L = \tfrac{1}{3}$$

Using the method of section, the virtual moment expressions are:

$$m(x) = \left[\tfrac{x}{3}\right]\text{ft.} \quad 0 \le x \le 10 \qquad m(x) = \left[10 - \frac{2x}{3}\right]\text{ft.} \quad 10 \le x \le 15$$



Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 *in*^{4} and E = 29(10^3) *ksi*).

Since the moment due to the virtual load is discontinuous, we have to break the integration up into two parts.

$$\delta_A = \int_0^{10} \frac{mM}{EI} dx + \int_{10}^{15} \frac{mM}{EI} dx$$

Substituting the moment expression into the virtual work equation and integrating yields the following:

$$\delta_A = \int_0^{10} \frac{x(15x - x^2)}{3EI} dx + \int_{10}^{15} \frac{(30 - 2x)(15x - x^2)}{3EI} dx$$

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 in^4 and E = 29(10^3) ksi).

$$\delta_A = \int_0^{10} \frac{x(15x - x^2)}{3EI} dx + \int_{10}^{15} \frac{(30 - 2x)(15x - x^2)}{3EI} dx$$

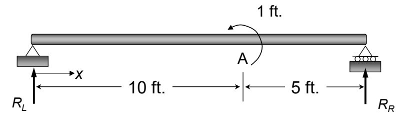
$$\delta_A = \frac{20x^3 - x^4}{12EI}\bigg|_0^{10} + \frac{900x^2 - 80x^3 + 2x^4}{12EI}\bigg|_{10}^{15}$$

$$\delta_A = \frac{(10{,}000 + 3{,}750)\,\text{k ft}^3}{12EI} = \frac{13{,}750\text{k ft}^3}{12EI}$$

$$\delta_A = \frac{13{,}750\text{k ft}^3}{12(29{,}000ksi)(1{,}000in^4)} \cdot \frac{1{,}728\text{in}^3}{\text{ft}^3} = 0.068 \text{ in.}$$

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 in^4 and E = 29(10^3) ksi).



$$\circlearrowleft^{+} \sum M_L = 0 = 1ft + R_R(15ft) \;\Rightarrow\; R_R = -\tfrac{1}{15} \;\Rightarrow\; R_L = \tfrac{1}{15}$$

Using the method of section, the virtual moment expressions are:

$$m_\theta(x) = \left[\tfrac{x}{15}\right]\text{ft} \quad 0 \le x \le 10 \qquad m_\theta(x) = \left[\frac{x - 15}{15}\right]\text{ft} \quad 10 \le x \le 15$$

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 in^4 and E = 29(10^3) ksi).

Since the moment due to the virtual couple is discontinuous, we have to break the integration into two parts.

$$\theta_A = \int_0^{10} \frac{m_\theta M}{EI} dx + \int_{10}^{15} \frac{m_\theta M}{EI} dx$$

Substituting the moment expression into the virtual work equation and integrating yields the following:

$$\theta_A = \int_0^{10} \frac{x(15x - x^2)}{15EI} dx + \int_{10}^{15} \frac{(x - 15)(15x - x^2)}{15EI} dx$$

Deflections

Example: Determine the displacement and slope at point A on the beam (I = 1,000 in^4 and E = 29(10^3) ksi).

$$\theta_A = \int_0^{10} \frac{x(15x - x^2)}{15EI} dx + \int_{10}^{15} \frac{(x - 15)(15x - x^2)}{15EI} dx$$

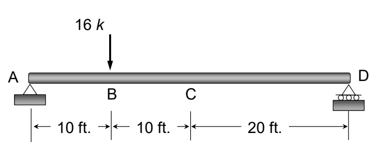
$$\theta_A = \frac{20x^3 - x^4}{60EI}\bigg|_0^{10} + \frac{-450x^2 + 40x^3 - x^4}{60EI}\bigg|_{10}^{15}$$

$$\theta_A = \frac{(10{,}000 - 1{,}875)\,\text{k ft}^2}{60EI} = \frac{8{,}125\text{k ft}^2}{60EI}$$

$$\theta_A = \frac{8{,}125\text{k ft}^2}{60(29{,}000ksi)(1{,}000in^4)} \cdot \frac{144\text{in}^2}{\text{ft}^2} = 0.0007 \textit{ radians}$$

Deflections

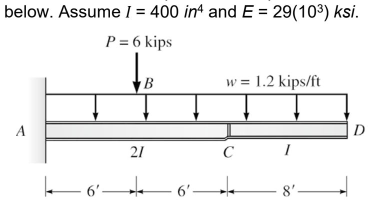
Example: Determine the displacement at point C on the beam shown below. Assume I = 240 in^4 and E = 29(10^3) ksi.



Notice that this beam must be divided into three sections to accommodate the real and virtual moment expressions

Deflections

Example: Determine the displacement at points *D* on the beam shown below. Assume I = 400 in^4 and E = 29(10^3) ksi.



Notice that this beam must be divided into three sections to accommodate the real and virtual moment expressions and the variation in the moment of inertia

End of Virtual Work - Beams

Any questions?