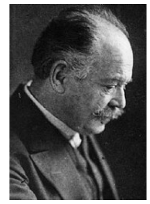
Deflections

Conjugate-Beam Method

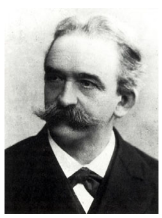
- The development of the conjugate beam method has been attributed to several structural engineers.
- Many credit **Heinrich Müller-Breslau** (1851-1925) for developing this method in 1865, while others say it was developed by **Christian Otto Mohr** (1835-1918).

Deflections



- Heinrich Franz Bernhard Müller was born in Wroclaw (Breslau) on 13 May 1851.
- In 1875, he opened a civil engineer's office in Berlin. Around this time, he decided to add the name of his hometown to his surname, becoming known as Müller-Breslau
- He founded the so-called "Berlin School" of structural theory.

Deflections



- Christian Otto Mohr was an enthusiast for graphical tools and developed the method for visually representing stress in 3D, known as Mohr's Circle.
- He also developed methods for truss displacements and for analyzing statically indeterminate structures.
- He founded the so-called “Dresden School" of applied mechanics and disagreed with Müller-Breslau throughout their careers.

Deflections

Conjugate-Beam Method

- A conjugate beam is an imaginary beam with the same dimensions (length) as the original beam.
- The load at any point on the conjugate beam is equal to the bending moment ***M*** at that point divided by *EI*.
- The conjugate-beam method is an engineering method to derive the slope and displacement of a beam.
- This method relies only on the principles of statics, so its application will be more familiar.

Deflections

The method is based on the similarity between the relationships for loading and shear, as well as shear and moment.

$$\frac{dV}{dx} = w(x) \qquad \frac{dM}{dx} = V \quad \Rightarrow \frac{d^2M}{dx^2} = w$$

$$V = \int w(x)dx \qquad M = \int\int w(x)dx\,dx$$

Deflections

- The previous expressions relate the internal shear and moment to the applied load.
- The slope and deflection of the elastic curve are related to the internal moment by the following expressions

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\theta = \int \frac{M}{EI}dx \qquad y = \int\int \frac{M}{EI}dx\,dx$$

Deflections

- Let's compare expressions for shear, ***V***, and the slope, θ

$$\frac{dV}{dx} = w \qquad \frac{d\theta}{dx} = \frac{M}{EI}$$

- What do you see?
- If you replace ***w*** with the term ***M/EI***, the expressions for shear force and slope are identical

Deflections

- Let's compare expressions for bending moment, ***M***, and the displacement, ***y***

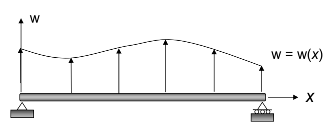
$$\frac{d^2M}{dx^2} = w \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

- What do you see?
- Just as before, if you replace ***w*** with the term ***M/EI***, the expressions for bending moment and displacement are identical

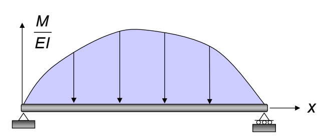
Deflections

- We will use this relationship to our advantage by constructing a beam of the same length as the real beam, referred to as the ***conjugate beam***.
- The conjugate beam is loaded with the ***M/EI*** diagram, simulating the external load ***w***.

Deflections



Real beam with applied loading. Determine the bending moment (draw the bending moment diagram)



Conjugate beam where the applied loading is bending moment from the real beam

Note the sign of loading ***w*** and the ***M/EI*** on the conjugate beam.

Deflections

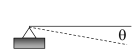
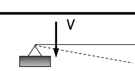
Therefore, the two theorems related to the conjugate beam method are:

- **Theorem 1: The slope at a point in the actual beam equals the shear at the corresponding point in the conjugate beam.**
- **Theorem 2: The displacement of a point in the real beam is equal to the moment at the corresponding point in the conjugate beam.**

Deflections

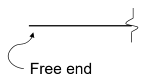
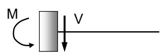
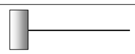
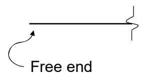
Conjugate-Beam Supports

When the conjugate beam is drawn, it is important that the shear and moment developed in it correspond to the slope and displacement conditions in the real beam.

Real Support	Conjugate Support
	
Pin or roller	Pin or roller
$\Delta = 0 \quad \theta \neq 0$	$M = 0 \quad V \neq 0$

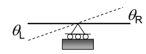
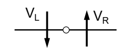
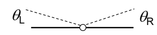
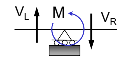
Deflections

Conjugate-Beam Supports

Real Support	Conjugate Support
 Free end $\Delta \neq 0$ $\theta \neq 0$	 Fixed end $M \neq 0$ $V \neq 0$
 Fixed end $\Delta = 0$ $\theta = 0$	 Free end $M = 0$ $V = 0$

Deflections

Conjugate-Beam Supports

Real Support	Conjugate Support
 Interior support $\Delta = 0$ $\theta_L = \theta_R \neq 0$	 Hinge $M = 0$ $V_L = V_R \neq 0$
 Hinge $\Delta \neq 0$ θ_L and θ_R may have different values	 Interior roller $M \neq 0$ V_L and V_R may have different values

Deflections

- As a rule, statically determinant real beams have statically determinant conjugate beams, and statically indeterminate beams become unstable conjugate beams.
- However, the ***M/EI*** loading may provide the necessary "equilibrium" to stabilize the conjugate beam.

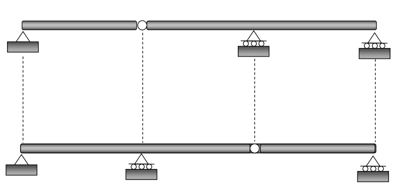
Deflections

Procedure for analysis

1. Construct the conjugate beam with the ***M/EI*** loading.
   Remember that when the ***M/EI*** diagram is positive, the loading is *upward*, and when it is negative, it is *downward*.
2. Use the equilibrium equations to solve for the reactions of the conjugate beam.
   This may be challenging if the moment diagram is complex.
3. Solve for the shear and moment at the point where the slope and displacement are desired.
   If the values are positive, the slope is counterclockwise, and the displacement is upward.

Deflections

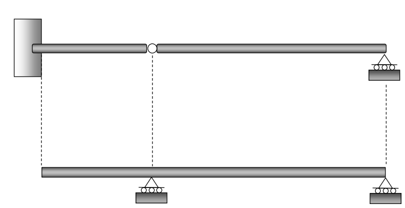
Draw the conjugate beam, including supports, for the following beams



Conjugate beam and supports

Deflections

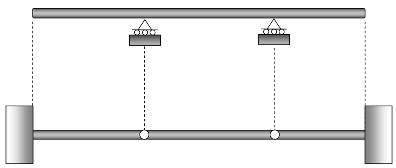
Draw the conjugate beam, including supports, for the following beams



Conjugate beam and supports

Deflections

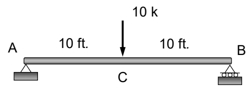
Draw the conjugate beam, including supports, for the following beams

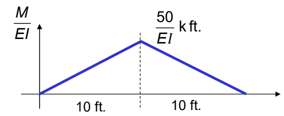


Conjugate beam and supports

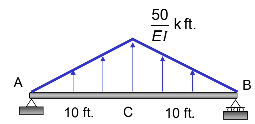
Deflections

Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 ksi and I = 300 in^4.





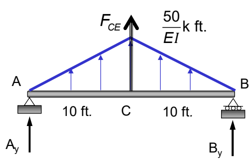
Construct the conjugate beam and apply the ***M/EI*** diagram as loading



Remember that a positive (+) bending moment is a positive (+) loading on the conjugate beam.

Deflections

Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 ksi and I = 300 in^4.



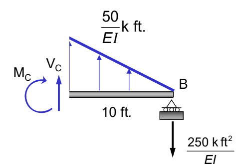
$$F_{CE} = \frac{1}{2}\left(\frac{50\text{k ft.}}{EI}\right)(20\text{ft.})$$

$$\circlearrowleft^{+} \sum M_A = 0 = \frac{1}{2}\left(\frac{50\text{k ft.}}{EI}\right)(20\text{ft.})(10\text{ft.}) + B_y(20\text{ft.}) \qquad B_y = -\frac{250\text{ k ft}^2}{EI}$$

$$^{+}\uparrow \sum F_y = 0 = \frac{1}{2}\left(\frac{50\text{k ft.}}{EI}\right)(20\text{ft.}) + B_y + A_y \qquad A_y = -\frac{250\text{ k ft}^2}{EI}$$

Deflections

Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 ksi and I = 300 in^4.

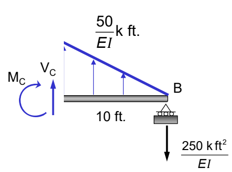


$$\circlearrowleft^{+} \sum M_C = 0 = -M_C + \frac{1}{2}\left(\frac{50\text{ k ft}}{EI}\right)(10\text{ft.})\left(\frac{10\text{ft.}}{3}\right) - \left(\frac{250\text{ k ft}^2}{EI}\right)(10\text{ft.})$$

$$M_C = -\frac{1{,}666.6\text{ k ft}^3}{EI} = -\frac{1{,}666.6\text{ k ft}^3}{(30{,}000\text{ksi})(300\text{in}^4)} \cdot \frac{1{,}728\text{ in}^3}{\text{ft}^3} = \boxed{-0.32\text{in.}}$$

Deflections

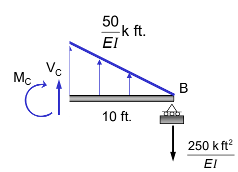
Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 ksi and I = 300 in^4.



$$^{+}\uparrow \sum F_y = 0 = \frac{1}{2}\left(\frac{50\text{ k ft.}}{EI}\right)(10\text{ft.}) - \left(\frac{250\text{ k ft}^2}{EI}\right) + V_C \qquad V_C = 0$$

Deflections

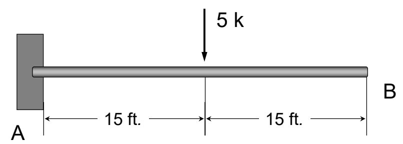
Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 ksi and I = 300 in^4.



- Therefore, the displacement of the beam at point C is equal to the moment at point C on the conjugate beam, and the slope is equal to the shear in the conjugate beam.
- In this problem, the displacement at point C is -0.32 in., and the slope is zero.

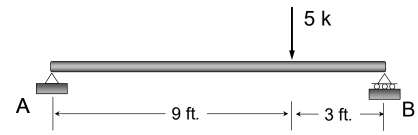
Deflections

Example: Determine the slope and the displacement at point B for the following beam. Assume that *E* = 29,000 ksi and *I* = 800 in^4.



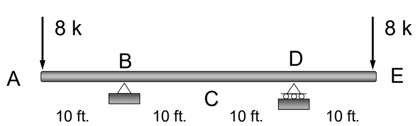
Deflections

Example: Determine the maximum displacement at the mid-span of the following beam. Assume that *E* = 30,000 ksi and *I* = 800 in^4.



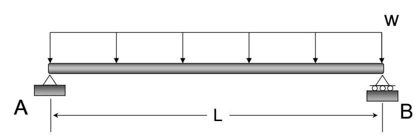
Deflections

Example: Determine the slope at point B and the dis-placement at point E for the following beam. Assume that *E* = 29,000 ksi and $I_{AB} = I_{DE}$ = 400 in^4, and I_{BD} = 800 in^4.



Deflections

Example: Determine the slope at A and the displacement at mid-span. Assume that *EI* is a constant.



Are there any disadvantages to the conjugate beam method for uniform or high-order loading functions?

End of Defections – Part 2

Any questions?