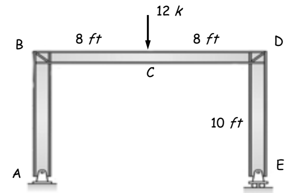
Virtual Work for Frames

- Applying the virtual work equations to a frame structure is as simple as separating the frame into a series of "beams" and summing the virtual work for each section.
- In addition, when evaluating the deformation of a frame structure, you may have to consider both bending and axial internal force components.

Virtual Work for Frames

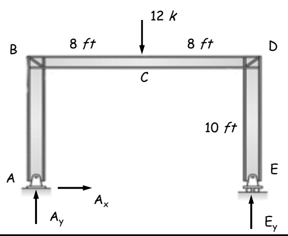
- Compute the the ***deflection*** at point *C* on the frame shown below. Include only the effects of bending in your virtual work equation (do not consider axial work).



E = 29,000 *ksi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

- The first step is to find the equation for moment in each section of the frame due to the ***real loads***.
- To do develop the moment expression we need the reaction a points A and E.



$$\sum M_A = 0 = -12k(8ft) + E_y(16ft)$$

$$E_y = 6k$$

$$\sum F_y = 0 = -12k + E_y + A_y$$

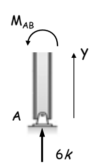
$$A_y = 6k$$

$$\sum F_x = 0 = A_x$$

Virtual Work for Frames

- The next step is to find the equation for moment in each section of the frame.
- Consider section AB

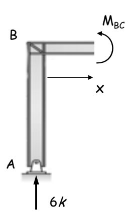
$$\sum M_{cut} = 0 = M_{AB} \qquad M_{AB} = 0$$



Virtual Work for Frames

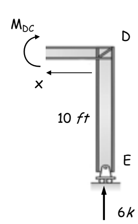
- Consider section BC

$$\sum M_{cut} = 0 = M_{BC} - 6k(x) \qquad M_{BC} = 6x$$



Virtual Work for Frames

- Consider section DC



$$\sum M_{cut} = 0 = -M_{DC} + 6k(x)$$

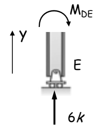
$$M_{DC} = 6x$$

Virtual Work for Frames

- Consider section ED

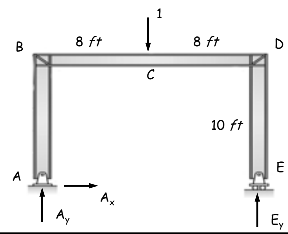
$$\sum M_{cut} = 0 = -M_{DE}$$

$$\boxed{M_{DE} = 0}$$



Virtual Work for Frames

- The next step is to find the equation for moment in each section of the frame due to the ***virtual load***.
- To do develop the moment expression we need the reaction a points *A* and E.



$$\sum M_A = 0 = -1(8ft) + E_y(16ft)$$

$$\boxed{E_y = 0.5}$$

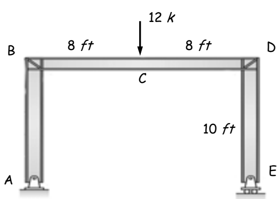
$$\sum F_y = 0 = -1 + E_y + A_y$$

$$\boxed{A_y = 0.5}$$

$$\sum F_x = 0 = A_x$$

Virtual Work for Frames

- In this problem, the virtual moments are the real moments divided by 12 (from superposition).



Section	M	m
AB	0	0
BC	6x	x/2
DC	6x	x/2
DE	0	0

Virtual Work for Frames

- The virtual work equations are:

$$\Delta_C = \int_A^B \frac{M_{AB} m_{AB}}{EI} dy + \int_B^C \frac{M_{BC} m_{BC}}{EI} dx + \int_D^C \frac{M_{DC} m_{DC}}{EI} dx + \int_D^E \frac{M_{DE} m_{DE}}{EI} dy$$

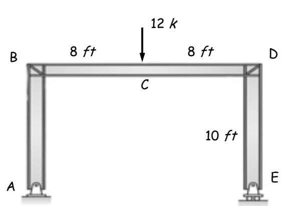
- Substituting the moment expressions into the virtual work equation and integrating yields the following:

$$\Delta_C = \int_0^8 \frac{(6x)x}{2EI} dx + \int_0^8 \frac{(6x)x}{2EI} dx = \int_0^8 \frac{6x^2}{EI} dx = \left.\frac{2x^3}{EI}\right|_0^8$$

$$= \frac{1,024\ kft^3}{EI} = \frac{1,024\ kft^3 (1,728 in^3/ft^3)}{(29,000 ksi)(3,500 in^4)} = \boxed{0.017 in}$$

Virtual Work for Frames

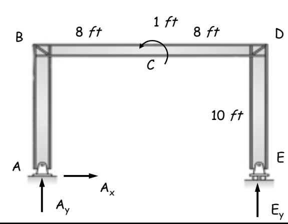
- Compute the the ***slope*** at point *C* on the frame shown below. Include only the effects of bending in your virtual work equation (do not consider axial work).



E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

- Find the moments in the frame due to a virtual couple
- First, find the reaction in the frame to the virtual couple



$$\sum M_A = 0 = 1ft + E_y(16ft)$$

$$\boxed{E_y = -\tfrac{1}{16}}$$

$$\sum F_y = 0 = E_y + A_y$$

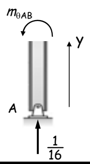
$$\boxed{A_y = \tfrac{1}{16}}$$

$$\sum F_x = 0 = A_x$$

Virtual Work for Frames

- The next step is to find the equation for moment in each section of the frame.
- Consider section AB

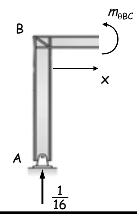
$$\sum M_{cut} = 0 = m_{\theta AB} \qquad \boxed{m_{\theta AB} = 0}$$



Virtual Work for Frames

- Consider section BC

$$\sum M_{cut} = 0 = m_{\theta BC} - \tfrac{1}{16}(x) \qquad \boxed{m_{\theta BC} = \frac{x}{16}}$$

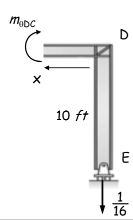


Virtual Work for Frames

- Consider section DC

$$\sum M_{cut} = 0 = -m_{\theta DC} - \tfrac{1}{16}(x)$$

$$\boxed{m_{\theta DC} = -\frac{x}{16}}$$

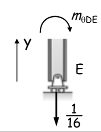


Virtual Work for Frames

- Consider section ED

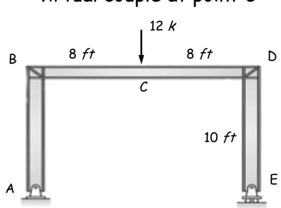
$$\sum M_{cut} = 0 = -m_{\theta DE}$$

$$\boxed{m_{\theta DE} = 0}$$



Virtual Work for Frames

- The following table lists the moment expression due to the real loading and the moment expression due to a virtual couple at point C



Section	M	m_θ
AB	0	0
BC	6x	x/16
DC	6x	-x/16
DE	0	0

Virtual Work for Frames

- The virtual work equations are:

$$\Delta_C = \int_A^B \frac{M_{AB} m_{\theta AB}}{EI} dy + \int_B^C \frac{M_{BC} m_{\theta BC}}{EI} dx + \int_D^C \frac{M_{DC} m_{\theta DC}}{EI} dx + \int_E^D \frac{M_{DE} m_{\theta DE}}{EI} dy$$

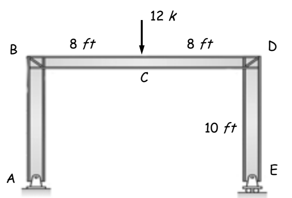
- Substituting the moment expressions into the virtual work equation and integrating yields the following:

$$\Delta_C = \int_0^8 \frac{(6x)x}{16EI} dx - \int_0^8 \frac{(6x)x}{16EI} dx = 0$$

- The slope at point C is zero

Virtual Work for Frames

- Repeat the previous example and include the effects of axial work.
- In order to compute the axial work, we need the axial force in the real and virtual loading systems

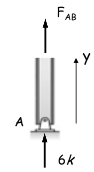


E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

- Find the axial force in each section of the frame.
- Consider section AB

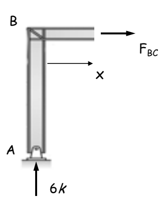
$$\sum F_y = 0 = F_{AB} + 6k \qquad F_{AB} = -6k$$



Virtual Work for Frames

- Consider section BC

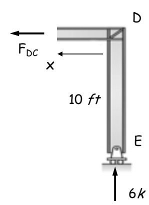
$$\sum F_x = 0 = F_{BC} \qquad F_{BC} = 0$$



Virtual Work for Frames

- Consider section DC

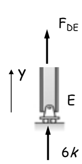
$$\sum F_x = 0 = -F_{DC} \qquad F_{DC} = 0$$



Virtual Work for Frames

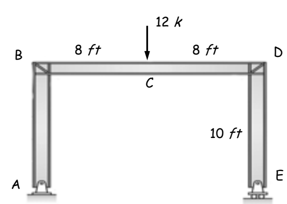
- Consider section ED

$$\sum F_y = 0 = F_{DE} + 6k \qquad F_{DE} = -6k$$



Virtual Work for Frames

- In this problem, the virtual axial forces are the real axial forces divided by 12 (from superposition).



Section	*N*	*n*
AB	-6*k*	-0.5
BC	0	0
DC	0	0
DE	-6*k*	-0.5

Virtual Work for Frames

- The virtual work equations for axial forces are:

$$\Delta_C = \sum \frac{nNL}{AE}$$

- Substituting the values for the axial forces into the virtual work equations yields the following:

$$\Delta_C = \frac{(-0.5)(-6k)(120in)}{AE} + \frac{(-0.5)(-6k)(120in)}{AE}$$

$$= \frac{720\ k\ in}{AE} = \frac{720\ k\ in}{(29{,}000 ksi)(35 in^2)} = \boxed{0.0007 in}$$

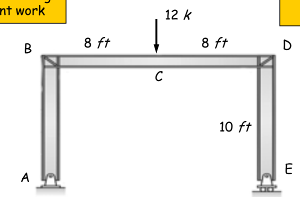
Virtual Work for Frames

- The displacement at point C due to bending moment work and axial force work is:

$$\Delta_C = 0.017in + 0.0007in = \boxed{0.0177in}$$

Δ from bending moment work

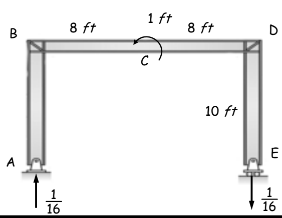
Δ from axial force work



E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

- Compute the axial forces in the frame due to the virtual couple.
- Recall we already have the frame reactions due to the virtual couple

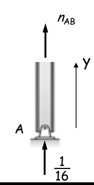


E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

- The next step is to find the axial force in each section of the frame.
- Consider section AB

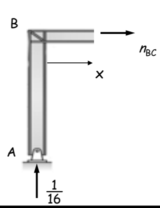
$$\sum F_y = 0 = n_{AB} + \tfrac{1}{16} \qquad \boxed{n_{AB} = -\tfrac{1}{16}}$$



Virtual Work for Frames

- Consider section BC

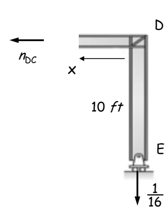
$$\sum F_x = 0 = n_{BC} \qquad \boxed{n_{BC} = 0}$$



Virtual Work for Frames

- Consider section DC

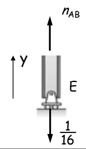
$$\sum F_x = 0 = -n_{DC} \qquad \boxed{n_{DC} = 0}$$



Virtual Work for Frames

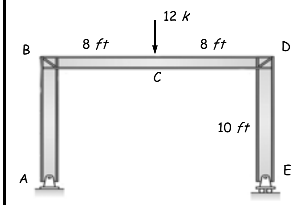
- Consider section ED

$$\sum F_y = 0 = n_{DE} - \tfrac{1}{16} \qquad n_{DE} = \tfrac{1}{16}$$



Virtual Work for Frames

- The real axial forces and the virtual axial forces due to a unit virtual couple are:



Section	N	n_θ
AB	-6*k*	-1/16
BC	0	0
DC	0	0
DE	-6*k*	1/16

Virtual Work for Frames

- The virtual work equations for axial forces are:

$$\Delta_C = \sum \frac{nNL}{AE}$$

- Substituting the values for the axial forces into the virtual work equations yields the following:

$$\Delta_C = \frac{(6k)(120in)}{16AE} - \frac{(6k)(120in)}{16AE} = 0$$

- The contribution to the slope at point *C* from the axial energy is slope is zero.
- The total slope at point *C* due to bending moment and axial force work is zero.

Virtual Work for Frames

- In problems involving both bending and axial deformation, be careful with the units.
- Also, note that the contribution of the axial deformation is 5% of the total deformation.
- This is more or less typical of the relative size of the bending and the axial effects in frame-deflection problems.
- Therefore, it is usually permissible to neglect the effect of axial deformation in such cases.

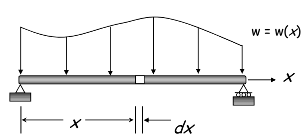
Virtual Work for Frames

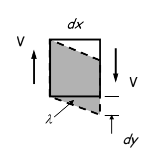
- The primary cause of deformation in beams and frames is due to ***bending strain***.
- However, in some structures additional deformation due to ***axial*** and ***shear forces***, ***torsion***, and perhaps ***temperature*** may be important.
- We have already discussed deformation due to bending moments and axial forces.
- In this section, we will consider the effect of shear, torsion, and temperature on the deformation of linear elastic structures.

Virtual Work for Frames

Virtual Strain Energy From Shear

- Consider the following beam and a small element *dx*

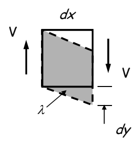




Virtual Work for Frames

Virtual Strain Energy From Shear

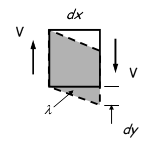
- The shearing deformation *dy* caused by the real loads is $dy = \lambda dx$, where λ is the shear strain.
- Since we are assuming the material is linear and elastic, then Hooke's law applies
- The shear strain is related to the shear stress by $\lambda = \tau/G$, where τ is the shear stress and *G* is the shearing modulus of elasticity.



Virtual Work for Frames

Virtual Strain Energy From Shear

- The shear stress may be calculated as $\tau = K(V/A)dx$, where *K* is a form factor that depends of the shape of the beam's cross-sectional area *A*.
- Combining these two expressions gives $dy = KV/(GA)\ dy$.
- The internal virtual work done by the virtual shear force *v* acting on the beam before it is deformed by the real loads is $dU_i = v\ dy$



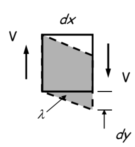
Virtual Work for Frames

Virtual Strain Energy From Shear

- Integrating the expression $dU_i = v\ dy$ over the entire beam gives:

$$U_{shear} = \int_0^L K\left(\frac{vV}{GA}\right)dx$$

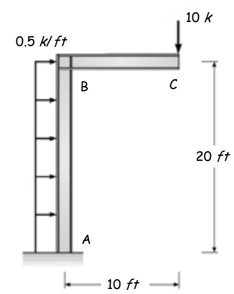
- Remember that *v* is the shear due to the virtual load and *V* is the shear due to the real loads.



Virtual Work for Frames

- Compute the vertical ***deflection*** and rotation at point *C* on the frame shown.
- Include the effects of bending and both axial and shear force in your virtual work equations.

E = 29,000 *ksi*
G = 12,000 *ksi*
I = 1,000 in^4
A = 25 in^2
K = 1.2



End of Virtual Work - Frames

Any questions?