

Deflections

- The geometric methods previously discussed are very good and quite straight-forward for simple loadings.
- However, these methods are very tedious for complex loadings.
- In cases like this, an **energy method** is the preferred technique.
- Energy methods are based on the principle of conservation of energy.

Deflections

- This principle states that the work done by all the external forces, U_e , acting on a structure is equal to the internal work or the strain energy, U_i , stored in the structure.

$$U_e = U_i$$

- Both shear and moment contribute to the deformation of beams.
- Typically, the effects of bending on deformation is much more significant than effects of shear

Deflections

- The procedure to compute a deflection component of a beam is similar to that for a truss
- Begin by applying a unit virtual load Q at the point where the deflection is to be computed
- Apply a unit couple at the point where slope is to be computed

Deflections

- Let's examine the following beam and use virtual work to compute the displacement at point C

M - moment due to real loads P
 m - moment due to virtual loads Q

Deflections

The change in slope due to the P system is:

$$d\theta = \frac{M}{EI} dx$$

The external work done by the virtual force W_Q moving through the distance δ_p is:

$$W_Q = \sum Q\delta_p$$

The virtual strain energy dU_Q stored in each element as the moment m moves through angle $d\theta$ is:

$$dU_Q = m d\theta$$

Deflections

The find the magnitude of dU_Q we must sum - or integrate - the energy:

$$U_Q = \int_{x=0}^{x=L} m d\theta$$

The principle of conservation of energy states the the external virtual work W_Q equals the virtual strain energy U_Q :

$$\sum Q\delta_p = \int_{x=0}^{x=L} m d\theta$$

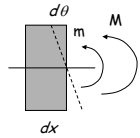
The $d\theta$ is known in terms of M , then the virtual work expression is:

$$\sum Q\delta_p = \int_{x=0}^{x=L} \frac{mM}{EI} dx$$

Deflections

Since the virtual load is a unit load the previous expression reduces to:

$$\delta_p = \int_{x=0}^{x=L} \frac{mM}{EI} dx$$



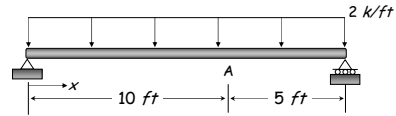
If the external work is done by a virtual moment M_Q moving through a slope θ_p , then the virtual work equation is:

$$\theta_p = \int_{x=0}^{x=L} \frac{m_0 M}{EI} dx$$

Where virtual moment m_0 is caused by a virtual couple

Deflections

- Example: Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).



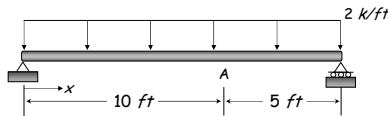
- The first step is to find the M expression for the real forces. Integrating the loading function we can get the shear force equation

$$V(x) = -\int w(x) dx = -\int 2 dx = -2x + C_1 \quad \begin{cases} V(0) = 15k \\ V(15) = -15k \end{cases}$$

$$V(x) = [15 - 2x]k$$

Deflections

- Example: Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).



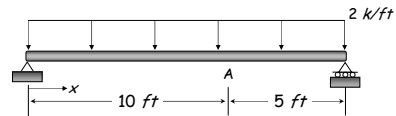
- The next step is to find the M equation from the shear equation

$$M(x) = \int V(x) dx = \int (15 - 2x) dx = 15x - x^2 + C_2 \quad \begin{cases} M(0) = 0 \\ M(15) = 0 \end{cases}$$

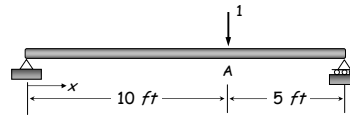
$$M(x) = [15x - x^2] kft$$

Deflections

- Example: Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).



- The next step is to find the virtual moment equation



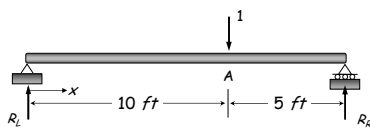
Deflections

- Example: Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).

$$\sum M_L = 0 = -1(10ft) + R_R(15ft) \Rightarrow R_R = \frac{2}{3} \Rightarrow R_L = \frac{1}{3}$$

- Using the method of section the virtual moment expressions are:

$$m(x) = \left[\frac{x}{3}\right] ft \quad 0 \leq x \leq 10 \quad m(x) = \left[10 - \frac{2x}{3}\right] ft \quad 10 \leq x \leq 15$$



Deflections

- Example: Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).

- Since the moment due to the virtual load is discontinuous, we have to break the integration up into two parts.

$$\delta_A = \int_0^{10} \frac{mM}{EI} dx + \int_{10}^{15} \frac{mM}{EI} dx$$

- Substituting the moment expression into the virtual work equation and integrating yields the following:

$$\delta_A = \int_0^{10} \frac{x(15x - x^2)}{3EI} dx + \int_{10}^{15} \frac{(30 - 2x)(15x - x^2)}{3EI} dx$$

Deflections

- Example:** Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).

$$\delta_A = \int_0^{10} \frac{x(15x - x^2)}{3EI} dx + \int_{10}^{15} \frac{(30 - 2x)(15x - x^2)}{3EI} dx$$

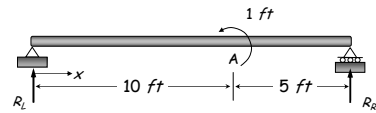
$$\delta_A = \frac{20x^3 - x^4}{12EI} \Big|_0^{10} + \frac{900x^2 - 80x^3 + 2x^4}{12EI} \Big|_{10}^{15}$$

$$\delta_A = \frac{(10,000 + 3,750) k ft^3}{12EI} = \frac{13,750 k ft^3}{12EI}$$

$$\delta_A = \frac{13,750 k ft^3}{12(29,000 \text{ ksi})(1,000 \text{ in}^4)} \cdot \frac{1,728 \text{ in}^3}{\text{ft}^3} = 0.068 \text{ in}$$

Deflections

- Example:** Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).



$$\sum M_L = 0 = 1ft + R_R(15ft) \Rightarrow R_R = -\frac{1}{15} \Rightarrow R_L = \frac{1}{15}$$

- Using the method of section the virtual moment expressions are:

$$m(x) = \left[\frac{x}{15} \right] ft \quad 0 \leq x \leq 10 \quad m(x) = \left[\frac{x-15}{15} \right] ft \quad 10 \leq x \leq 15$$

Deflections

- Example:** Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).
- Since the moment due to the virtual couple is discontinuous, we have to break the integration up into two parts.

$$\theta_A = \int_0^{10} \frac{m_v M}{EI} dx + \int_{10}^{15} \frac{m_v M}{EI} dx$$

- Substituting the moment expression into the virtual work equation and integrating yields the following:

$$\theta_A = \int_0^{10} \frac{x(15x - x^2)}{15EI} dx + \int_{10}^{15} \frac{(x-15)(15x - x^2)}{15EI} dx$$

Deflections

- Example:** Determine the displacement and slope at point A on the beam ($I = 1,000 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).

$$\theta_A = \int_0^{10} \frac{x(15x - x^2)}{15EI} dx + \int_{10}^{15} \frac{(x-15)(15x - x^2)}{15EI} dx$$

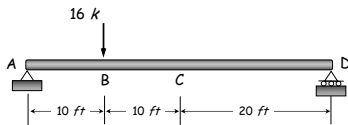
$$\theta_A = \frac{20x^3 - x^4}{60EI} \Big|_0^{10} + \frac{-450x^2 + 40x^3 - x^4}{60EI} \Big|_{10}^{15}$$

$$\theta_A = \frac{(10,000 - 1,875) k ft^3}{60EI} = \frac{8,150 k ft^3}{60EI}$$

$$\theta_A = \frac{8,125 k ft^3}{60(29,000 \text{ ksi})(1,000 \text{ in}^4)} \cdot \frac{144 \text{ in}^3}{\text{ft}^3} = 0.0007 \text{ radians}$$

Deflections

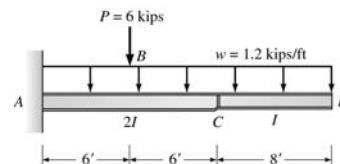
- Example:** Determine the displacement at point C on the beam shown below ($I = 240 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).



- Notice that this beam must be divided into three sections to accommodate the real and virtual moment expressions

Deflections

- Example:** Determine the displacement at points C and D on the beam shown below ($I = 400 \text{ in}^4$, and $E = 29(10^3) \text{ ksi}$).



- Notice that this beam must be divided into three sections to accommodate the real and virtual moment expressions and the variation in the moment of inertia

End of Virtual Work - Beams

Any questions?

