Virtual Work for Frames

- Applying the virtual work equations to a frame structure is as simple as separating the frame into a series of “beams” and summing the virtual work for each section.
- In addition, when evaluating the deformation of a frame structure, you may have to consider both bending and axial internal force components.

Virtual Work for Frames

The first step is to find the equation for moment in each section of the frame due to the real loads.

To do develop the moment expression we need the reaction a points A and E.

\[ \sum \Delta M_A = 0 = 12k(8ft) + E_f(16ft) \]

Virtual Work for Frames

Consider section AB

\[ \sum \Delta M_A = 0 = M_{AB} - 6k \]

Virtual Work for Frames

Consider section BC

\[ \sum \Delta M_A = 0 = M_{BC} - 6k(x) \]

Virtual Work for Frames

Consider section DC

\[ \sum \Delta M_A = 0 = M_{DC} - 6k(x) \]

Virtual Work for Frames

Compute the deflection at point C on the frame shown below.

Include only the effects of bending in your virtual work equation (no axial work).
Virtual Work for Frames

Consider section ED

\[ \sum M_{cut} = -M_{OC} \]

\[ M_{OC} = 0 \]

Virtual Work for Frames

In this problem, the virtual moments are the real moments divided by 12 (from superposition).

<table>
<thead>
<tr>
<th>Section</th>
<th>M</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>6x</td>
<td>0.5x</td>
</tr>
<tr>
<td>DC</td>
<td>6x</td>
<td>0.5x</td>
</tr>
<tr>
<td>DE</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Virtual Work for Frames

The virtual work equations are:

\[ \Delta_C = \int M_{ab}m_{ab}dy + \int M_{bc}m_{bc}dx + \int M_{cc}m_{cc}dx + \int M_{de}m_{de}dy \]

Substituting the moment expressions into the virtual work equation and integrating yields the following:

\[ \Delta_C = \frac{1.024 \text{kft}^2}{EI} + \frac{1.024 \text{kft}^2 (1.728 \text{in}^2 / \text{ft}^3)}{(29,000 \text{ksi})(3,500 \text{in}^4)} = 0.017 \text{in} \]

Virtual Work for Frames

Compute the slope at point C on the frame shown below. Include only the effects of bending in your virtual work equation (no axial work).

\[ E = 29,000 \text{ psi} \]
\[ I = 3,500 \text{ in}^4 \]
\[ A = 35 \text{ in}^2 \]

Virtual Work for Frames

Find the moments in the frame due to a virtual couple. First, find the reaction in the frame to the virtual couple.

\[ \sum M_{cut} = 0 = m_{ab} \]

\[ m_{ab} = 0 \]

Virtual Work for Frames

The next step is to find the equation for moment in each section of the frame. Consider section AB

\[ \sum M_{cut} = 0 = m_{ab} \]

\[ m_{ab} = 0 \]
Virtual Work for Frames

Consider section BC

\[ \sum M_{int} = 0 = m_{abc} - \frac{4}{8}(x) \]

\[ m_{abc} = \frac{x}{16} \]

Virtual Work for Frames

Consider section DC

\[ \sum M_{int} = 0 = -m_{abc} - \frac{4}{8}(x) \]

\[ m_{abc} = -\frac{x}{16} \]

Virtual Work for Frames

Consider section ED

\[ \sum M_{int} = 0 = -m_{abc} \]

\[ m_{abc} = 0 \]

Virtual Work for Frames

The following table lists the moment expression due to the real loading and the moment expression due to a virtual couple at point C

<table>
<thead>
<tr>
<th>Section</th>
<th>M</th>
<th>( m_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>6x</td>
<td>( x/16 )</td>
</tr>
<tr>
<td>DC</td>
<td>6x</td>
<td>( -x/16 )</td>
</tr>
<tr>
<td>DE</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Virtual Work for Frames

The virtual work equations are:

\[ \theta_c = \frac{1}{4} M_{ax} m_{ax} dy + \frac{1}{5} M_{bc} m_{bc} dx + \frac{1}{6} M_{dc} m_{dc} dy + \frac{1}{7} M_{ec} m_{ec} dy \]

Substituting the moment expressions into the virtual work equation and integrating yields the following:

\[ \theta_c = \frac{5}{16} (6x) dx - \frac{5}{16} (6x) dx = 0 \]

The slope at point C is zero

Virtual Work for Frames

Repeat the previous example and include the effects of axial work.

In order to compute the axial work, we need the axial force in the real and virtual loading systems

\[ E = 29,000 \text{ psi} \]

\[ I = 3,500 \text{ in}^4 \]

\[ A = 35 \text{ in}^2 \]
Virtual Work for Frames

Find the axial force in each section of the frame. Consider section AB

\[ \sum F_y = 0 = F_{AB} + 6k \]

\[ F_{AB} = -6k \]

Virtual Work for Frames

Consider section BC

\[ \sum F_y = 0 = F_{BC} \]

\[ F_{BC} = 0 \]

Virtual Work for Frames

Consider section DC

\[ \sum F_y = 0 = -F_{DC} \]

\[ F_{DC} = 0 \]

Virtual Work for Frames

Consider section ED

\[ \sum F_y = 0 = F_{DE} + 6k \]

\[ F_{DE} = -6k \]

Virtual Work for Frames

In this problem, the virtual axial forces are the real axial forces divided by 12 (from superposition).

<table>
<thead>
<tr>
<th>Section</th>
<th>N</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>-6k</td>
<td>-0.5</td>
</tr>
<tr>
<td>BC</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DC</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DE</td>
<td>-6k</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Virtual Work for Frames

The virtual work equations for axial forces are:

\[ \Delta_c = \sum \frac{F_{NL} \cdot L}{AE} \]

Substituting the values for the axial forces into the virtual work equations yields the following:

\[ \Delta_c = \frac{(-0.5)(-6k)(120in)}{AE} + \frac{(-0.5)(-6k)(120in)}{AE} \]

\[ \Delta_c = \frac{720k \text{ in}}{AE} = \frac{720k \text{ in}}{(29,000 \text{ksi})(35in^2)} = 0.0007 \text{ in} \]
Virtual Work for Frames

The displacement at point C due to bending moment work and axial force work is:

\[ \Delta_c = 0.017\text{in} + 0.0007\text{in} = 0.0177\text{in} \]

\( \Delta \) from bending moment work

\( \Delta \) from axial force work

\[ E = 29,000 \text{ psi} \]

\[ I = 3,500 \text{ in}^4 \]

\[ A = 35 \text{ in}^2 \]

The next step is to find the axial force in each section of the frame.

Consider section AB

\[ \uparrow \sum F_y = 0 = n_{uA} \uparrow + \]

\[ n_{uA} = -\frac{7}{16} \]

Consider section BC

\[ \sum F_x = 0 = n_{uC} \]

\[ n_{uC} = 0 \]

Consider section DC

\[ \sum F_y = 0 = -n_{uC} \]

\[ n_{uC} = 0 \]

Consider section ED

\[ \sum F_x = 0 = n_{uE} \uparrow \]

\[ n_{uE} = \frac{1}{16} \]
Virtual Work for Frames

The real axial forces and the virtual axial forces due to a unit virtual couple are:

<table>
<thead>
<tr>
<th>Section</th>
<th>$N$</th>
<th>$n_i$</th>
</tr>
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<tbody>
<tr>
<td>AB</td>
<td>-6k</td>
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</tr>
<tr>
<td>DE</td>
<td>-6k</td>
<td>1/16</td>
</tr>
</tbody>
</table>

Substituting the values for the axial forces into the virtual work equations yields the following:

$$\theta_C = \sum \frac{n_i NL}{AE}$$

The contribution to the slope at point C from the axial energy is slope is zero.
The total slope at point C due to bending moment and axial force work is zero.

Virtual Work for Frames

- In problems involving both bending and axial deformation, be careful with the units.
- Also, note that the contribution of the axial deformation is 5% of the total deformation.
- This is more or less typical of the relative size of the bending and the axial effects in frame-deflection problems.
- Therefore, it is usually permissible to neglect the effect of axial deformation in such cases.

Virtual Work for Frames

Virtual Strain Energy From Shear

Consider the following beam and a small element $dx$

The shearing deformation $dy$ caused by the real loads is $dy = \gamma dx$, where $\gamma$ is the shear strain.

Since we are assuming the material is linear and elastic, then Hooke's law applies

The shear strain is related to the shear stress by $\gamma = \tau/G$, where $\tau$ is the shear stress and $G$ is the shearing modulus of elasticity.
Virtual Work for Frames

Virtual Strain Energy From Shear

- The shear stress may be calculated as \( \tau = K(V/A)dx \), where \( K \) is a form factor that depends on the shape of the beam's cross-sectional area \( A \).
- Combining these two expressions gives \( dy = KV/(GA) dx \).
- The internal virtual work done by the virtual shear force \( v \) acting on the beam before it is deformed by the real loads is \( dU = v dy \).

Virtual Work for Frames

Virtual Strain Energy From Torsion

- For example, consider a circular cross-section where no wrapping of the section occurs.
- For non-circular sections a more rigorous analysis is required.

Virtual Work for Frames

Virtual Strain Energy From Torsion

- This torque causes a shear strain: \( \gamma = \frac{c d\theta}{dx} \).
- For a linear-elastic response: \( \gamma = \frac{\tau}{G} \)
Virtual Work for Frames

Virtual Strain Energy From Torsion

The angle of twist:

\[
d\theta = \frac{\tau}{Gc} dx = \frac{T}{GJ} dx
\]

Virtual Work for Frames

Virtual Strain Energy From Torsion

If a virtual load is applied to the structure that causes an internal virtual torque \( t \), then after applying the real loads, virtual strain energy will be:

\[
dU_v = t d\theta = \frac{tT}{GJ} dx
\]

Virtual Work for Frames

Virtual Strain Energy From Temperature

Integrating the virtual strain over the length of the member yields:

\[
U_v = \frac{tTL}{GJ}
\]

Virtual Work for Frames

Virtual Strain Energy From Temperature

If a virtual load is applied to the structure that causes an internal virtual torque \( m \), then after applying the real loads, virtual strain energy will be:

\[
dU_{\text{temp}} = \int_0^L \frac{m\alpha \Delta T_m}{c} dx
\]

Virtual Work for Frames

Virtual Strain Energy From Temperature

Consider a structure member is subjected to a temperature difference across its depth.

For discussion, we will choose the most common case of a beam having a neutral axis located at the mid-depth \( c \) of the beam.

First compute the amount of rotation of a differential element \( dx \) of the beam caused by the thermal gradient.
Virtual Work for Frames

- Unless otherwise stated, in this course we will consider only beam and frame deflections due to bending.
- The additional deflections caused by the shear and axial force alter the deflections by only a few percent and are generally ignored for even "small" two- and three-member frames of one-story height.

End of Virtual Work - Frames

Any questions?