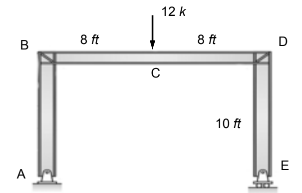
Virtual Work for Frames

- Applying the virtual work equations to a frame structure is as simple as separating the frame into a series of “beams” and summing the virtual work for each section.
- In addition, when evaluating the deformation of a frame structure, you may have to consider both bending and axial internal force components.

Virtual Work for Frames

Compute the ***deflection*** at point C on the frame shown below.

Include only the effects of bending in your virtual work equation (no axial work).

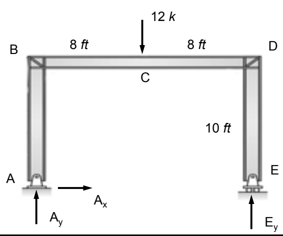


E = 29,000 *ksi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

The first step is to find the equation for moment in each section of the frame due to the ***real loads***.

To do develop the moment expression we need the reaction a points A and E.



$$\circlearrowleft^{+}\sum M_A = 0 = -12k(8ft) + E_Y(16ft)$$

$$\boxed{E_y = 6k}$$

$$+\uparrow \sum F_y = 0 = -12k + E_Y + A_y$$

$$\boxed{A_y = 6k}$$

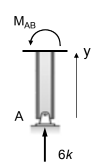
$$\xrightarrow{+} \sum F_x = 0 = A_x \qquad \boxed{A_x = 0}$$

Virtual Work for Frames

The next step is to find the equation for moment in each section of the frame.

Consider section AB

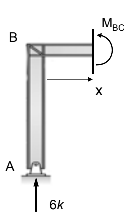
$$\circlearrowleft^{+}\sum M_{cut} = 0 = M_{AB} \qquad \boxed{M_{AB} = 0}$$



Virtual Work for Frames

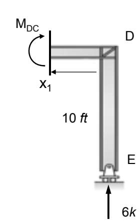
Consider section BC

$$\circlearrowleft^{+}\sum M_{cut} = 0 = M_{BC} - 6k(x) \qquad \boxed{M_{BC} = 6x}$$



Virtual Work for Frames

Consider section DC



$$\circlearrowleft^{+}\sum M_{cut} = 0 = -M_{DC} + 6k(x_1)$$

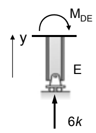
$$\boxed{M_{DC} = 6x_1}$$

Virtual Work for Frames

Consider section ED

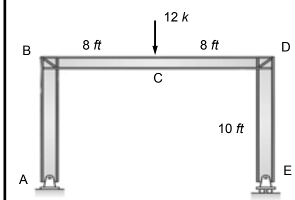
$$\circlearrowleft^{+} \sum M_{cut} = 0 = -M_{DE}$$

$$\boxed{M_{DE} = 0}$$



Virtual Work for Frames

In this problem, the virtual moments are the real moments divided by 12 (from superposition).



Section	M	m
AB	0	0
BC	6x	0.5x
DC	$6x_1$	$0.5x_1$
DE	0	0

Virtual Work for Frames

The virtual work equations are:

$$\Delta_C = \int_A^B \frac{M_{AB} m_{AB}}{EI} dy + \int_B^C \frac{M_{BC} m_{BC}}{EI} dx + \int_D^C \frac{M_{DC} m_{DC}}{EI} dx + \int_D^E \frac{M_{DE} m_{DE}}{EI} dy$$

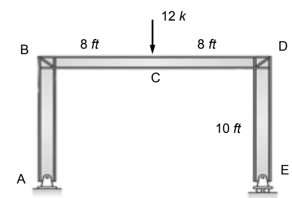
Substituting the moment expressions into the virtual work equation and integrating yields the following:

$$\Delta_C = \int_0^8 \frac{(6x)x}{2EI} dx + \int_0^8 \frac{(6x_1)x_1}{2EI} dx_1 = \int_0^8 \frac{6x^2}{EI} dx = \left.\frac{2x^3}{EI}\right|_0^8$$

$$= \frac{1{,}024\ kft^3}{EI} = \frac{1{,}024\ kft^3 (1{,}728 in^3 / ft^3)}{(29{,}000 ksi)(3{,}500 in^4)} = \boxed{0.017 in}$$

Virtual Work for Frames

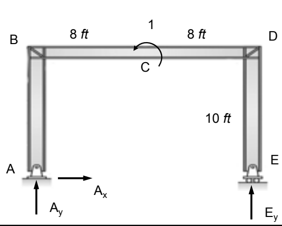
Compute the ***slope*** at point C on the frame shown below. Include only the effects of bending in your virtual work equation (no axial work).



E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

Find the moments in the frame due to a virtual couple. First, find the reaction in the frame to the virtual couple.



$$\circlearrowleft^{+} \sum M_A = 0 = 1 + E_Y(16ft)$$

$$\boxed{E_y = -\tfrac{1}{16}}$$

$$+\uparrow \sum F_y = 0 = E_Y + A_y$$

$$\boxed{A_y = \tfrac{1}{16}}$$

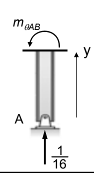
$$\overset{+}{\rightarrow} \sum F_x = 0 = A_x$$

Virtual Work for Frames

The next step is to find the equation for moment in each section of the frame.

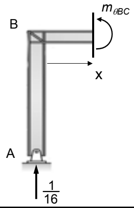
Consider section AB

$$\circlearrowleft^{+} \sum M_{cut} = 0 = m_{\theta AB} \qquad \boxed{m_{\theta AB} = 0}$$



Virtual Work for Frames

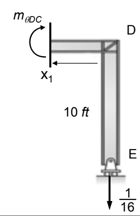
Consider section BC



$$\circlearrowleft^{+} \sum M_{cut} = 0 = m_{\theta BC} - \tfrac{1}{16}(x) \qquad \boxed{m_{\theta BC} = \frac{x}{16}}$$

Virtual Work for Frames

Consider section DC



$$\circlearrowleft^{+} \sum M_{cut} = 0 = -m_{\theta DC} - \tfrac{1}{16}(x_1)$$

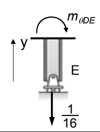
$$\boxed{m_{\theta DC} = -\frac{x_1}{16}}$$

Virtual Work for Frames

Consider section ED

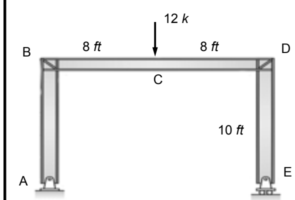
$$\circlearrowleft^{+} \sum M_{cut} = 0 = -m_{\theta DE}$$

$$\boxed{m_{\theta DE} = 0}$$



Virtual Work for Frames

The following table lists the moment expression due to the real loading and the moment expression due to a virtual couple at point C



Section	M	m_θ
AB	0	0
BC	6x	x/16
DC	$6x_1$	$-x_1/16$
DE	0	0

Virtual Work for Frames

The virtual work equations are:

$$\theta_C = \int_A^B \frac{M_{AB} m_{\theta AB}}{EI} dy + \int_B^C \frac{M_{BC} m_{\theta BC}}{EI} dx + \int_D^C \frac{M_{DC} m_{\theta DC}}{EI} dx + \int_E^D \frac{M_{DE} m_{\theta DE}}{EI} dy$$

Substituting the moment expressions into the virtual work equation and integrating yields the following:

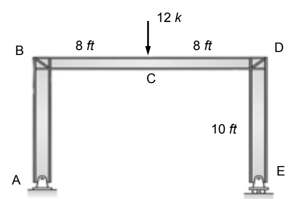
$$\theta_C = \int_0^8 \frac{(6x)x}{16EI} dx - \int_0^8 \frac{(6x_1)x_1}{16EI} dx_1 \quad = 0$$

The slope at point C is zero

Virtual Work for Frames

Repeat the previous example and include the effects of axial work.

In order to compute the axial work, we need the axial force in the real and virtual loading systems

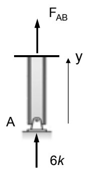


E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

Find the axial force in each section of the frame.
Consider section AB

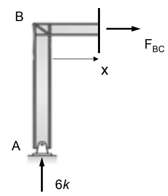
$$\overset{+}{\uparrow}\sum F_y = 0 = F_{AB} + 6k \qquad \boxed{F_{AB} = -6k}$$



Virtual Work for Frames

Consider section BC

$$\overset{+}{\rightarrow}\sum F_x = 0 = F_{BC} \qquad \boxed{F_{BC} = 0}$$

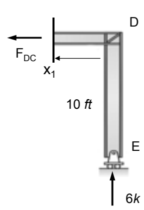


Virtual Work for Frames

Consider section DC

$$\overset{+}{\rightarrow}\sum F_x = 0 = -F_{DC}$$

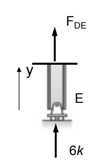
$$\boxed{F_{DC} = 0}$$



Virtual Work for Frames

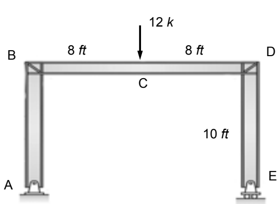
Consider section ED

$$\overset{+}{\uparrow}\sum F_y = 0 = F_{DE} + 6k \qquad \boxed{F_{DE} = -6k}$$



Virtual Work for Frames

In this problem, the virtual axial forces are the real axial forces divided by 12 (from superposition).



Section	N	n
AB	-6k	-0.5
BC	0	0
DC	0	0
DE	-6k	-0.5

Virtual Work for Frames

The virtual work equations for axial forces are:

$$\Delta_C = \sum \frac{nNL}{AE}$$

Substituting the values for the axial forces into the virtual work equations yields the following:

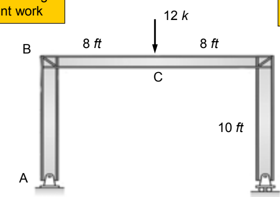
$$\Delta_C = \frac{(-0.5)(-6k)(120in)}{AE} + \frac{(-0.5)(-6k)(120in)}{AE}$$

$$= \frac{720\ k\ in}{AE} = \frac{720\ k\ in}{(29{,}000ksi)(35in^2)} = \boxed{0.0007in}$$

Virtual Work for Frames

The displacement at point C due to bending moment work and axial force work is:

$$\Delta_C = \underbrace{0.017in}_{\Delta\text{ from bending moment work}} + \underbrace{0.0007in}_{\Delta\text{ from axial force work}} = \boxed{0.0177in}$$

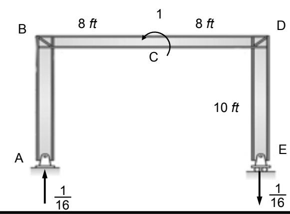


E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

Compute the axial forces in the frame due to the virtual couple.

Recall we already have the frame reactions due to the virtual couple



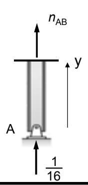
E = 29,000 *psi*
I = 3,500 in^4
A = 35 in^2

Virtual Work for Frames

The next step is to find the axial force in each section of the frame.

Consider section AB

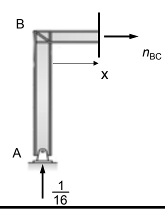
$$+\uparrow \sum F_y = 0 = n_{AB} + \tfrac{1}{16} \qquad \boxed{n_{AB} = -\tfrac{1}{16}}$$



Virtual Work for Frames

Consider section BC

$$\overset{+}{\rightarrow} \sum F_x = 0 = n_{BC} \qquad \boxed{n_{BC} = 0}$$

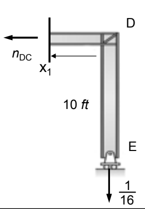


Virtual Work for Frames

Consider section DC

$$\overset{+}{\rightarrow} \sum F_x = 0 = -n_{DC}$$

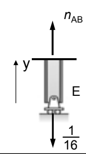
$$\boxed{n_{DC} = 0}$$



Virtual Work for Frames

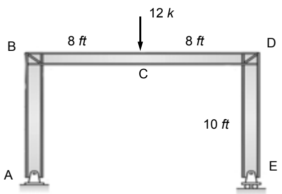
Consider section ED

$$+\uparrow \sum F_y = 0 = n_{DE} - \tfrac{1}{16} \qquad \boxed{n_{DE} = \tfrac{1}{16}}$$



Virtual Work for Frames

The real axial forces and the virtual axial forces due to a unit virtual couple are:



Section	N	n_θ
AB	-6k	-1/16
BC	0	0
DC	0	0
DE	-6k	1/16

Virtual Work for Frames

The virtual work equations for axial forces are:

$$\Delta_C = \sum \frac{nNL}{AE}$$

Substituting the values for the axial forces into the virtual work equations yields the following:

$$\Delta_C = \frac{(6k)(120in)}{16AE} - \frac{(6k)(120in)}{16AE} = 0$$

The contribution to the slope at point C from the axial energy is slope is zero.

The total slope at point C due to bending moment and axial force work is zero.

Virtual Work for Frames

- In problems involving both bending and axial deformation, be careful with the units.
- Also, note that the contribution of the axial deformation is 5% of the total deformation.
- This is more or less typical of the relative size of the bending and the axial effects in frame-deflection problems.
- Therefore, it is usually permissible to neglect the effect of axial deformation in such cases.

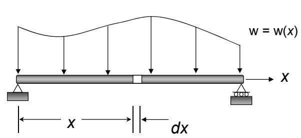
Virtual Work for Frames

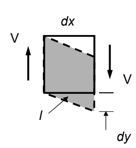
- The primary cause of deformation in beams and frames is due to ***bending strain***.
- However, in some structures additional deformation due to ***axial*** and ***shear forces***, ***torsion***, and perhaps ***temperature*** may be important.
- We have already discussed deformation due to bending moments and axial forces.
- In this section, we will consider the effect of shear, torsion, and temperature on the deformation of linear elastic structures.

Virtual Work for Frames

Virtual Strain Energy From Shear

Consider the following beam and a small element dx

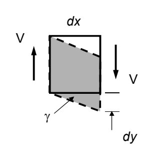




Virtual Work for Frames

Virtual Strain Energy From Shear

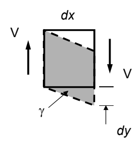
- The shearing deformation dy caused by the real loads is $dy = \gamma dx$, where γ is the shear strain.
- Since we are assuming the material is linear and elastic, then Hooke's law applies
- The shear strain is related to the shear stress by $\gamma = \tau/G$, where τ is the shear stress and G is the shearing modulus of elasticity.



Virtual Work for Frames

Virtual Strain Energy From Shear

- The shear stress may be calculated as $\tau = K(V/A)dx$, where K is a form factor that depends of the shape of the beam's cross-sectional area A.
- Combining these two expressions gives $dy = KV/(GA)\,dx$.
- The internal virtual work done by the virtual shear force $\boldsymbol{v}$ acting on the beam before it is deformed by the real loads is $\boldsymbol{dU_i = v\,dy}$

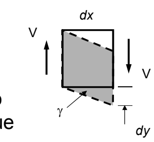


Virtual Work for Frames

Virtual Strain Energy From Shear

- Integrating the expression $dU_i = v\,dy$ over the entire beam gives:

$$U_{shear} = \int_0^L K\left(\frac{vV}{GA}\right)dx$$



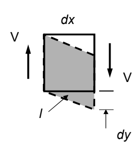
- Remember that v is the shear due to the virtual load and V is the shear due to the real loads.

Virtual Work for Frames

Virtual Strain Energy From Shear

- Integrating the expression $dU_i = v\,dy$ over the entire beam gives:

$$U_{shear} = \int_0^L K\left(\frac{vV}{GA}\right)dx$$



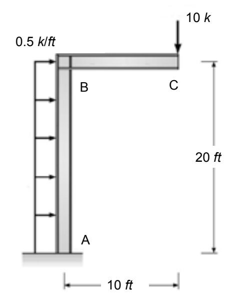
- The form factor K is based on the cross-sectional area:
  - $\boldsymbol{K}$ = 1.2 for rectangular sections
  - $\boldsymbol{K}$ = 10/9 for circular sections
  - $\boldsymbol{K} \approx$ 1 for I-beams, where A is the area of the web

Virtual Work for Frames

Compute the vertical ***deflection*** and rotation at point C on the frame shown.

Include the effects of bending moment and both axial and shear forces in your virtual work equations.

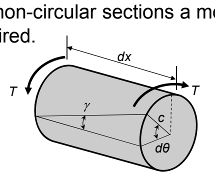
E = 29,000 *ksi*
G = 12,000 *ksi*
I = 1,000 in^4
A = 25 in^2
K = 1.2



Virtual Work for Frames

Virtual Strain Energy From Torsion

- For example, consider a circular cross-section where no wrapping of the section occurs.
- For non-circular sections a more rigorous analysis is required.

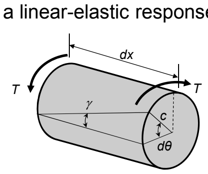


Virtual Work for Frames

Virtual Strain Energy From Torsion

- This torque causes a shear strain: $\gamma = \dfrac{c\,d\theta}{dx}$
- For a linear-elastic response: $\gamma = \dfrac{\tau}{G}$

$$\tau = \frac{Tc}{J}$$

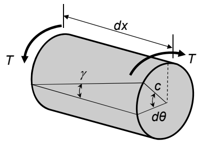


Virtual Work for Frames

Virtual Strain Energy From Torsion

The angle of twist:

$$d\theta = \frac{\gamma}{c}dx = \frac{\tau}{Gc}dx = \frac{T}{GJ}dx$$

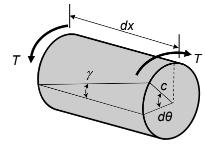


Virtual Work for Frames

Virtual Strain Energy From Torsion

If a virtual load is applied to the structure that causes an internal virtual torque ***t***, then after applying the real loads, virtual strain energy will be:

$$dU_t = t\,d\theta = \frac{tT}{GJ}dx$$

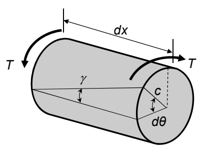


Virtual Work for Frames

Virtual Strain Energy From Torsion

Integrating the virtual strain over the length of the member yields:

$$U_t = \frac{tTL}{GJ}$$



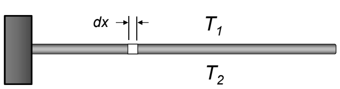
Virtual Work for Frames

Virtual Strain Energy From Temperature

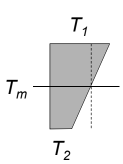
- Consider a structure member is subjected to a temperature difference across its depth.
- For discussion, we will choose the most common case of a beam having a neutral axis located at the mid-depth *c* of the beam
- First compute the amount of rotation of a differential element *dx* of the beam caused by the thermal gradient.

Virtual Work for Frames

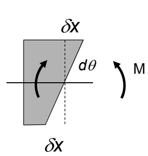
Virtual Strain Energy From Temperature



$$\delta x = \alpha\,\Delta T_m\,dx$$



$$T_m = \frac{T_1 + T_2}{2}$$



Virtual Work for Frames

Virtual Strain Energy From Temperature

If a virtual load is applied to the structure that causes an internal virtual torque ***m***, then after applying the real loads, virtual strain energy will be:

$$dU_{temp} = \int_0^L \frac{m\alpha\,\Delta T_m}{c}dx$$

Virtual Work for Frames

- Unless otherwise stated, in this course we will consider only beam and frame deflections due to bending.
- The additional deflections caused by the shear and axial force alter the deflections by only a few percent and are generally ignored for even “small” two- and three-member frames of one-story height.

End of Virtual Work - Frames

Any questions?