Deflections

- The ***conjugate-beam method*** was developed by in Heinrich Müller-Breslau in 1865.
- The method is based on the similarity between the relationships for loading and shear, and shear and moment.

$$\frac{dV}{dx} = -w(x) \qquad \frac{dM}{dx} = V \Rightarrow \frac{d^2M}{dx^2} = -w$$

$$V = -\int w(x)dx \qquad M = -\int\int w(x)dx\,dx$$

Deflections

- The previous expressions relate the internal shear and moment to the applied load.
- The slope and deflection of the elastic curve are related to the internal moment by the following expressions

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\theta = \int \frac{M}{EI}dx \qquad y = \int\int \frac{M}{EI}dx\,dx$$

Deflections

- Let's compare expressions for shear, ***V***, and the slope, θ

$$\frac{dV}{dx} = -w(x) \qquad \frac{d\theta}{dx} = \frac{M}{EI}$$

- What do you see?
- If you replace ***w*** with the term –***M/EI*** the expressions for shear force and slope are identical

Deflections

- Let's compare expressions for bending moment, ***M***, and the displacement, ***y***

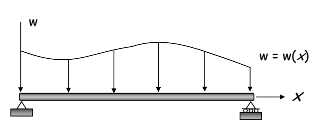
$$\frac{d^2M}{dx^2} = -w \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

- What do you see?
- Just as before, if you replace ***w*** with the term –***M/EI*** the expressions for bending moment and displacement are identical

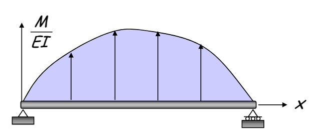
Deflections

- We will use this relationship to our advantage by constructing a beam with the same length as the real beam referred to as the ***conjugate beam***.
- The conjugate beam is loaded with the ***M/EI*** diagram, simulating the external load ***w***.

Deflections



Real beam with applied loading. Determine the bending moment (draw the bending moment diagram)



Conjugate beam where the applied loading is bending moment from the real beam

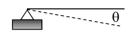
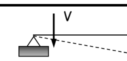
Note the sign of loading ***w*** and the ***M/EI*** on the conjugate beam.

Deflections

- Therefore, the two theorems related to the conjugate beam method are:
  - **Theorem 1: The slope at a point in the real beam is equal to the shear at the corresponding point in the conjugate beam.**
  - **Theorem 2: The displacement of a point in the real beam is equal to the moment at the corresponding point in the conjugate beam.**

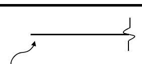
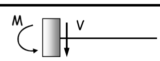
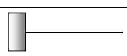
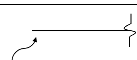
Deflections

- ***Conjugate-Beam Supports***
- When the conjugate beam is drawn, it is important that the shear and moment developed in the conjugate beam correspond to the slope and displacement conditions in the real beam.

Real Support	Conjugate Support
 Pin or roller $\Delta = 0 \quad \theta \neq 0$	 Pin or roller $M = 0 \quad V \neq 0$

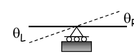
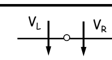
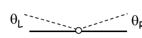
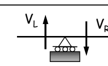
Deflections

- ***Conjugate-Beam Supports***

Real Support	Conjugate Support
 Free end $\Delta \neq 0 \quad \theta \neq 0$	 Fixed end $M \neq 0 \quad V \neq 0$
 Fixed end $\Delta = 0 \quad \theta = 0$	 Free end $M = 0 \quad V = 0$

Deflections

- ***Conjugate-Beam Supports***

Real Support	Conjugate Support
 Interior support $\Delta = 0 \quad \theta_L = \theta_R \neq 0$	 Hinge $M = 0 \quad V_L = V_R \neq 0$
 Hinge $\Delta \neq 0$ θ_L and θ_R may have different values	 Interior roller $M \neq 0$ V_L and V_R may have different values

Deflections

- As a rule, statically determinant real beams have statically determinant conjugate beams and statically indeterminate beams become unstable conjugate beams.
- However, the ***M/EI*** loading may provide the necessary "equilibrium" to hold the conjugate beam stable.

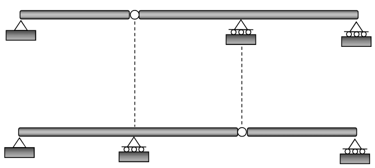
Deflections

Procedure for analysis

1. Construct the conjugate beam with the ***M/EI*** loading.
   Remember when the ***M/EI*** diagram is positive the loading is *upward* and when the *M/EI* diagram is negative the loading is *downward*.
2. Use the equations of equilibrium to solve for the reactions of the conjugate beam.
   This may be difficult if the moment diagram is complex.
3. Solve for the shear and moment at the point or points where the slope and displacement are desired.
   If the values are positive, the slope is counterclockwise and the displacement is upward.

Deflections

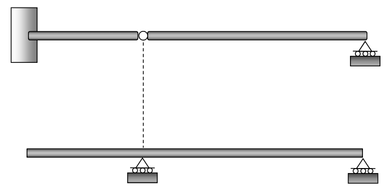
- **Draw the conjugate beam, including supports, for the following beams**



Conjugate beam and supports

Deflections

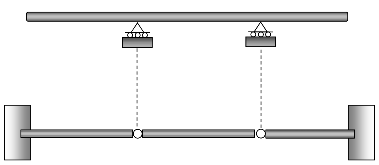
- **Draw the conjugate beam, including supports, for the following beams**



Conjugate beam and supports

Deflections

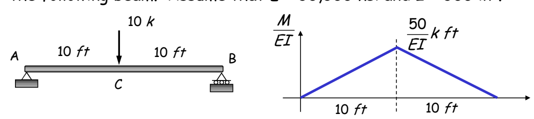
- **Draw the conjugate beam, including supports, for the following beams**



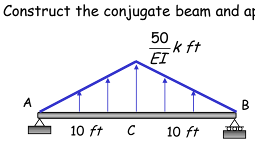
Conjugate beam and supports

Deflections

Example: Determine the slope and the displacement at point *C* for the following beam. Assume that E = 30,000 *ksi* and I = 300 in^4.



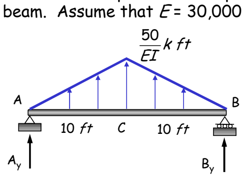
Construct the conjugate beam and apply the ***M/EI*** diagram as loading



Remember positive (+) bending moment is a negative (-) loading on the conjugate beam.

Deflections

Example: Determine the slope and the displacement at point *C* for the following beam. Assume that E = 30,000 *ksi* and I = 300 in^4.

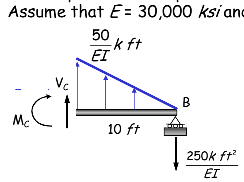


$$\sum M_A = 0 = \frac{1}{2}\left(\frac{50k\ ft}{EI}\right)(20ft)(10ft) + B_y(20ft) \qquad B_y = -\frac{250k\ ft^2}{EI}$$

$$\sum F_y = 0 = \frac{1}{2}\left(\frac{50k\ ft}{EI}\right)(20ft) + B_y + A_y \qquad A_y = -\frac{250k\ ft^2}{EI}$$

Deflections

Example: Determine the slope and the displacement at point *C* for the following beam. Assume that E = 30,000 *ksi* and I = 300 in^4.

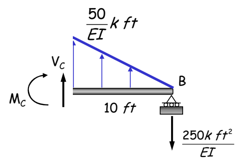


$$\sum M_C = 0 = -M_C + \frac{1}{2}\left(\frac{50k\ ft}{EI}\right)(10ft)\left(\frac{10ft}{3}\right) - \left(\frac{250k\ ft^2}{EI}\right)(10ft)$$

$$M_C = -\frac{1{,}666.6k\ ft^3}{EI} = -\frac{1{,}666.6k\ ft^3}{(30{,}000ksi)(300in^4)} \cdot \frac{1{,}728in^3}{ft^3} = \boxed{-0.32in}$$

Deflections

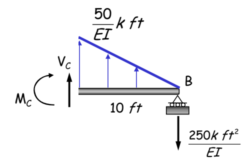
Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 *ksi* and I = 300 in^4.



$$\sum F_y = 0 = \frac{1}{2}\left(\frac{50k\,ft}{EI}\right)(10ft) - \left(\frac{250k\,ft^2}{EI}\right) + V_C \qquad V_C = 0$$

Deflections

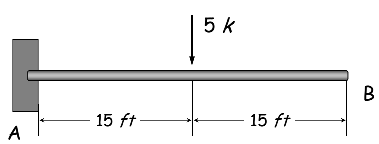
Example: Determine the slope and the displacement at point C for the following beam. Assume that E = 30,000 *ksi* and I = 300 in^4.



- Therefore, the displacement of the beam at point C is equal to the moment at point C on the conjugate beam and the slope is equal to the shear in the conjugate beam.
- In this problem. the displacement at point C is -0.32 *in* and the slope is zero.

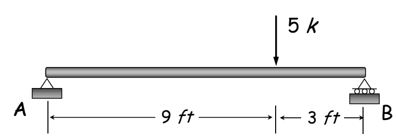
Deflections

- ***Example:*** Determine the slope and the displacement at point B for the following beam. Assume that E = 29,000 *ksi* and I = 800 in^4 (see page 93 in notes).



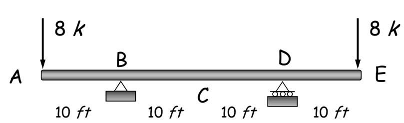
Deflections

- ***Example:*** Determine the maximum displacement at the mid-span of the following beam. Assume that E = 30,000 *ksi* and I = 800 in^4 (see page 94 in notes).



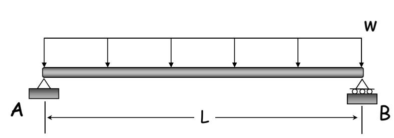
Deflections

- ***Example:*** Determine the slope at point B and the displacement at point E for the following beam. Assume that E = 29,000 *ksi* and $I_{AB} = I_{DE}$ = 400 in^4, and I_{BD} = 800 in^4. (see page 95 in notes).



Deflections

- ***Example:*** Determine the slope at A and the displacement at mid-span.



- Are there any disadvantages to the conjugate beam method for uniform or high-order loading functions?

End of Defections – Part 2

Any questions?