Principle of Virtual Work

- The principle of virtual work was developed by John Bernoulli in 1717 and is sometimes referred to as the unit-load method.
- Johann Bernoulli (1667–1748; also known as Jean or John) was a Swiss mathematician and was one of the many prominent mathematicians in the Bernoulli family.
- He is known for his contributions to infinitesimal calculus and educated Leonhard Euler in his youth.

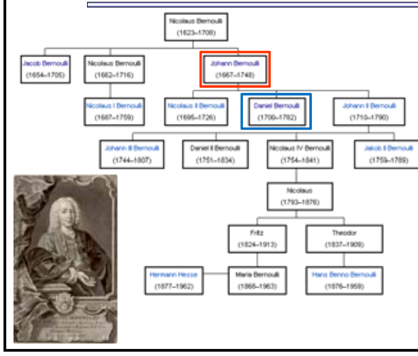
Principle of Virtual Work

- In 1738, Johann and his son Daniel nearly simultaneously published separate works on hydrodynamics.
- Daniel is said to have had a bad relationship with his father
  - Upon both of them entering and tying for first place in a scientific contest at the University of Paris, Johann, unable to bear the "shame" of being compared Daniel's equal, banned Daniel from his house.

Principle of Virtual Work

- In 1738, Johann and his son Daniel nearly simultaneously published separate works on hydrodynamics.
- Daniel is said to have had a bad relationship with his father
  - Johann Bernoulli also plagiarized some key ideas from Daniel's book *Hydrodynamica* in his own book *Hydraulica* which he backdated to before *Hydrodynamica*.
  - Despite Daniel's attempts at reconciliation, his father carried the grudge until his death.

Principle of Virtual Work



Truss – Virtual Work

- Virtual work is a procedure for computing a single component of deflection at any point on a structure.
- To compute a component of deflection by the method of virtual work, the designer applies a force to structure at the point and in the direction of the desired displacement.
- The force is called the ***dummy load*** or the ***virtual load***.
- The force system created by the virtual loads is called the ***Q-system.***

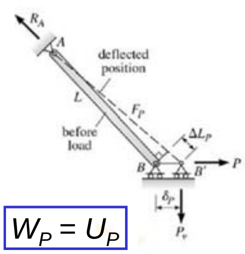
Truss – Virtual Work

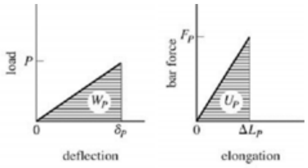
- The force system created by the ***actual loads*** is called the ***P-system***.
- As the structure deforms under the actual loads, ***external virtual work*** W_Q is done by the virtual loads as they move through real displacements.
- Due to the conversation of energy an equivalent quantity of ***virtual strain energy*** U_Q is stored in the structure.

$$W_Q = U_Q$$

Truss – Virtual Work

Consider the method of virtual work applied to one-bar truss, as shown below.





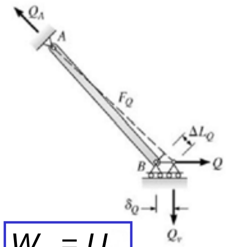
$\boldsymbol{W_P}$ - real work done by $\boldsymbol{P}$

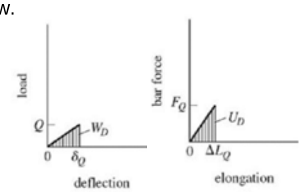
$\boldsymbol{U_P}$ - real strain energy stored in AB due to $\boldsymbol{P}$

$\boxed{W_P = U_P}$

Truss – Virtual Work

Now consider the forces and displacements produced by the virtual load, as shown below.





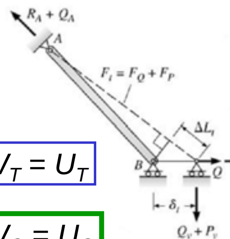
$\boldsymbol{W_D}$ - real work done by $\boldsymbol{Q}$

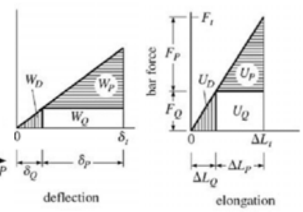
$\boldsymbol{U_D}$ - real strain energy stored in AB due to $\boldsymbol{Q}$

$\boxed{W_D = U_D}$

Truss – Virtual Work

Now consider the forces and displacements produced by the virtual load and actual loads acting together.





$\boldsymbol{W_T}$ – total work done by $\boldsymbol{Q}$ and $\boldsymbol{P}$

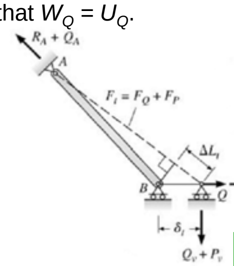
$\boldsymbol{U_T}$ - total strain energy due to $\boldsymbol{Q}$ and $\boldsymbol{P}$

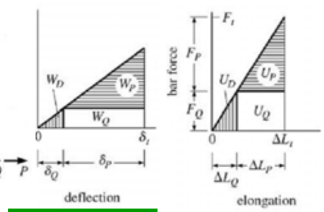
$\boxed{W_T = U_T}$

$\boxed{W_Q = U_Q}$

Truss – Virtual Work

By the principle of conservation of energy, it follows that $W_Q = U_Q$.

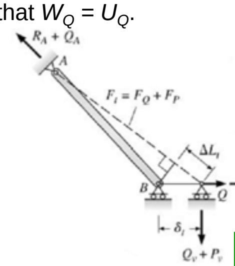


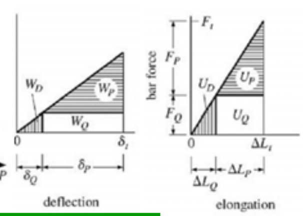


$\boxed{W_Q = Q\delta_P}$

Truss – Virtual Work

By the principle of conservation of energy, it follows that $W_Q = U_Q$.

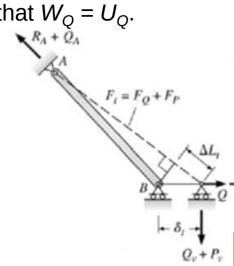


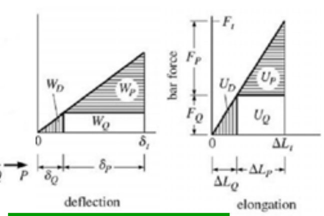


$\boxed{U_Q = F_Q \Delta L_P}$

Truss – Virtual Work

By the principle of conservation of energy, it follows that $W_Q = U_Q$.





$\boxed{Q\delta_P = F_Q \Delta L_P}$

Truss – Virtual Work

- By summing the energy expression for each member in a truss, we get:

$$\sum Q\delta_P = \sum F_Q \Delta L_P$$

- The bar elongation ΔL_P can be compute is terms of the actual load ***P*** and the properties of the section.

$$\sum Q\delta_P = \sum F_Q \Delta L_P = \sum F_Q\left(\frac{F_P L}{AE}\right) = \boxed{\sum\left(\frac{nNL}{AE}\right)}$$

Where ***n*** is the force in each member of the truss due to the virtual load and ***N*** is the force in each member of the truss due to the real loads.

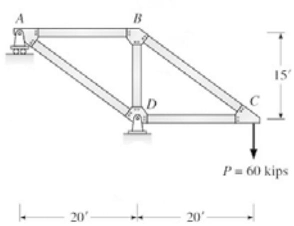
Truss – Virtual Work

- Typically, the virtual load ***Q*** is assumed to be 1 and dimensionless.
- Therefore, the virtual work expression for the a single displacement component in the direction of the applied virtual load ***Q*** is:

$$\boxed{\delta_P = \sum\left(\frac{nNL}{AE}\right)}$$

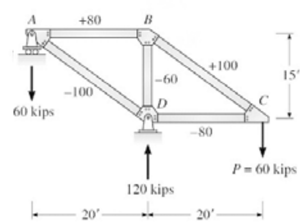
Truss – Virtual Work Example 1

Example: Compute the horizontal displacement at joint B in the truss shown below. Assume the E = 30,000 *kip/in*^{2}, the area of bars AD and BC = 5 *in*^{2}; the area of all other bars = 4 *in*^{2}.



Truss – Virtual Work Example 1

Example: Compute the horizontal displacement at joint B

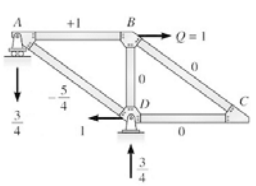


- The value and direction of each real force is indicated on the truss.
- A (+) sign is tension and (–) sign is compression.

Real Force System

Truss – Virtual Work Example 1

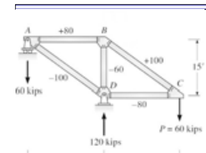
Example: Compute the horizontal displacement at joint B

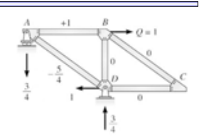


- The value and direction of each virtual force is indicated on the truss.
- A (+) sign is tension and (–) sign is compression.

Virtual Force System

Truss – Virtual Work Example 1





Member	*N (kips)*	*n*	*L (in)*	*A (in^2)*	*NnL/A (kip in/in^2)*
AB	80	1	240	4	4,800
BC	100	0	300	5	0
CD	-80	0	240	4	0
AD	-100	-1.25	300	5	7,500
BD	-60	0	180	4	0
					12,300 *(kip/in)*

Truss – Virtual Work Example 1

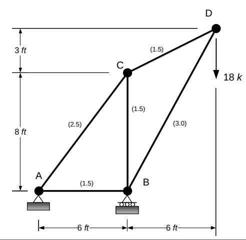
Therefore the horizontal displacement a joint B may be computed as:

$$\delta_{Bx} = \frac{12{,}300\ kip/in}{30{,}000\ kip/in^2} = \boxed{0.41 in \rightarrow}$$

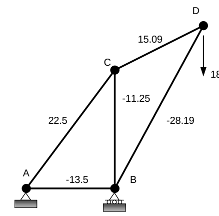
Member	N (kips)	n	L (in)	A (in^2)	NnL/A (kip in/in^2)
AB	80	1	240	4	4,800
BC	100	0	300	5	0
CD	-80	0	240	4	0
AD	-100	-1.25	300	5	7,500
BD	-60	0	180	4	0
					12,300 (kip/in)

Truss – Virtual Work Example 2

Example: Compute the magnitude and direction of the displacement at joint D in the truss shown below. Assume the E = 29,000 kip/in^2 and that the area of each bars is given in parenthesis (in^2).



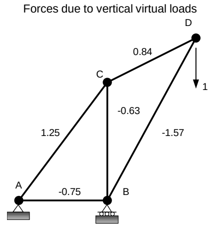
Forces due to real loads



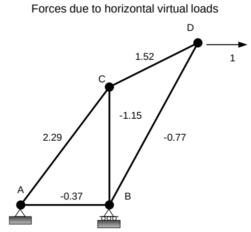
Truss – Virtual Work Example 2

Example: Compute the magnitude and direction of the displacement at joint D in the truss shown below. Assume the E = 29,000 kip/in^2 and that the area of each bars is given in parenthesis (in^2).

Forces due to vertical virtual loads



Forces due to horizontal virtual loads



Truss – Virtual Work Example 2

Example: Compute the magnitude and direction of the displacement at joint D in the truss shown below. Assume the E = 29,000 kip/in^2 and that the area of each bars is given in parenthesis (in^2).

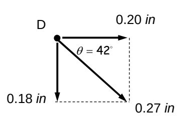
Member	N (kips)	n_V	n_H	L (in)	A (in^2)	Nn_VL/A (kip/in)	Nn_HL/A (kip/in)
AB	-13.5	-0.75	-0.37	72	1.5	486.0	239.76
AC	22.5	1.25	2.29	120	2.5	1,350.0	2,473.20
BC	-11.25	-0.63	-1.15	96	1.5	453.6	828.0
BD	-28.19	-1.57	-0.77	150.35	3.0	2,218.08	1,087.92
CD	15.09	0.84	1.52	84.49	1.5	713.97	1,231.2
						5,188.2 (kip/in)	**5,860.08** (kip/in)

Truss – Virtual Work Example 2

Therefore the total displacement a joint D may be computed as:

$$\delta_{D_H} = \frac{5{,}860.08\ kip/in}{29{,}000\ kip/in^2} = \boxed{0.20\ in \rightarrow}$$

$$\delta_{D_V} = \frac{5{,}188.2\ kip/in}{29{,}000\ kip/in^2} = \boxed{0.18\ in \downarrow}$$



$$\delta_D = \sqrt{(0.18in)^2 + (0.20in)^2} = \boxed{0.27in}$$

$$\theta = \tan^{-1}\left(\frac{0.18in}{0.20in}\right) = \boxed{42^\circ}$$

End of Virtual Work - Trusses

Any questions?