

## Deflections

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### Conjugate-Beam Method

- The development of the conjugate beam method has been attributed to several structural engineers.
- Many credit **Heinrich Müller-Breslau** (1851-1925) with the development of this method in 1865, while others, say the method was developed by **Christian Otto Mohr** (1835-1918).

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- Heinrich Franz Bernhard Müller was born in Wroclaw (Breslau) on 13 May 1851.
- In 1875 he opened a civil engineer's office in Berlin. Around this time he decided to add the name of his hometown to his surname, becoming known as Müller-Breslau.
- He founded the so-called "Berlin School" of structural theory.



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- Christian Otto Mohr was an enthusiast for graphical tools and developed the method for visually representing stress in 3D known as Mohr's Circle.
- He also developed methods for truss displacements and for analyzing statically indeterminate structures.
- He founded the so-called "Dresden School" of applied mechanics and has differences of opinion with Müller-Breslau throughout their careers.



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### Conjugate-Beam Method

- Conjugate beam is defined as the imaginary beam with the same dimensions (length) as that of the original beam but load at any point on the conjugate beam is equal to the bending moment at that point divided by  $EI$ .
- The conjugate-beam method is an engineering method to derive the slope and displacement of a beam.
- This method relies only on the principles of statics, so its application will be more familiar.

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The method is based on the similarity between the relationships for loading and shear, and shear and moment.

$$\frac{dV}{dx} = w(x) \quad \frac{dM}{dx} = V \quad \Rightarrow \quad \frac{d^2M}{dx^2} = w$$

$$V = \int w(x) dx \quad M = \int \int w(x) dx dx$$

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- The previous expressions relate the internal shear and moment to the applied load.
- The slope and deflection of the elastic curve are related to the internal moment by the following expressions

$$\frac{d\theta}{dx} = \frac{M}{EI} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\theta = \int \frac{M}{EI} dx \quad y = \int \int \frac{M}{EI} dx dx$$

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➤ Let's compare expressions for shear,  $V$ , and the slope,  $\theta$

$$\frac{dV}{dx} = w \qquad \frac{d\theta}{dx} = \frac{M}{EI}$$

➤ What do you see?  
 ➤ If you replace  $w$  with the term  $M/EI$  the expressions for shear force and slope are identical

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➤ Let's compare expressions for bending moment,  $M$ , and the displacement,  $y$

$$\frac{d^2M}{dx^2} = w \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

➤ What do you see?  
 ➤ Just as before, if you replace  $w$  with the term  $M/EI$  the expressions for bending moment and displacement are identical

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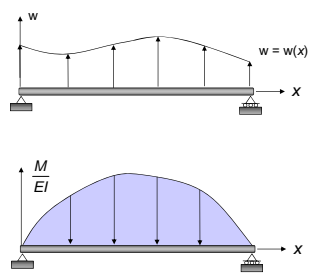
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➤ We will use this relationship to our advantage by constructing a beam with the same length as the real beam referred to as the **conjugate beam**.

➤ The conjugate beam is loaded with the  $M/EI$  diagram, simulating the external load  $w$ .

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**Real beam** with applied loading. Determine the bending moment (draw the bending moment diagram)

**Conjugate beam** where the applied loading is bending moment from the real beam

Note the sign of loading  $w$  and the  $M/EI$  on the conjugate beam.

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Therefore, the two theorems related to the conjugate beam method are:

➤ **Theorem 1: The slope at a point in the real beam is equal to the shear at the corresponding point in the conjugate beam.**

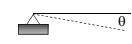
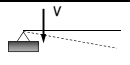
➤ **Theorem 2: The displacement of a point in the real beam is equal to the moment at the corresponding point in the conjugate beam.**

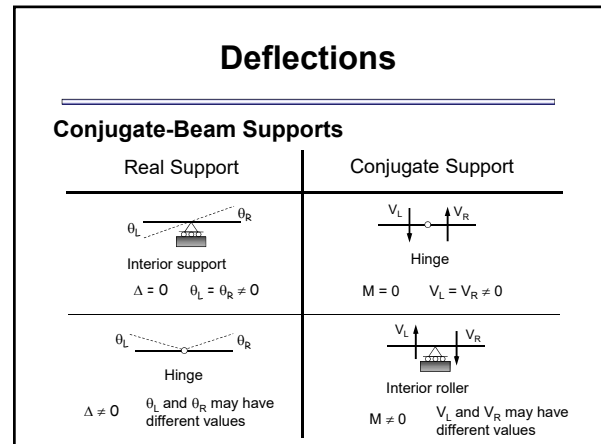
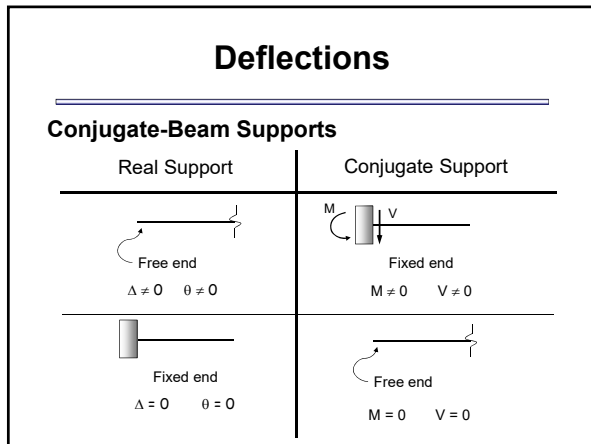
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#### Conjugate-Beam Supports

When the conjugate beam is drawn, it is important that the shear and moment developed in the conjugate beam correspond to the slope and displacement conditions in the real beam.

Real Support	Conjugate Support
 Pin or roller	 Pin or roller
$\Delta = 0 \quad \theta \neq 0$	$M = 0 \quad V \neq 0$



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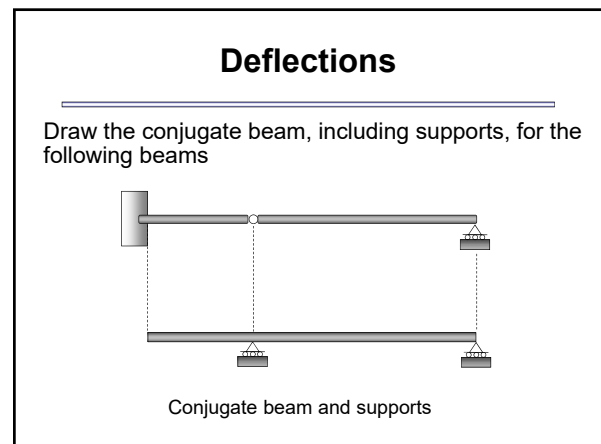
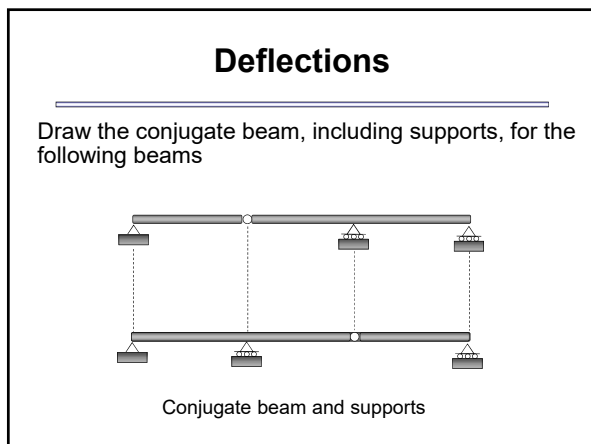
- As a rule, statically determinant real beams have statically determinant conjugate beams and statically indeterminate beams become unstable conjugate beams.
- However, the  $M/EI$  loading may provide the necessary "equilibrium" to hold the conjugate beam stable.

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#### Procedure for analysis

1. Construct the conjugate beam with the  $M/EI$  loading. Remember when the  $M/EI$  diagram is positive the loading is *upward* and when the  $M/EI$  diagram is negative the loading is *downward*.
2. Use the equations of equilibrium to solve for the reactions of the conjugate beam. This may be difficult if the moment diagram is complex.
3. Solve for the shear and moment at the point or points where the slope and displacement are desired. If the values are positive, the slope is counterclockwise and the displacement is upward.



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Draw the conjugate beam, including supports, for the following beams

Conjugate beam and supports

### Deflections

**Example:** Determine the slope and the displacement at point C for the following beam. Assume that  $E = 30,000 \text{ ksi}$  and  $I = 300 \text{ in}^4$ .

Construct the conjugate beam and apply the  $M/EI$  diagram as loading

Remember positive (+) bending moment is a positive (+) loading on the conjugate beam.

### Deflections

**Example:** Determine the slope and the displacement at point C for the following beam. Assume that  $E = 30,000 \text{ ksi}$  and  $I = 300 \text{ in}^4$ .

$$F_{CE} = \frac{1}{2} \left( \frac{50 \text{ k ft}}{EI} \right) (20 \text{ ft.})$$

$$\sum M_A = 0 = \frac{1}{2} \left( \frac{50 \text{ k ft}}{EI} \right) (20 \text{ ft.})(10 \text{ ft.}) + B_y (20 \text{ ft.}) \quad B_y = -\frac{250 \text{ k ft}^2}{EI}$$

$$\sum F_y = 0 = \frac{1}{2} \left( \frac{50 \text{ k ft}}{EI} \right) (20 \text{ ft.}) + B_y + A_y \quad A_y = -\frac{250 \text{ k ft}^2}{EI}$$

### Deflections

**Example:** Determine the slope and the displacement at point C for the following beam. Assume that  $E = 30,000 \text{ ksi}$  and  $I = 300 \text{ in}^4$ .

$$\sum M_C = 0 = -M_C + \frac{1}{2} \left( \frac{50 \text{ k ft}}{EI} \right) (10 \text{ ft.}) \left( \frac{10 \text{ ft.}}{3} \right) - \left( \frac{250 \text{ k ft}^2}{EI} \right) (10 \text{ ft.})$$

$$M_C = -\frac{1,666.6 \text{ k ft}^3}{EI} = -\frac{1,666.6 \text{ k ft}^3}{(30,000 \text{ ksi})(300 \text{ in}^4)} \cdot \frac{1,728 \text{ in}^3}{\text{ft}^3} = \boxed{-0.32 \text{ in.}}$$

### Deflections

**Example:** Determine the slope and the displacement at point C for the following beam. Assume that  $E = 30,000 \text{ ksi}$  and  $I = 300 \text{ in}^4$ .

$$\sum F_y = 0 = \frac{1}{2} \left( \frac{50 \text{ k ft}}{EI} \right) (10 \text{ ft.}) - \left( \frac{250 \text{ k ft}^2}{EI} \right) + V_C \quad V_C = 0$$

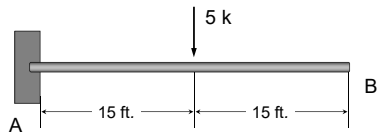
### Deflections

**Example:** Determine the slope and the displacement at point C for the following beam. Assume that  $E = 30,000 \text{ ksi}$  and  $I = 300 \text{ in}^4$ .

- Therefore, the displacement of the beam at point C is equal to the moment at point C on the conjugate beam and the slope is equal to the shear in the conjugate beam.
- In this problem, the displacement at point C is  $-0.32 \text{ in}$  and the slope is zero.

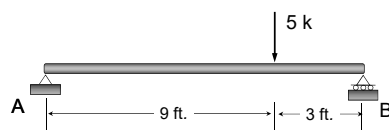
### Deflections

**Example:** Determine the slope and the displacement at point B for the following beam. Assume that  $E = 29,000$  ksi and  $I = 800$  in<sup>4</sup>.



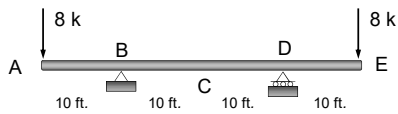
### Deflections

**Example:** Determine the maximum displacement at the mid-span of the following beam. Assume that  $E = 30,000$  ksi and  $I = 800$  in<sup>4</sup>.



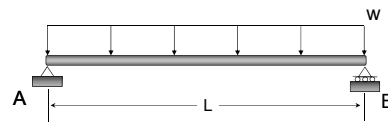
### Deflections

**Example:** Determine the slope at point B and the displacement at point E for the following beam. Assume that  $E = 29,000$  ksi and  $I_{AB} = I_{DE} = 400$  in<sup>4</sup>, and  $I_{BD} = 800$  in<sup>4</sup>.



### Deflections

**Example:** Determine the slope at A and the displacement at mid-span.



Are there any disadvantages to the conjugate beam method for uniform or high-order loading functions?

### End of Deflections – Part 2

Any questions?

