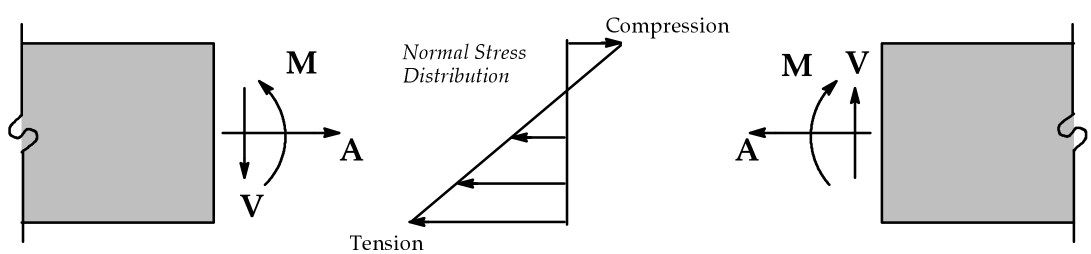
Internal Loadings Developed in Structural Members

Before a structural member can be sized or designed, the forces and moments that act on the member must be determined. In this section, we will develop methods to calculate the force and moment at any point along the member's axis.

Internal Loadings At A Specified Point

In general, the internal load at a specified point can be determined by the *method of sections.* A FBD of a cut section of a member is used to isolate internal loadings. Typically, coplanar structures are subject to internal loadings that may consist of an axial force **A**, a shear force **V**, and the bending moment **M**. These three types of loadings are used to represent the *stress distribution* acting over the member cross–section.



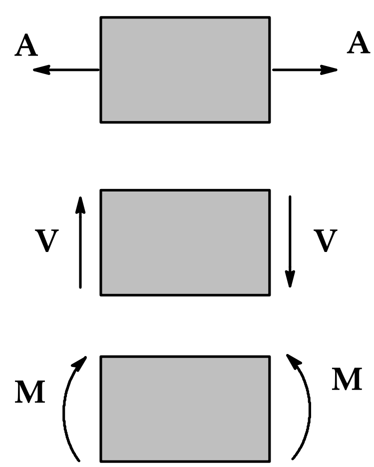
Internal Loadings

- **Sign Convention** – Choice of a sign convention is arbitrary, however, we will use the following convention to keep things consistent.

Axial Force – positive (+) if the force **A** causes the segement to be in tension or elongate.

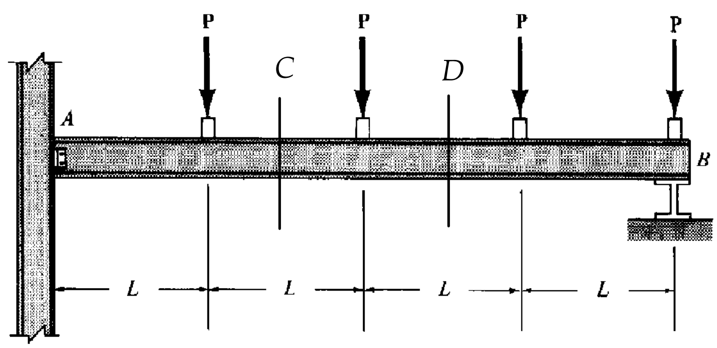
Shear Force – positive (+) if forces **V** tend to rotate the segement in a clockwise manner.

Bending Moment – positive (+) if the moment **M** tends to bend the segment concave upward, as if the segment is smiling or forms a bowl.



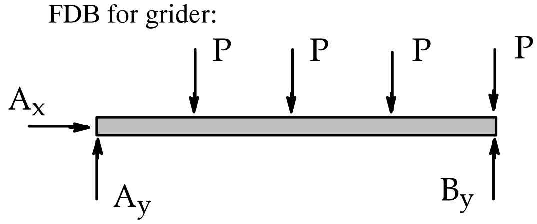
- **Procedure for analysis** – the following is a procedure for determining the internal forces in a member using the method of sections:
  1. Before the member is "cut" or sectioned, determine the support reactions for the structure.
  2. Keeping all external loadings in their exact locations, make an imaginary "cut" through the member at the point where the internal loading is desired. Draw the corresponding free-body diagram of one of the "cut" segments indicating the unknown reactions **A**, **V**, and **M** acting in their positive (+) directions.
  3. Apply the three equations of equilibrium. In most cases, the moment equation should be summed at the cut section about the *centroid* of the member cross–section to eliminate the unknowns **A** and **V**.

Example: Consider the following beam:



Determine the shear and moment in the floor girder at points *C* and *D* (both points are at the center of each span *L*.

FDB for grider:



$$\sum M_A = 0 = -P(L) - P(2L) - P(3L) - P(4L) + B_y(4L)$$

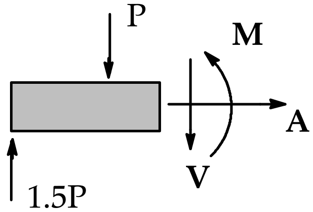
$$\boxed{B_y = 2.5P}$$

$$\sum F_y = 0 = A_y + B_y - 4P$$

$$\boxed{A_y = 1.5P}$$

$$\sum F_x = 0 = A_x$$

Cut the member at section *C*

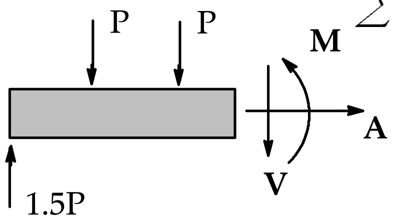


$$\sum M_{cut} = 0 = \boldsymbol{M} + P\left(\frac{L}{2}\right) - 1.5P\left(\frac{3L}{2}\right) \qquad \boxed{\boldsymbol{M} = 1.75PL}$$

$$\sum F_y = 0 = 1.5P - P - \boldsymbol{V} \qquad \boxed{\boldsymbol{V} = 0.5P}$$

$$\sum F_x = 0 = \boldsymbol{A} \qquad \boxed{\boldsymbol{A} = 0}$$

Cut the member at section *D*

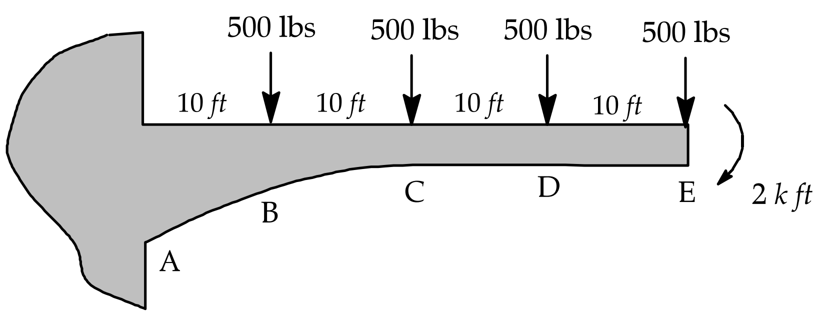


$$\sum M_{cut} = 0 = \boldsymbol{M} + P\left(\frac{L}{2} + \frac{3L}{2}\right) - 1.5P\left(\frac{5L}{2}\right) \qquad \boxed{\boldsymbol{M} = -1.75PL}$$

$$\sum F_y = 0 = 1.5P - 2P - \boldsymbol{V} \qquad \boxed{\boldsymbol{V} = -0.5P}$$

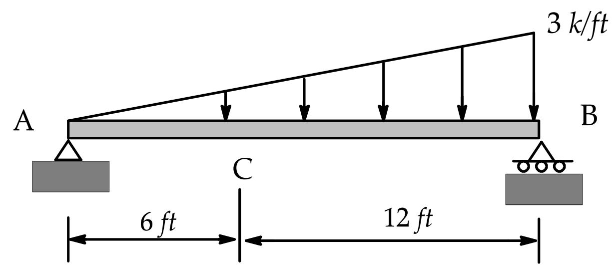
$$\sum F_x = 0 = \boldsymbol{A} \qquad \boxed{\boldsymbol{A} = 0}$$

Example: Consider the following beam:



Determine the internal shear and moment in the cantilever beam shown above at a section passing through point C.

Example: Consider the following beam:

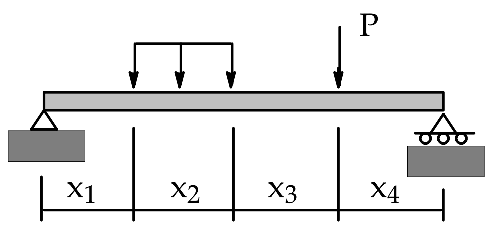


Determine the internal shear and moment at a section passing through point C.

Shear And Moment Functions

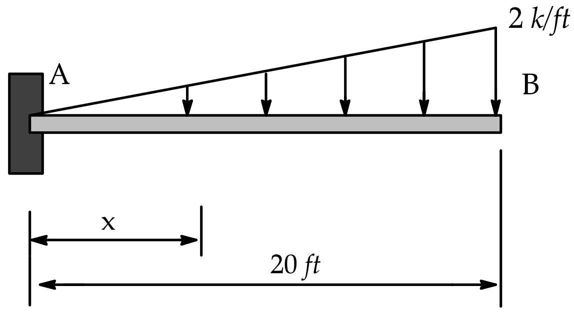
Beams are structural members which carry lateral loading (perpendicular to the bending axis). Therefore, to design a beam, a detailed knowledge of the variation of the axial force, **A**, shear force, **V**, and the bending moment, **M**, through out the member is required. Typically, axial force is not considered since: (1) in most cases the loading is perpendicular to the beam; and (2), the beam's resistance to shear and bending moment is more critical.

The variation of the shear and moment along the beam may be written as a function of the position, x. In general, the shear and moment functions are discontinuous at points where the type and magnitude of the loading changes. Therefore, the variation of the internal shear and moment should be determined for each region between any two discontinuities of loading. In the beam below, there are four discontinuous regions denoted by the lengths x_1, x_2, x_3, and x_4.



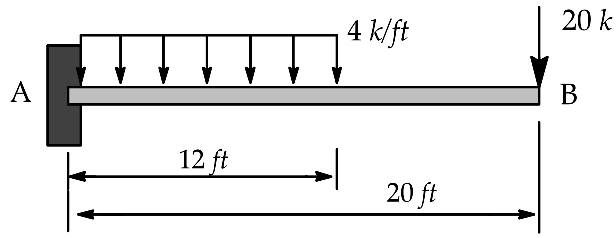
- **Procedure for analysis** – the following is a procedure for determining the variation of shear and moment in a member using the method of sections:
  1. Determine the support reactions for the structure.
  2. Keeping all external loadings in their exact locations, make a imaginary "cut" through the member at a point within the region where the shear and moment functions are desired. Draw the corresponding free-body diagram of one of the "cut" segments indicating the unknown reactions **V** and **M** acting in their positive (+) directions.
  3. Apply the equations of equilibrium. The moment equation should be summed at the cut section.

Example: Consider the following structure:



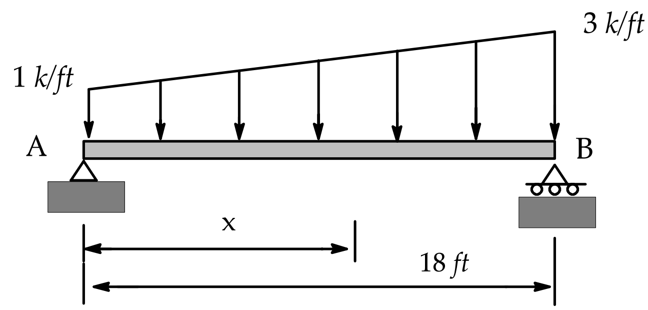
Determine the internal shear and moment as a function of x.

Example: Consider the following structure:



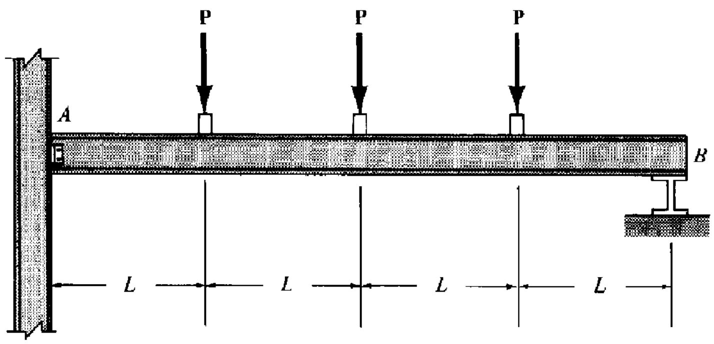
Determine the internal shear and moment as a function of x.

Example: Consider the following structure:



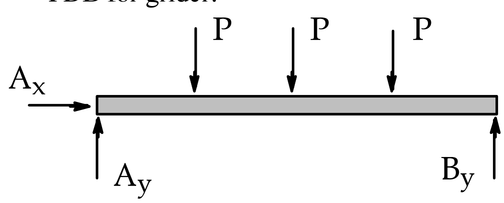
Determine the internal shear and moment as a function of x.

Example: Consider the following structure:



Determine the variation of the shear and moment in the floor girder as a function of x.

FDB for grider:



$$\sum M_A = 0 = -P(L) - P(2L) - P(3L) + B_y(4L)$$

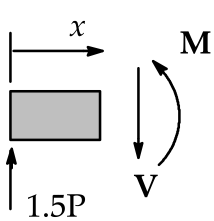
$$\boxed{B_y = 1.5P}$$

$$\sum F_y = 0 = A_y + B_y - 3P$$

$$\boxed{A_y = 1.5P}$$

$$\sum F_x = 0 = A_x$$

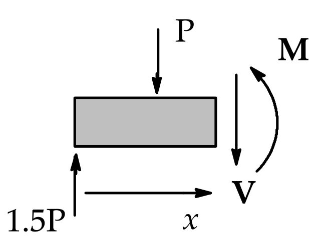
Cut the member at section where $0 < x < L$



$$\sum M_{cut} = 0 = \boldsymbol{M} - 1.5Px \qquad \boxed{\boldsymbol{M} = 1.5Px}$$

$$\sum F_y = 0 = 1.5P - \boldsymbol{V} \qquad \boxed{\boldsymbol{V} = 1.5P}$$

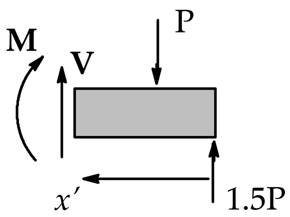
Cut the member at section $L < x < 2L$



$$\sum M_{cut} = 0 = \boldsymbol{M} - 1.5Px + P(x - L) \qquad \boxed{\boldsymbol{M} = P\left(\frac{x}{2} + L\right)}$$

$$\sum F_y = 0 = 1.5P - P - \boldsymbol{V} \qquad \boxed{\boldsymbol{V} = 0.5P}$$

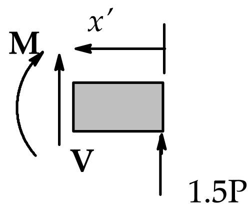
Cut the member at section $2L < x < 3L$, where $x' = 4L - x$



$$\sum M_{cut} = 0 = -\boldsymbol{M} + 1.5Px' - P(x' - L) \qquad \boxed{\boldsymbol{M} = P\left(\frac{x'}{2} + L\right)}$$

$$\sum F_y = 0 = 1.5P - P + \boldsymbol{V} \qquad \boxed{\boldsymbol{V} = -0.5P}$$

Cut the member at section 3L < *x* < 4L, where $x' = 4L - x$



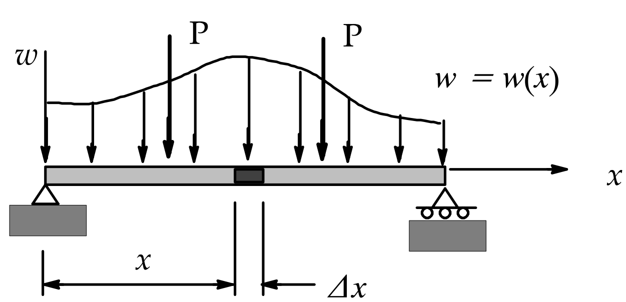
$$\sum M_{cut} = 0 = -\boldsymbol{M} + 1.5Px' \qquad \boxed{\boldsymbol{M} = 1.5Px'}$$

$$\sum F_y = 0 = 1.5P + \boldsymbol{V} \qquad \boxed{\boldsymbol{V} = -1.5P}$$

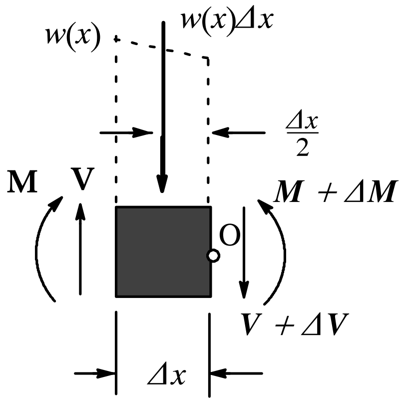
Shear And Moment Diagrams For A Beam

If the variation of **V** and **M** are written as functions of position, *x*, and plotted, the resulting graphs are called the *shear diagram* and the *moment diagram*. As we learned in the last section, developing the shear and moment functions for complex beams can be quite tedious. Therefore, in this section we will develop a simpler method for constructing shear and moment diagrams.

To develop shear and moment diagrams we need to derive the relationship between loading, shear force, and bending moment. Consider the beam shown below subjected to an arbitrary loading. We will assume that *distributed loadings will be positive (+) if they act downward.*



Let's draw a free body diagram of the small segment of length Δx and apply the equations of equilibrium. Since the segment is chosen at a point *x* where there is **no** concentrated forces or moments, the result of this analysis will **not** apply to points of concentrated loading.



$$\sum F_y = 0 = \mathbf{V} - w(x)\Delta x - (\mathbf{V} + \Delta \mathbf{V})$$

$$\boxed{\Delta \mathbf{V} = -w(x)\Delta x}$$

$$\sum \mathbf{M}_o = 0 = -\mathbf{M} + (\mathbf{M} + \Delta \mathbf{M}) - \mathbf{V}\Delta x + w(x)\Delta x\left(\frac{\Delta x}{2}\right)$$

$$\boxed{\Delta \mathbf{M} = \mathbf{V}\Delta x - w(x)\frac{(\Delta x)^2}{2}}$$

Dividing both sides of the Δ**V** and Δ**M** expressions by Δx and taking the limit as Δx tends to 0 gives:

$\frac{d\mathbf{V}}{dx} = -w(x)$			$\frac{d\mathbf{M}}{dx} = \mathbf{V}$		
Slope of Shear Curve	=	–Intensity of the Distributed Loading	Slope of Moment Curve	=	Intensity of the Shear

- The slope of the shear diagram at a point is equal to the (negative) intensity of the distributed loading *w(x)* at that point.
- The slope of the moment diagram at a point is equal to the intensity of the shear at that point.

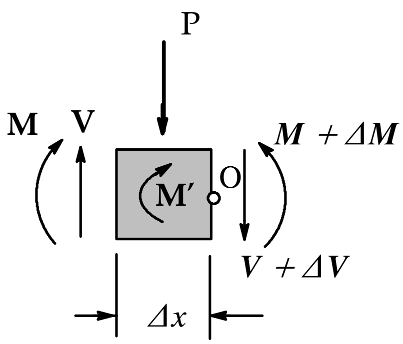
If we multiply both sides of each of the above expressions by *dx* and integrate:

$\Delta \mathbf{V} = -\int w(x)dx$			$\Delta \mathbf{M} = \int \mathbf{V}(x)dx$		
Change in Shear	=	–Area under the Distributed Loading	Change in Moment	=	Area under the Shear Diagram

- The change in shear between any two points is equal to the (negative) area under the loading curve between the points.

- The change in moment between any two points is equal to the area under the shear diagram between the points.

Note: Recall that the derivation did not consider the effect of concentrated forces, since there is a discontinuous jump in the shear diagram at a concentrated force. Similarly, the above expression does not account for discontinuities in moment. Let's consider the case where a concentrated force and/or a couple are applied to the segment.



$$\sum F_y = 0 = \mathbf{V} - (\mathbf{V} + \Delta \mathbf{V}) - P$$

$$\boxed{\Delta V = -P}$$

$$\sum \mathbf{M}_o = 0 = -\mathbf{M} + (\mathbf{M} + \Delta \mathbf{M}) - \mathbf{V}\Delta x - \mathbf{M}'$$

$$\boxed{\Delta \mathbf{M}_{\Delta x \to 0} = \mathbf{M}'}$$

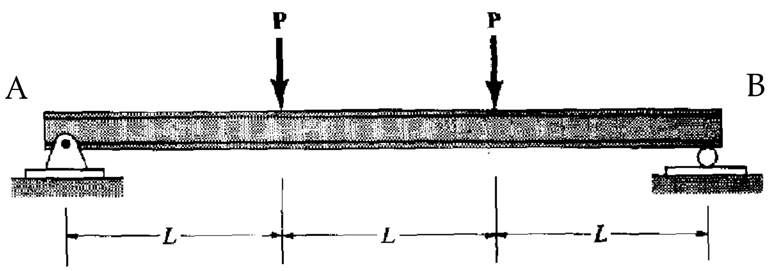
Therefore, when a force P acts downward (+) on a beam, Δ**V** is negative so the "jump" in the shear diagrams is downward. Likewise, if P acts upward (–), the "jump" is upward. When a couple acts clockwise, the resulting moment Δ**M** is positive, so the "jump" in the moment diagram is upward (+), and when the couple acts counterclockwise, the "jump" is downward (–).

- **Procedure for analysis** – the following is a procedure for constructing the shear and moment diagrams for a beam

  1. Determine the support reactions for the structure.
  2. To construct the shear diagram, first, establish the **V** and *x* axes and plot the value of the shear at each end of the beam. Since the $d\mathbf{V}/dx = -w$, the slope of the shear diagram at any point is equal to the (negative) intensity of the applied distributed loading. To determine a value of the shear at any point, one could use the method of sections or remember that the *change in the shear force is equal to the (negative) area under the distributed loading*. If the distributed loading is a curve of degree *n*, the shear will be a curve of degree *n+1*.
  3. To construct the moment diagram, first, establish the **M** and *x* axes and plot the value of the moment at each end of the beam. Since the $d\mathbf{M}/dx = \mathbf{V}$, the slope of the moment diagram at any

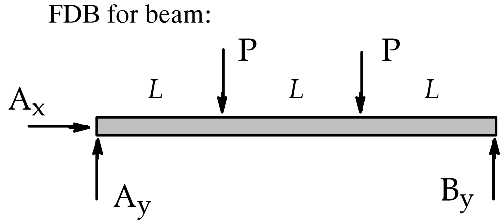
point is equal to the intensity of the shear. To determine a value of the moment at any point, one could use the method of sections or recall that the *change in the moment is equal to the area under the shear diagram*. If the shear is a curve of degree n, the moment will be a curve of degree $n+1$.

Example: Let's draw the shear and moment diagrams for the following beam:



First, solve the external reactions of the beam:

FDB for beam:



$$\sum M_A = 0 = -P(L) - P(2L) + B_y(3L)$$

$$\boxed{B_y = P}$$

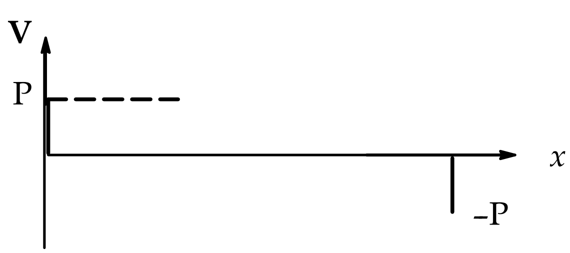
$$\sum F_y = 0 = A_y + B_y - 2P$$

$$\boxed{A_y = P}$$

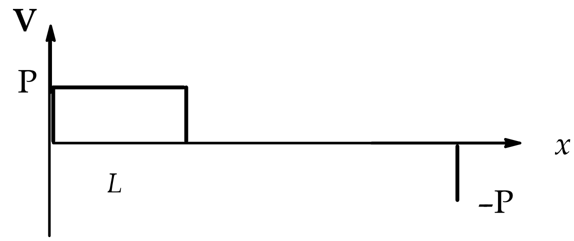
$$\sum F_x = 0 = A_x$$

Establish the **V** and x axes and plot the value of the shear at each end. In this case, the values are: at $x = 0$, **V** = P; and at $x = 3L$, **V** = –P.

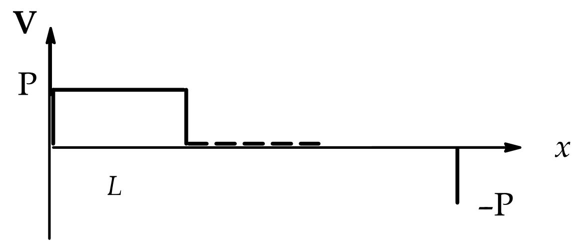
The slope of the shear diagram over the interval $0 < x < L$ is the equal to the loading. In this case $w(x) = 0$.



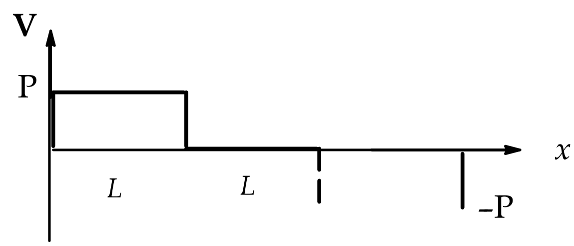
At a point x = L, a concentrated load P is applied. The shear diagram is discontinuous and “jumps” downward (recall Δ**V** = –P).



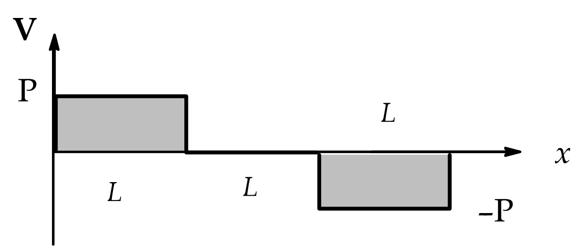
The slope of the shear diagram over the interval L < x < 2L is zero since, $w(x)$ = 0.



At 2L, P is applied and the shear diagram “jumps” downward (recall Δ**V** = –P).



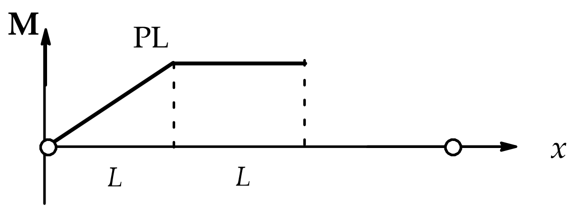
The slope of the shear diagram over the interval 2L < x < 3L is zero since, $w(x)$ = 0. The resulting shear diagram matches the shear at the right end determined from the equilibrium equations.



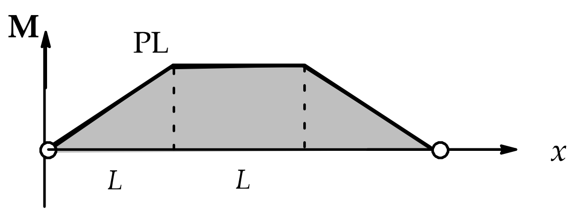
To construct the moment diagram, establish the **M** and x axes and plot the value of the moment at each end, in this case, the values are equal to zero.

The slope of the moment diagram over the interval $0 < x < L$ is the equal to value of the shear; in this case **V** = P. This indicates a positive slope of constant value. The change in moment is equal to the area under the shear diagram, in this case, Δ**M** = PL.

The slope of the moment diagram over the interval $L < x < 2L$ is the equal to value of the shear; in this case **V** = 0.

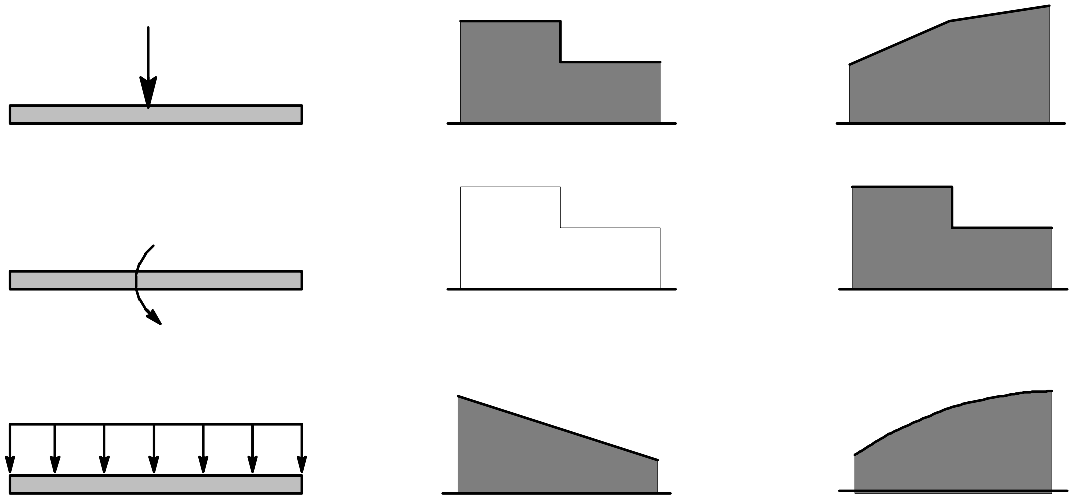


The slope of the moment diagram over the interval $2L < x < 3L$ is the equal to value of the shear, **V** = –P.

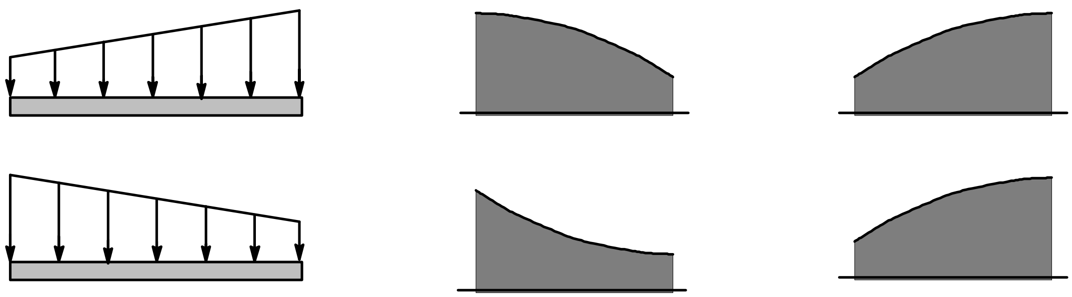


Shear And Moment Diagrams For Selected Loadings

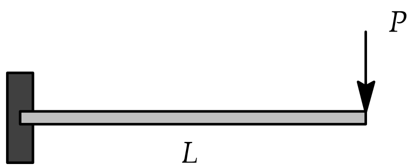
Loading | Shear Diagram $\frac{dV}{dx} = -w$ | Moment Diagram $\frac{dM}{dx} = V$

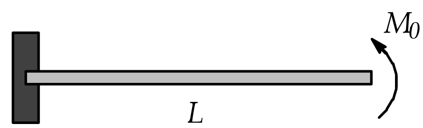


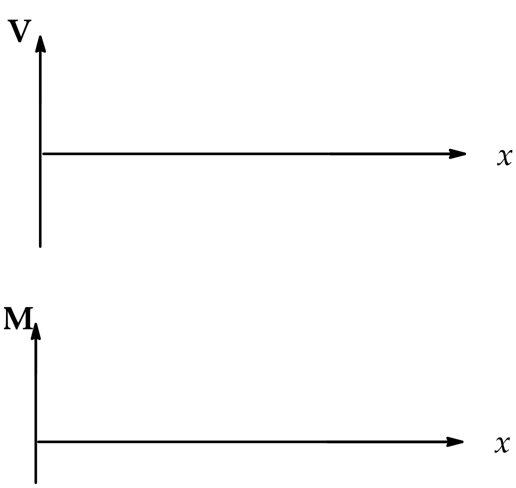
Loading | Shear Diagram $\frac{dV}{dx} = -w$ | Moment Diagram $\frac{dM}{dx} = V$

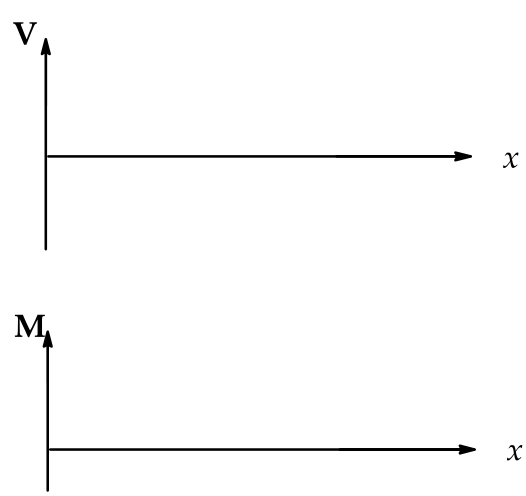


Example: Draw the shear and moment diagrams for the beams shown below:

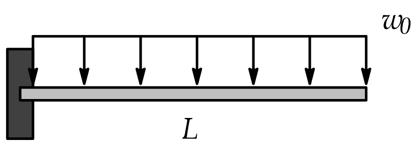


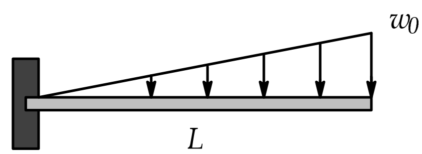




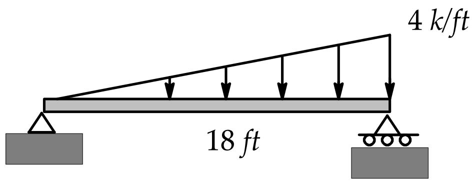


Example: Draw the shear and moment diagrams for the beams shown below:

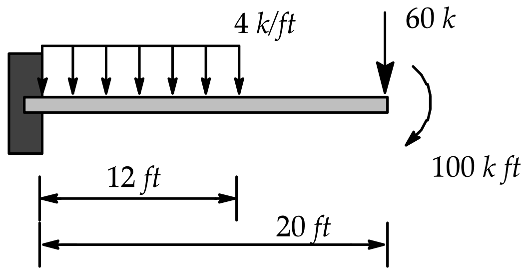




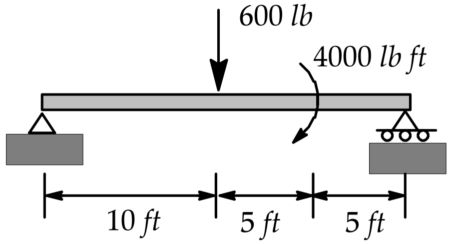
Example: Draw the shear and moment diagrams for the beam shown below:



Example: Draw the shear and moment diagrams for the beam shown below:



Example: Draw the shear and moment diagrams for the beam shown below:

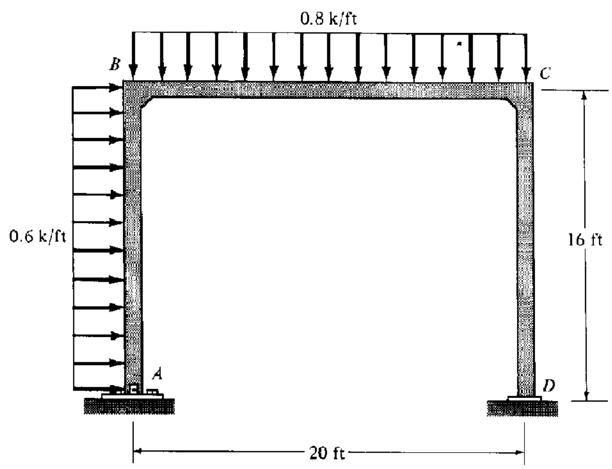


Shear And Moment Diagrams For A Frame

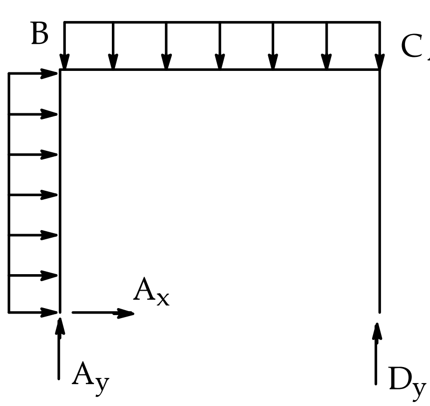
A *frame* is a structure composed of several members that are either fixed- or pin-connected at their ends. It is often necessary to draw shear and moment diagrams to design frames.

- **Procedure for analysis** – the following is a procedure for constructing the shear and moment diagrams for a frame
  1. Determine the support reactions for the frame, if possible.
  2. Determine the support reactions **A**, **V**, and **M** at the end of each member using the method of sections.
  3. Construct both shear and moment diagrams just as before. We will use the following sign convention: *always draw the moment diagram on the compression side of the member.*

Example: Let's draw the shear and moment diagrams for the following frame:



First, find as many external reactions as possible.



$$\sum M_A = 0 = -0.6(16)(8) - 0.8(20)(10) + D_y(20)$$

$$\boxed{D_y = 11.84\ k}$$

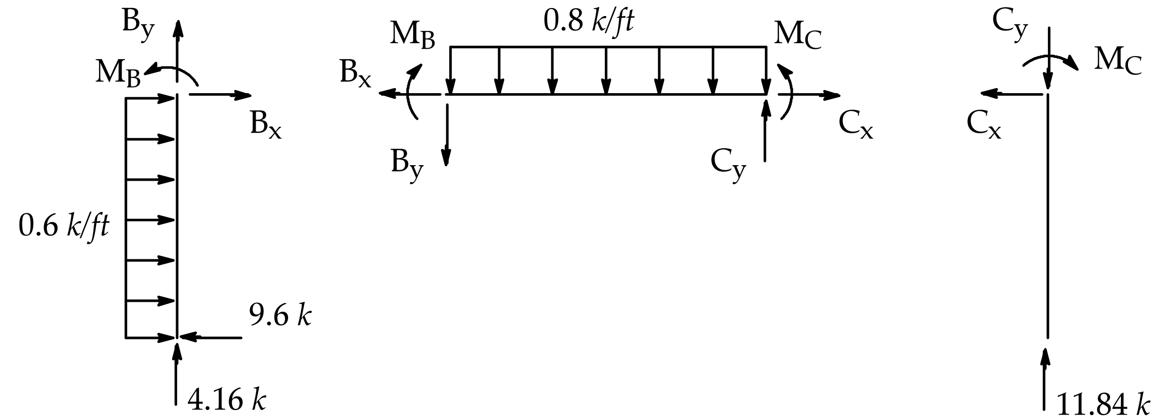
$$\sum F_y = 0 = A_y + D_y - 0.8(20)$$

$$\boxed{A_y = 4.16\ k}$$

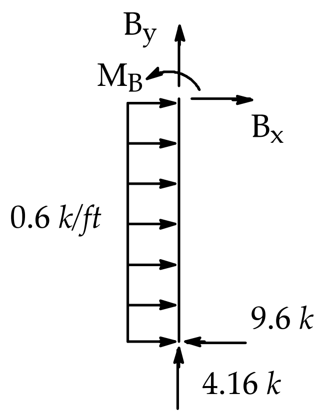
$$\sum F_x = 0 = A_x + 0.6(16)$$

$$\boxed{A_x = -9.6\ k}$$

Second, cut the frame into its component members and find the internal reactions.



Next, solve the equations of equilibrium for each member. Let's start with member AB.



$$\sum M_B = 0 = M_B + 0.6(16)(8) - 9.6(16)$$

$$\boxed{M_B = 76.8\ k\,ft}$$

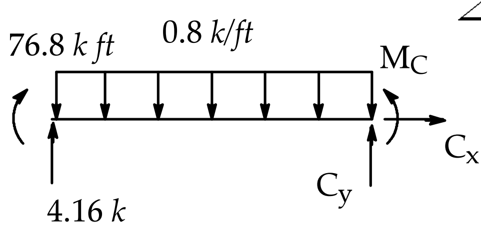
$$\sum F_y = 0 = A_y + B_y$$

$$\boxed{B_y = -4.16\ k}$$

$$\sum F_x = 0 = -9.6 + 0.6(16) + B_x$$

$$\boxed{B_x = 0}$$

Solve the equations of equilibrium for member BC.



$$\sum M_C = 0 = M_C + 0.8(20)(10) - 4.16(20) - 76.8$$

$$\boxed{M_C = 0}$$

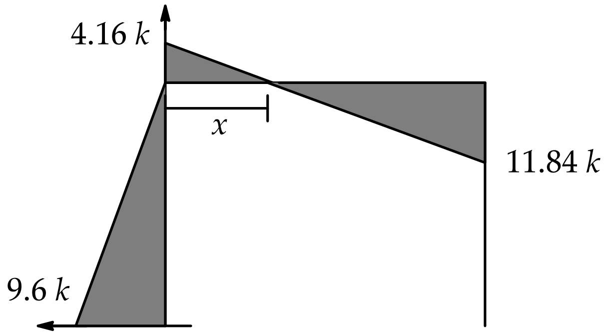
$$\sum F_y = 0 = C_y + 4.16 - 0.8(20)$$

$$\boxed{C_y = 11.84\ k}$$

$$\sum F_x = 0 = C_x$$

$$\boxed{C_x = 0}$$

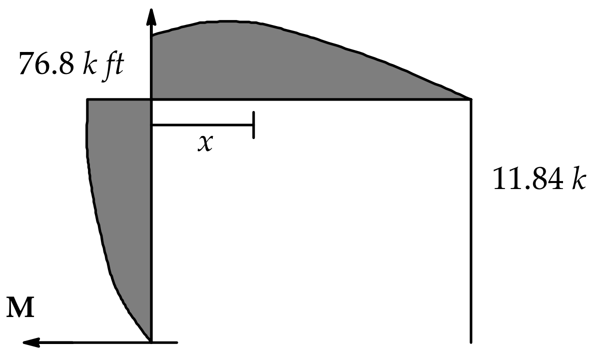
Next, let's draw the shear and moment diagrams (remember to draw the diagram on the compression side of the member)



$$V = 4.16 - 0.8x$$

$$V(x) = 0 = 4.16 - 0.8x$$

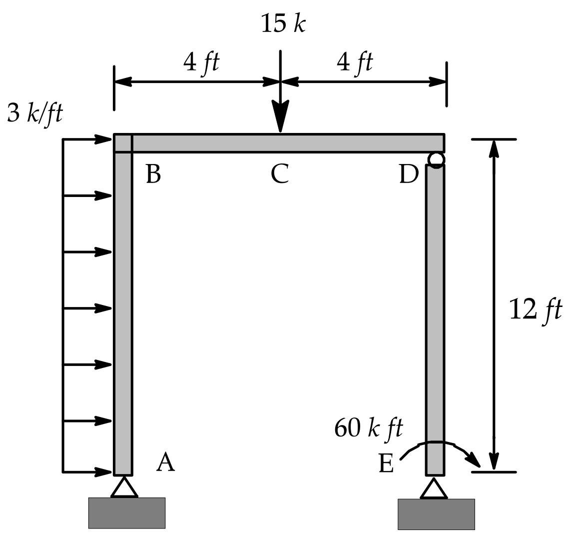
at $x = 5.2\ ft$

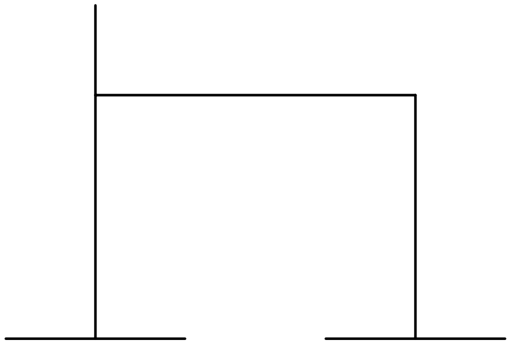


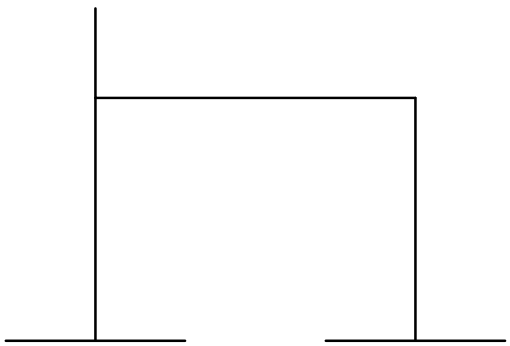
M_{max} occurs at $x = 5.2$

$M_{max} = 76.8 + 4.16(5.2)(0.5) = 87.6\ k\ ft$

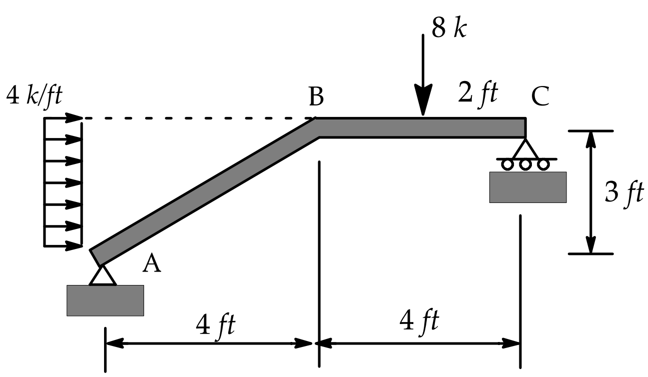
Example: Let's draw the shear and moment diagrams for the following frame:

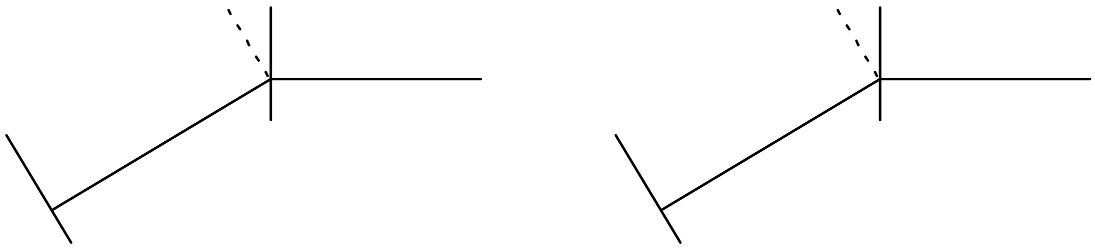






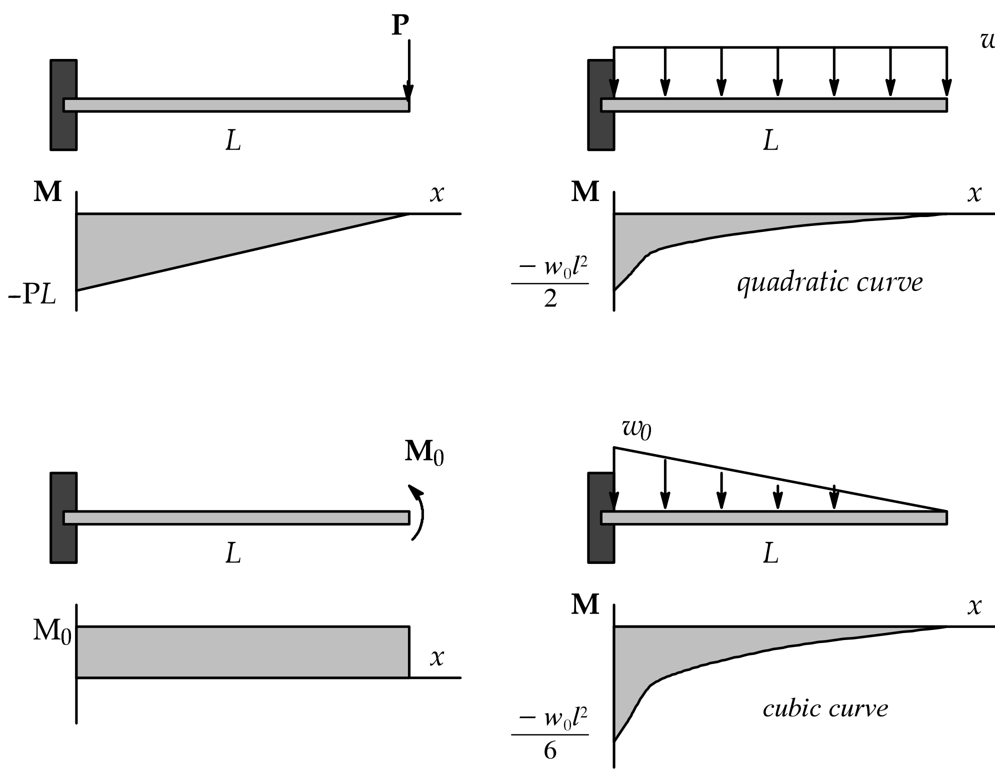
Example: Let's draw the shear and moment diagrams for the following frame:



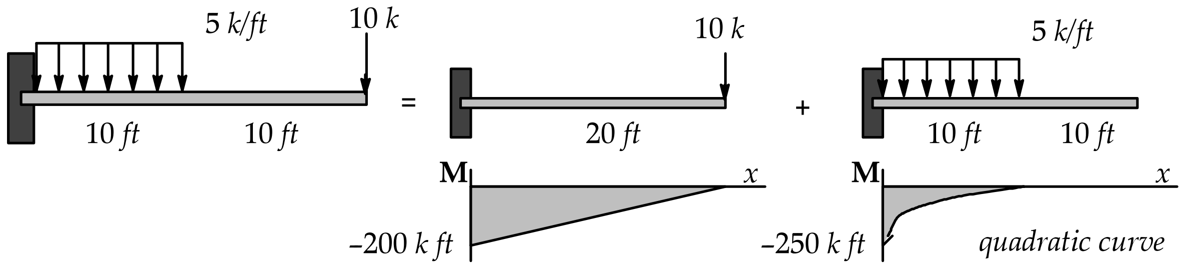


Moment Diagrams Constructed By The Method of Superposition

We have learned how to construct a moment diagram from either writing the moment as a function of x or from the slope relationship with the shear diagram. If the beam or frame is linearly elastic, we can use the principles of superposition to construct moment diagrams from a series of parts rather than from a single complex shape. Most loadings on beams and frames in structural analysis can be formed as a combination of the following loadings:



Let's use the method of superposition to find the moment diagram for the following beam:



Example: Draw the moment diagram for the following beam using superposition: