ENGINEERING PROBABILITY AND STATISTICS

DISPERSION, MEAN, MEDIAN, AND MODE VALUES

If $X_1, X_2, \ldots, X_n$ represent the values of a random sample of n items or observations, the *arithmetic mean* of these items or observations, denoted $\overline{X}$, is defined as

$$\overline{X} = (1/n)(X_1 + X_2 + \ldots + X_n) = (1/n)\sum_{i=1}^{n} X_i$$

$\overline{X} \rightarrow \mu$ for sufficiently large values of n.

The *weighted arithmetic mean* is

$$\overline{X}_w = \frac{\sum w_i X_i}{\sum w_i}, \text{ where}$$

X_i = the value of the ith observation, and
w_i = the weight applied to X_i.

The *variance* of the population is the *arithmetic mean* of the *squared deviations from the population mean*. If μ is the arithmetic mean of a discrete population of size N, the *population variance* is defined by

$$\sigma^2 = (1/N)\left[(X_1 - \mu)^2 + (X_2 - \mu)^2 + \ldots + (X_N - \mu)^2\right]$$
$$= (1/N)\sum_{i=1}^{N}(X_i - \mu)^2$$

The *standard deviation* of the population is

$$\sigma = \sqrt{(1/N)\sum(X_i - \mu)^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

The *sample standard deviation* is

$$s = \sqrt{[1/(n-1)]\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$$

The *sample coefficient of variation* = $CV = s/\overline{X}$

The *sample geometric mean* = $\sqrt[n]{X_1 X_2 X_3 \ldots X_n}$

The *sample root-mean-square value* = $\sqrt{(1/n)\sum X_i^2}$

When the discrete data are rearranged in increasing order and n is odd, the median is the value of the $\left(\frac{n+1}{2}\right)^{\text{th}}$ item

When n is even, the median is the average of the $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ items.

The *mode* of a set of data is the value that occurs with greatest frequency.

The *sample range R* is the largest sample value minus the smallest sample value.

PERMUTATIONS AND COMBINATIONS

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken r at a time* is

   $$P(n, r) = \frac{n!}{(n-r)!}$$

   nPr is an alternative notation for $P(n,r)$

2. The number of different *combinations* of n distinct objects *taken r at a time* is

   $$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{[r!(n-r)!]}$$

   nCr and $\binom{n}{r}$ are alternative notations for $C(n,r)$

3. The number of different *permutations* of n objects *taken n at a time*, given that n_i are of type i, where $i = 1, 2, \ldots, k$ and $\sum n_i = n$, is

   $$P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \ldots n_k!}$$

SETS

DeMorgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Associative Law

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

LAWS OF PROBABILITY

Property 1. General Character of Probability

The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

Property 2. Law of Total Probability

$$P(A + B) = P(A) + P(B) - P(A, B), \text{ where}$$

$P(A + B)$ = the probability that either A or B occur alone or that both occur together,

$P(A)$ = the probability that A occurs,

$P(B)$ = the probability that B occurs, and

$P(A, B)$ = the probability that both A and B occur simultaneously.

Property 3. Law of Compound or Joint Probability

If neither $P(A)$ nor $P(B)$ is zero,

$P(A, B) = P(A)P(B \mid A) = P(B)P(A \mid B)$, where

$P(B \mid A) =$ the probability that B occurs given the fact that A has occurred, and

$P(A \mid B) =$ the probability that A occurs given the fact that B has occurred.

If either $P(A)$ or $P(B)$ is zero, then $P(A, B) = 0$.

Bayes Theorem

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

where $P(A_j)$ is the probability of event A_j within the population of A

$P(B_j)$ is the probability of event B_j within the population of B

PROBABILITY FUNCTIONS

A random variable X has a probability associated with each of its possible values. The probability is termed a discrete probability if X can assume only discrete values, or

$X = x_1, x_2, x_3, \ldots, x_n$

The *discrete probability* of any single event, $X = x_i$, occurring is defined as $P(x_i)$ while the *probability mass function* of the random variable X is defined by

$f(x_k) = P(X = x_k), k = 1, 2, \ldots, n$

Probability Density Function

If X is continuous, the *probability density function, f,* is defined such that

$$P(a \le X \le b) = \int_a^b f(x)dx$$

Cumulative Distribution Functions

The *cumulative distribution function, F,* of a discrete random variable X that has a probability distribution described by $P(x_i)$ is defined as

$$F(x_m) = \sum_{k=1}^{m} P(x_k) = P(X \le x_m), m = 1, 2, \ldots, n$$

If X is continuous, the *cumulative distribution function, F,* is defined by

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

which implies that $F(a)$ is the probability that $X \le a$.

Expected Values

Let X be a discrete random variable having a probability mass function

$$f(x_k), k = 1, 2, \ldots, n$$

The expected value of X is defined as

$$\mu = E[X] = \sum_{k=1}^{n} x_k f(x_k)$$

The variance of X is defined as

$$\sigma^2 = V[X] = \sum_{k=1}^{n} (x_k - \mu)^2 f(x_k)$$

Let X be a continuous random variable having a density function $f(X)$ and let $Y = g(X)$ be some general function. The expected value of Y is:

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

The mean or expected value of the random variable X is now defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

while the variance is given by

$$\sigma^2 = V[X] = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$$

The standard deviation is given by

$$\sigma = \sqrt{V[X]}$$

The coefficient of variation is defined as σ/μ.

Sums of Random Variables

$$Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$$

The expected value of Y is:

$$\mu_y = E(Y) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$$

If the random variables are statistically *independent*, then the variance of Y is:

$$\sigma_y^2 = V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2) + \ldots + a_n^2 V(X_n)$$
$$= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \ldots + a_n^2 \sigma_n^2$$

Also, the standard deviation of Y is:

$$\sigma_y = \sqrt{\sigma_y^2}$$

Binomial Distribution: $p(X=x)=\binom{n}{x} p^x q^{n-x}$

$$E(X)=np$$
$$\sigma^2=npq$$

Negative Binomial: $p(n)=\binom{n-1}{x-1} p^x q^{n-x}$

$$E(n)=\frac{x}{p}$$
$$\sigma^2=\frac{xq}{p^2}$$

Geometric Distribution: $p(n)=p\,q^{n-1}$

$$E(n)=\frac{1}{p}$$
$$\sigma^2=\frac{q}{p^2}$$

Hypergeometric Distribution: $P(X=x)=\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

$$E(X)=n(\frac{K}{N})$$
$$\sigma^2=np(1-p)\left(\frac{N-n}{N-1}\right)$$

Poisson Distribution: $P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!}$

$$E(X)=\lambda$$
$$\sigma^2=\lambda$$

Continuous Distributions

Uniform:

$$F(x) = \frac{x-a}{b-a}$$

$$E(X) = \mu = \frac{a+b}{2}$$

$$V(X) = \sigma^2 = \frac{1}{12}(b-a)^2$$

Exponential:

$$F(x) = 1 - e^{-\lambda x}$$

$$E(X) = \mu = \frac{1}{\lambda}$$

$$V(X) = \sigma^2 = \frac{1}{\lambda^2}$$

Standard Normal:

$$Z = \frac{X-\mu}{\sigma}$$

$$F(z) = P(Z \leq z)$$

Normal Approx. to the Binomial:

$$Z = \frac{a-np}{\sqrt{npq}}$$

<u>Simple Linear Regression</u>:

$$\hat{a}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i x_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i}$$

$$MSR = \frac{SSR}{k} = \frac{\sum_{i=1}^{n} (\hat{y}_i - \bar{y}_i)^2}{k}$$

$$\hat{a}_0 = \bar{y} - \hat{a}_1 \bar{x}$$

$$MSE = s_{y|x}^2 = \frac{SSE}{n-k-1} = \frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n-k-1}$$

$$r^2 = \frac{SSR}{SST}$$

$$\text{C.I.} = \hat{y}_i \pm t_{\alpha/2, n-2}\, s_{y|x} \sqrt{\frac{1}{n} + \frac{(x_i - \bar{x})^2}{SSX}}$$

$$s_y = \sqrt{\frac{SST}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1}}$$

$$s_{y|x} = \sqrt{\frac{SSE}{n-2}} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n-2}}$$

***t* Test for the Slope**

The null and alternative hypotheses are:

H_o: $a_1 = 0$ (There is no linear relationship between *Y* and *X*)

H_a: $a_1 \neq 0$ (There is a linear relationship between *Y* and *X*)

T.S.: $t = \frac{\bar{x} - \mu_o}{s/\sqrt{n}} = \frac{\hat{a}_1}{s_{y|x}/\sqrt{SSX}}$ *where* $SSX = \sum_{i=1}^{n} (x_i - \bar{x})^2$

R.R.: $|t| > t_{\alpha/2, n-2}$

***F* Test for the Slope**

H$_o$: $a_1 = 0$ (There is no linear relationship between Y and X)

H$_a$: $a_1 \neq 0$ (There is a linear relationship between Y and X)

T.S.: $F = \dfrac{MSR}{MSE}$

R.R.: $F > F_{\alpha,k,n-k-1}$

<u>Multiple Regression</u>:

Coefficient of *multiple* determination:

$$R^2 = \frac{SSR}{SST} =$$

and the standard error of the y estimate:

$$s_{y|x} = \sqrt{\frac{SSE}{n-k-1}}$$

Hypothesis Testing

The null and alternative hypotheses are as follows:

H$_o$: $a_1 = a_2 = 0$ (There is no linear relationship)

H$_a$: At least one $a_i \neq 0$ (There is at least one linear relationship)

The appropriate test statistic is:

T.S.: $F = \dfrac{MSR}{MSE}$

R.R.: $F > F_{\alpha,k,n-k-1}$

***t*-test for individual regressor variables x_i:**

H$_o$: $a_i = 0$ (delivery distance does not contribute anything to the regression)

H$_a$: $a_i \neq 0$ (delivery distance does contribute something to the regression)

The appropriate test statistic is:

T.S.: $t = \frac{\hat{a}_i}{s_{y|x} / \sqrt{SSX_i}}$ *where* $SSX_i = \sum_{j=1}^{n} \left(x_{ij} - \overline{x}_i \right)^2$

R.R.: $|t| > t_{\alpha/2, n-k-1}$

Confidence Interval for Regression Coefficient

$$\text{C.I.} = \hat{a}_i \pm s_i t_{0.025, n-k-1}$$

where:

$$s_i = \frac{s_{y|x}}{\sqrt{SSX_i}}$$

Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x)\,dx$$

are:

Euler's or Forward Rectangular Rule

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

Trapezoidal Rule

for $n = 1$

$$\int_a^b f(x)\,dx \approx \Delta x\left[\frac{f(a) + f(b)}{2}\right]$$

for $n > 1$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(a + k\Delta x) + f(b)\right]$$

Simpson's Rule/Parabolic Rule (n must be an even integer)

for $n = 2$

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

for $n \ge 4$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[\begin{array}{l} f(a) + 2\sum\limits_{k=2,4,6,\ldots}^{n-2} f(a + k\Delta x) \\ \\ +4\sum\limits_{k=1,3,5,\ldots}^{n-1} f(a + k\Delta x) + f(b) \end{array}\right]$$

with $\Delta x = (b - a)/n$

n = number of intervals between data points

Numerical Solution of Ordinary Differential Equations

<u>Euler's Approximation</u>

Given a differential equation

$dx/dt = f(x, t)$ with $x(0) = x_o$

At some general time $k\Delta t$

$x[(k + 1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t), k\Delta t]$

which can be used with starting condition x_o to solve recursively for $x(\Delta t)$, $x(2\Delta t)$, …, $x(n\Delta t)$.

The method can be extended to nth order differential equations by recasting them as n first-order equations.

In particular, when $dx/dt = f(x)$

$x[(k + 1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t)]$

which can be expressed as the recursive equation

$x_{k+1} = x_k + \Delta t\,(dx_k/dt)$

Refer to the **ELECTRICAL AND COMPUTER ENGINEERING** section for additional information on Laplace transforms and algebra of complex numbers.

Interpolation:

1st order:

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

2nd order:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

3rd order:

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

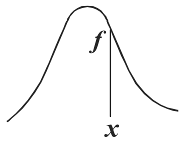
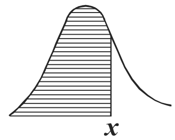
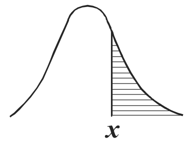
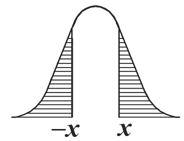
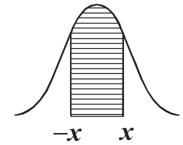
$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

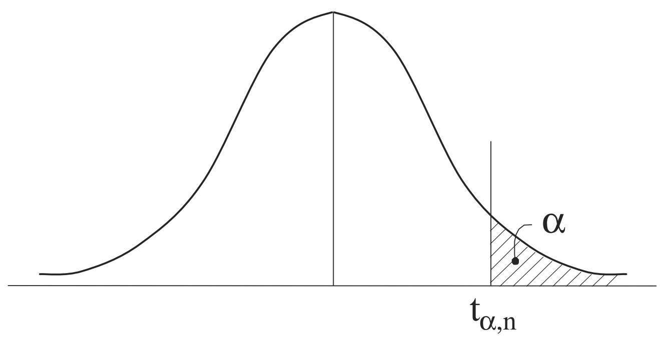
$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$b_3 = \frac{\left(\frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}\right) - \left(\frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}\right)}{x_3 - x_0}$$

UNIT NORMAL DISTRIBUTION

					
x	***f*(*x*)**	***F*(*x*)**	***R*(*x*)**	**2*R*(*x*)**	***W*(*x*)**
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

STUDENT'S *t*-DISTRIBUTION

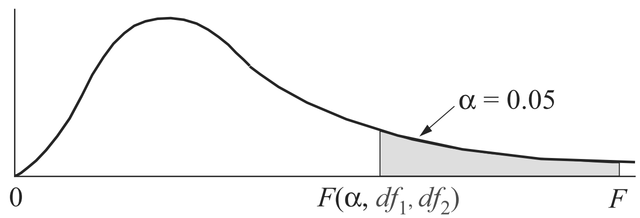


VALUES OF $t_{\alpha,n}$

n	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	n
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	1
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	2
3	0.765	0.978	1.350	1.638	2.353	3.182	4.541	5.841	3
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	4
5	**0.727**	**0.920**	**1.156**	**1.476**	**2.015**	**2.571**	**3.365**	**4.032**	**5**
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	6
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	7
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	8
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	9
10	**0.700**	**0.879**	**1.093**	**1.372**	**1.812**	**2.228**	**2.764**	**3.169**	**10**
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	11
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	12
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	13
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	14
15	**0.691**	**0.866**	**1.074**	**1.341**	**1.753**	**2.131**	**2.602**	**2.947**	**15**
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	16
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	17
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	18
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	19
20	**0.687**	**0.860**	**1.064**	**1.325**	**1.725**	**2.086**	**2.528**	**2.845**	**20**
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	21
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	22
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	23
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	24
25	**0.684**	**0.856**	**1.058**	**1.316**	**1.708**	**2.060**	**2.485**	**2.787**	**25**
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	26
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	27
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	28
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	29
30	**0.683**	**0.854**	**1.055**	**1.310**	**1.697**	**2.042**	**2.457**	**2.750**	**30**
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	∞

CRITICAL VALUES OF THE *F* DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of *F* corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1																		
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	**∞**
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**1.92**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00