Problem sets are a decoy to lure you away from potential exam material.

Centroids

- When we dealt with distributed loads, we found the magnitude of the force generated by the loading as the area under the loading curve.
- I gave you the location of the line of action of the force for both a rectangular shape and a right-triangular shape.
Centroids

- In this meeting, we are going to find out just why that line of action was located where it was.
- The line of action was located through the centroidial axis of the loading diagram.
- If we took a centroidial axis in every direction, their intersection point would be known as the centroid.

Centroids

- By common practice, we refer to the centroidal axis as the centroid but to keep the confusion down we will often speak of a x-centroid or a y-centroid referring to the coordinate along that axis where the centroidal axis intersects the coordinate axis.
Centroids

- We are going to look at two mathematical techniques for locating this centroidal axis or centroid.
- First we will look at what a centroid means.

Consider that we have a series of rectangular loads along an axis.
We would like to replace this loading with a single point force for analysis purposes.

We would label each load.
Then find the area of each loading, giving us the force which is located at the center of each area.

The force generated by each loading is equal to the area under its loading diagram so

\[ F_n = A_{L_n} \]
The force generated by each loading is equal to the area under the loading diagram so

\[ F_n = A_{L_n} \]

The total force generated by all these forces is just their sum

\[ F = \sum F_n = \sum A_{L_n} \]
But we want to replace these forces with a single force with the same net effect on the system.

\[ F = \sum F_n = \sum A_{I_n} \]

That would mean that it would have to produce the same moment about any point on the system.

\[ F = \sum F_n = \sum A_{I_n} \]
If we choose the origin for the moment center, the moment of each of these forces about the origin is equal to the x-distance to the line of action of the force times the force.

For example, for the force (area) $F_3$

$$M_3 = x_3 F_3$$
The moment for any force is

\[ M_n = x_n F_n \]

Which can be replaced by

\[ M_n = x_n A_{L_n} \]
And the total moment of all the forces (areas) about the origin is

\[ M_{total} = \sum_{i=1}^{n} x_i A_{L_i} \]
The value $\bar{x}$ is known as the x-centroid of the loading

$$M = \bar{x}F$$

If we substitute the sums developed earlier for the total force and the total moment

$$M = \bar{x}F$$

$$\sum x_i F_i = \bar{x} \sum F_i$$
Centroids

And isolating the centroid we have

\[
\bar{x} = \frac{\sum x_i F_i}{\sum F_i} \quad \text{or} \quad \frac{\sum x_i A_i}{\sum A_i}
\]

A general definition for the x-centroid of a series of n areas would be

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}
\]
Centroids

- If the areas represent forces, the centroid represents the center of gravity

\[
\overline{x} = \frac{\sum_{i=1}^{n} x_i F_i}{\sum_{i=1}^{n} F_i}
\]

Centroids

- If the areas represent masses, the centroid represents the center of mass

\[
\overline{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}
\]
If the areas are just areas, the centroid represents the center of area or centroid:

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}
\]

This is the general formulation for finding the x-centroid of \( n \) areas:

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}
\]
The same type of formula could be found for finding the y centroid:

\[
\bar{y} = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i}
\]

Remember that the \( x_i \) is the x-distance to the centroid of the \( i^{th} \) area:

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i}
\]
So far, we have been able to describe the forces (areas) using rectangles and triangles.

Now we have to extend that to loadings and areas that are described by mathematical functions.

For example:
Centroids from Functions

- This is a distributed load that at any $x$ has a load intensity of $w_0x^2$

$w_0$ is a proportionality constant that will have units to make sure that the units of the product $w_0x^2$ will be in force per length units.
If we had this type of loading over a distance L, how would we find the equivalent point force and its location?

We could generate a series of rectangles to lay over the curve.
Centroids from Functions

- Each rectangle will have some width $Dx$

And a height based on where the rectangle is positioned
Generalizing for any rectangle

So for n-rectangles we would have

\[ A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} w_0 x_i^2 \Delta x \]
The total moment generated by these areas would be

\[ M = \sum_{i=1}^{n} x_i A_i = \sum_{i=1}^{n} x_i \left( w_0 x_i^2 \right) \Delta x \]

And the location of the centroid would be

\[ M = A \bar{x} \]

\[ \sum_{i=1}^{n} x_i A_i = \bar{x} \sum_{i=1}^{n} A_i \]

\[ \sum_{i=1}^{n} x_i \left( w_0 x_i^2 \right) \Delta x = \bar{x} \sum_{i=1}^{n} \left( w_0 x_i^2 \right) \Delta x \]

\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i \left( w_0 x_i^2 \right) \Delta x}{\sum_{i=1}^{n} \left( w_0 x_i^2 \right) \Delta x} \]
The general form would be

\[ \sum_{i=1}^{n} x_i A_i = \bar{x} \sum_{i=1}^{n} A_i \]

\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \]

So for any loading that we can break up into n individual loadings with known centroids, the centroid of the composite would be equal to

\[ \sum_{i=1}^{n} x_i A_i = \bar{x} \sum_{i=1}^{n} A_i \]

\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \]
Centroids from Functions

- If we reduce the width of the rectangles to a differential size, the summation becomes an integral.

\[
X = \frac{\int_0^L x_i q(x_i) \, dx}{\int_0^L q(x_i) \, dx}
\]

- \(q(x)\) represents a general loading function.

- If we can define the height of the loading diagram at any point \(x\) by the function \(q(x)\), then we can generalize our summations of areas by the quotient of the integrals.

\[
X = \frac{\int_0^L x_i q(x_i) \, dx}{\int_0^L q(x_i) \, dx}
\]
Centroids from Functions

- This is the general form for the integral to locate the centroid

\[ \bar{x} = \frac{\int_A xq(x) \, dx}{\int_A q(x) \, dx} \]

- It isn't always quite that simple
- You have to be careful in
  - Knowing the height of your rectangular section
  - Knowing the limits of integration
  - Making the correct integration
An Example

- We need to locate the x and y centroids of the shape between the curves.
An Example

- We can start with the x-centroid

First we will sketch a representative rectangle
An Example

- Determine the height of the rectangle

- Determine the width of the rectangle
An Example

- So the area of the rectangle becomes

\[ A = \left( \sqrt{4x} - \frac{x^2}{4} \right) dx \]

An Example

- The moment arm is the distance from the moment center (in this case the origin)
An Example

- The limits of integration will be the beginning and ending points of $x$.

So when we set up the integral form for the centroid we have

$$
\bar{x} = \frac{\int_{0}^{4m} x \left( \sqrt{4x} - \frac{x^2}{4} \right) dx}{\int_{0}^{4m} \left( \sqrt{4x} - \frac{x^2}{4} \right) dx}
$$
An Example

Integrating

\[
\overline{x} = \frac{\int_{0}^{4m} \left( \frac{3}{2}x^2 - \frac{x^3}{4} \right) dx}{\int_{0}^{4m} \left( \frac{1}{2}x^2 - \frac{x^2}{4} \right) dx} = \frac{\frac{4}{5} \left( 4m \right)^{\frac{5}{2}} - \frac{4}{16} \left( 4m \right)^{\frac{4}{2}}}{\frac{4}{3} \left( 4m \right)^{\frac{3}{2}} - \frac{4}{12} \left( 4m \right)^{\frac{2}{2}}}
\]

Substituting at the upper and lower limits

\[
\overline{x} = \left( \frac{4}{5} \left( 4m \right) - \frac{(4m)^{\frac{5}{2}}}{16} \right) - \left( \frac{4}{5} \left( 0m \right) - \frac{(0m)^{\frac{5}{2}}}{16} \right)
\]

\[
- \left( \frac{4}{3} \left( 4m \right) - \frac{(4m)^{\frac{3}{2}}}{12} \right) + \left( \frac{4}{3} \left( 0m \right) - \frac{(0m)^{\frac{3}{2}}}{12} \right)
\]
An Example

For the y centroid, we need a rectangle that goes from left to right

\[
\bar{y} = \frac{\int_{0}^{4m} y \left( x_{\text{right}} - x_{\text{left}} \right) dy}{\int_{0}^{4m} \left( x_{\text{right}} - x_{\text{left}} \right) dy}
\]

\[
\bar{x} = 1.8m
\]
An Example

For this problem

\[ \bar{y} = \frac{\int_{0}^{4m} y \left( \sqrt{4} y - \frac{y^2}{4} \right) dy}{\int_{0}^{4m} \left( \sqrt{4} y - \frac{y^2}{4} \right) dy} \]

Evaluate the integrals and substitute the limits

\[ \bar{y} = 1.8m \]
Points to Remember

- Draw the rectangle you are going to use
- Be careful that you take the correct distance from the correct axis
- You may want to always use x or y as the variable of integration; be very careful here, it is only to the center of the differential side that you can assume the moment arm goes to