Dot Product
Components With Respect to an Arbitrary Axis

“It is not enough to have a good mind. The main thing is to use it well.”
-Rene Descartes

Objectives
- Understand the computation of the dot product between two vectors
- Understand the use of the dot product to calculate the angle between two lines/vectors
- Understand the use of the dot product to develop the components of a vector(force) parallel to and perpendicular to a line

Tools
- Basic Trigonometry
- Pythagorean Theorem
- Cartesian Representation of Vectors
- Position Vectors
Dot Product

- Vector multiplication comes in two distinct flavors
  - The DOT or scalar Product
  - The CROSS or vector Product
- We will limit ourselves to talking about the DOT product in this presentation

The result of taking the DOT product between two vectors is a scalar
- The units of the DOT product are the product of the units of the two vectors

For example if we have a force vector \( F \) with units of Newtons and a position vector \( r \) with units of meters then \( F \cdot r \) would have units of Newton-meters
- The Dot product itself is a scalar
Dot Product

- Forming a DOT product

\[ \vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} \]
\[ \vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k} \]
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \]

- Notice that the product doesn’t have any vectors on the right hand side

\[ \vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} \]
\[ \vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k} \]
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \]

- Also notice that it is only coefficients of like unit vectors that are multiplied together

\[ \vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} \]
\[ \vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k} \]
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \]
Dot Product

- The DOT product is also defined as

\[ \vec{A} \cdot \vec{B} = AB \cos(\Theta) \]

Where \( \Theta \) is the angle between the two vectors

\[
\vec{A} = A_i \hat{i} + A_j \hat{j} + A_k \hat{k} \\
\vec{B} = B_i \hat{i} + B_j \hat{j} + B_k \hat{k} \\
\vec{A} \cdot \vec{B} = A_i B_i + A_j B_j + A_k B_k
\]

- Remember that the magnitudes of the two vectors are always positive so the sign of the cosine of the angle will be determined by the sign of the dot product

\[ \vec{A} \cdot \vec{B} = AB \cos(\Theta) \]

\[
\vec{A} = A_i \hat{i} + A_j \hat{j} + A_k \hat{k} \\
\vec{B} = B_i \hat{i} + B_j \hat{j} + B_k \hat{k} \\
\vec{A} \cdot \vec{B} = A_i B_i + A_j B_j + A_k B_k
\]

- This leads to one use of the dot product: finding the angle between two vectors or lines (which we can define as position vectors)

\[
\frac{\vec{A} \cdot \vec{B}}{AB} = \cos(\Theta) \\
\vec{A} \cdot \vec{B} = A_i B_i + A_j B_j + A_k B_k
\]
Calculating Angle between Two Lines

Given:

\[ A = \{6, 8, 0\} f t \]
\[ B = \{0, 10, 4\} f t \]
\[ C = \{8, 0, 10\} f t \]

Required: Angle \( \theta \)

Solution

\[ \frac{\overrightarrow{r}_{AC} \otimes \overrightarrow{r}_{AC}}{(r_{AC})(r_{AC})} = \cos(\Theta) \]

\[ A = \{6, 8, 0\} f t \]
\[ B = \{0, 10, 4\} f t \]
\[ C = \{8, 0, 10\} f t \]
Calculating Angle between Two Lines

Solution:

\[
\frac{\vec{r}_{AB} \cdot \vec{r}_{AC}}{\|\vec{r}_{AB}\| \|\vec{r}_{AC}\|} = \cos(\theta)
\]

\[
\vec{r}_{AB} = \{B - A\}
\]

\[
\vec{r}_{AC} = \{C - A\}
\]

\[
\vec{r}_{AB} = \{6, 8, 0\}
\]

\[
\vec{r}_{AC} = \{0, 10, 4\}
\]

\[
\|\vec{r}_{AB}\| = \sqrt{6^2 + 8^2 + 0^2} = 10
\]

\[
\|\vec{r}_{AC}\| = \sqrt{0^2 + 10^2 + 4^2} = \sqrt{116}
\]

\[
\vec{r}_{AB} \cdot \vec{r}_{AC} = (6 \cdot 0) + (8 \cdot 10) + (0 \cdot 4) = 80
\]

\[
\cos(\theta) = \frac{80}{10 \cdot \sqrt{116}} = \frac{8}{\sqrt{29}}
\]

\[
\theta = \cos^{-1}\left(\frac{8}{\sqrt{29}}\right) \approx 51.37^\circ
\]
Calculating Angle between Two Lines

Solution:

\[
\frac{12 \cdot 7}{(7.48 \cdot 8)(12.96 \cdot 1)} = \cos(\Theta)
\]

\[
0.124 = \cos(\Theta)
\]

\[
\Theta = 82.89^\circ
\]
Dot Product

- Looking at the component along the line of action of $B$

$$\vec{A} \cdot \vec{B} = AB \cos(\Theta)$$

- Isolating the component magnitude

$$\frac{\vec{A} \cdot \vec{B}}{B} = A \cos(\Theta)$$

- Using the definition of a unit vector

$$\frac{\vec{A} \cdot \vec{B}}{B} = \vec{A} \cdot \vec{u}_B = A \cos(\Theta)$$
Dot Product and Projections

And defining the component as a vector we have
\[ \overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{A} \cdot \overrightarrow{u_B} = A \cos(\Theta) \]
\[ \overrightarrow{A}_{B} = (\overrightarrow{A} \cdot \overrightarrow{u_B}) \overrightarrow{u_B} \]
\[ \overrightarrow{A} = \overrightarrow{A}_{B} + \overrightarrow{A}_{\perp B} \]
\[ \overrightarrow{A}_{\perp B} = \overrightarrow{A} - (\overrightarrow{A} \cdot \overrightarrow{u_B}) \overrightarrow{u_B} \]

Components

Given: System as shown in previous example problem

\[ \overrightarrow{z} \cdot \overrightarrow{F_{AB}} = 12 \text{ lb} \]

Required: Components of \( \overrightarrow{F_{AB}} \) in the direction of \( \overrightarrow{AB} \) and perpendicular to the direction of \( \overrightarrow{AC} \)
Components

Solution:

\[ \vec{F}_{\text{net}} = (\vec{F}_{\text{AB}} \otimes \vec{u}_{\text{AC}})'_{\text{AC}} \]

\[ \vec{F}_{\text{AB}} = \vec{F}_{\text{AB}} \vec{u}_{\text{AB}} \]

\[ \vec{u}_{\text{AB}} = \frac{\vec{r}_{\text{AB}}}{r_{\text{AB}}} \]

\[ \vec{u}_{\text{AC}} = -0.80\hat{i} + 0.27\hat{j} + 0.53\hat{k} \]

\[ \vec{F}_{\text{AB}} = 12b (-0.80\hat{i} + 0.27\hat{j} + 0.53\hat{k}) \]

\[ \vec{F}_{\text{AB}} = (-9.62\hat{i} + 3.21\hat{j} + 6.41\hat{k})b \]

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Components

Solution:

\[ \vec{u}_{\text{AC}} = \frac{\vec{r}_{\text{AC}}}{r_{\text{AC}}} \]

\[ \vec{u}_{\text{AC}} = \frac{(2\hat{i} - 8\hat{j} + 10\hat{k})}{12.96\hat{i}} \]

\[ \vec{u}_{\text{AC}} = 0.15\hat{i} - 0.62\hat{j} + 0.77\hat{k} \]

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Components

\[ \vec{F}_{\text{net}} = (\vec{F}_{\text{AB}} \otimes \vec{u}_{\text{AC}})'_{\text{AC}} \]

\[ \vec{u}_{\text{AC}} = 0.15\hat{i} - 0.62\hat{j} + 0.77\hat{k} \]

\[ \vec{F}_{\text{AB}} = (-9.62\hat{i} + 3.21\hat{j} + 6.41\hat{k})b \]

\[ \vec{F}_{\text{net}} = \left[ (-9.62\hat{i} + 3.21\hat{j} + 6.41\hat{k}) \otimes 0.15\hat{i} - 0.62\hat{j} + 0.77\hat{k} \right] \]

\[ \vec{F}_{\text{net}} = \left[ 0.15\hat{i} - 0.62\hat{j} + 0.77\hat{k} \right] \]

\[ \vec{F}_{\text{net}} = 1.48b (0.15\hat{i} - 0.62\hat{j} + 0.77\hat{k}) \]

\[ \vec{F}_{\text{net}} = (0.23\hat{i} - 0.92\hat{j} + 1.15\hat{k})b \]

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Example Problem 2-116