Whitney Rectangular Stress Distribution

The computation of flexural strength M_n based on the approximately parabolic stress distribution shown in Figure 1 may be done using given values of $k_2/(k_1k_3)$. However, it is desirable to have a simple method in which basic static equilibrium is used.

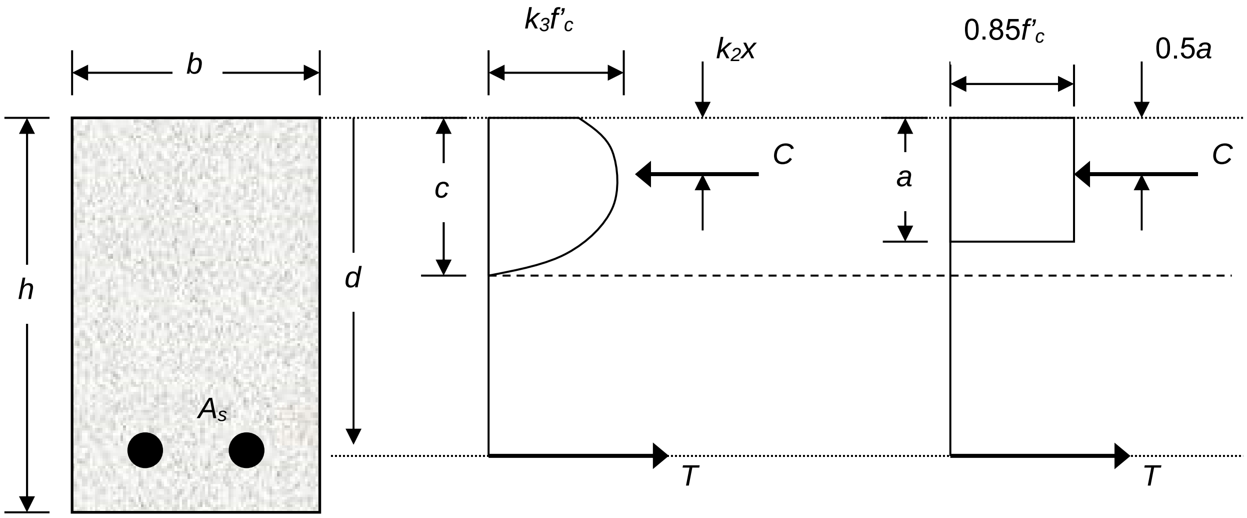

Figure 1. Definition of Whitney Rectangular Stress Distribution

In the 1930s, Whitney (1937) proposed the use of a rectangular compressive stress distribution to replace the parabolic stress distribution. As shown in Figure 1, an average stress of $0.85f'_c$ is used with a rectangle of depth $a = \beta_1 c$. Whitney determined that β_1 should be 0.85 for concrete with f'_c > 4,000 psi, and 0.05 less for each 1,000 psi of f'_c in excess of 4,000 psi. The value of β_1 may not be taken less than 0.65 (ACI 2011). The concrete below the neutral axis is ignored and the total tension force T is due the reinforcing steel. The Whitney stress block is used to estimate the compression force C.

The bending strength M_n using the equivalent rectangle is obtained from Figure 1 as follows

$$C = 0.85f'_c ba \tag{1}$$

$$T = A_s f_y \tag{2}$$

where the use of f_y is the yield stress of the steel reinforcement (assumes that the steel yields prior to crushing of the concrete). Equating $C = T$ gives:

$$a = \frac{A_s f_y}{0.85f'_c b} \tag{3}$$

The bending strength is computed as the tensile force multiplied by the distance between the forces

$$M_n = A_s f_y \left(d - \frac{a}{2} \right) \tag{4}$$

Combining the above Equations (3) and (4) gives:

$$M_n = A_s f_y \left(d - 0.59 \frac{A_s f_y}{f_c' b} \right) \tag{5}$$

The ACI Code explicitly accepts the Whitney rectangle (ACI 2011). For the loading conditions in this reinforced concrete beam competition, the ultimate force due to tensile would be:

$$P_{tension} = \frac{A_s f_y}{4} \left(d - 0.59 \frac{A_s f_y}{f_c' b} \right) \tag{6}$$

Shear Model

The design of shear reinforcement is based on the assumption that the shear force must not exceed the total shear capacity of the beam (ACI 2011). When shear reinforcement is used, the shear capacity of a beam cross-section can be estimated as

$$V_n = V_s + V_c \tag{7}$$

where V_n is the shear force in the beam, V_s is the shear capacity supplied by the reinforcement, and V_c is the shear strength of the concrete. The shear capacity of the reinforcement, which is assumed uniformly spaced across the diagonal crack, is

$$V_n = \frac{A_v f_y d}{s} + 2\sqrt{f'_c}\, bd \tag{8}$$

where A_v is the area of steel reinforcement in shear for each stirrup crossing the diagonal crack and s is spacing of the stirrups. For the loading conditions in this reinforced concrete beam competition, the ultimate force due to shear would be:

$$P_{shear} = 2\left(\frac{A_v f_y d}{s} + 2\sqrt{f'_c}\, bd \right) \tag{9}$$

Reinforcement Ratio for Rectangular Beams

The reinforcement ratio ρ (often called reinforcement percentage), may be conveniently used to represent the relative amount of tension reinforcement in a beam. Thus using the dimensions of Figure 1, the reinforcement ratio is:

$$\rho = \frac{A_s}{bd} \tag{10}$$

Rewriting Equation (10) expresses the reinforcement ratio in terms of the *c/d* ratio.

$$\rho = 0.85 \beta_1 \frac{c}{d} \frac{f_c'}{f_y} \tag{11}$$

For beams controlled by tensile failure, *c/d* < 0.375 (ACI 2011).

Compression Model

If the $c/d > 0.6$ then beam failure is controlled by compression (ACI 2011). For an overly reinforced beam, the stress in the tensile steel f_{steel} when the concrete reaches its ultimate strain is:

$$f_{steel} = 87{,}000\,psi\left(\frac{d-c}{c}\right) \tag{12}$$

If $f_{steel} < f_y$ or $c/d > 0.6$, then the maximum moment in compression is:

$$M_{compression} = A_s\left(\frac{d-c}{c}\right)\left(d-\frac{a}{2}\right)87{,}000\,psi \tag{13}$$

For the loading conditions in this reinforced concrete beam competition, the ultimate force due to compression would be:

$$P_{compression} = \frac{A_s}{4}\left(\frac{d-c}{c}\right)\left(d-\frac{a}{2}\right)87{,}000\,psi \tag{14}$$

Estimation of Ultimate Beam Strength

The ultimate strength, S, of the reinforced concrete beam may be estimated based the value of c/d as follows (ACI 2011):

$$\begin{aligned}
&\text{If } \frac{c}{d} \le 0.375 && S = Minimun\left(P_{tension}, P_{shear}\right)\\
&0.375 < \frac{c}{d} < 0.6 && S = Minimun\left(P_{tension}, P_{shear}, P_{compression}\right)\\
&\frac{c}{d} \ge 0.6 && S = Minimun\left(P_{compression}, P_{shear}\right)
\end{aligned} \tag{15}$$

Estimation of Beam Weight and Cost

The estimated weight W of a simply reinforced rectangular concrete beam is

$$W = V_{beam}\,\gamma_{concrete} + A_s L\left(\gamma_{steel} - \gamma_{concrete}\right) \tag{16}$$

where V_{beam} is the volume of the beam, L is the length of the beam, $\gamma_{concrete}$ is the unit weight of concrete (typically 145 lb./ft.^{3}), and γ_{steel} is the unit weight of steel (490 lb./ft.^{3}).

The total cost estimate for a reinforced concrete beam C_{beam} is

$$C_{beam} = C_{steel} + C_{concrete} \tag{17}$$

where C_{steel} is the cost of the steel reinforcement and $C_{concrete}$ is the cost of the concrete. Table 1 lists the unit cost for a reinforced concrete beam for this competition.

Table 1. Reinforced Concrete Material Cost

Material	Cost
Portland Type I cement	$116/ton
Coarse aggregate	$18/ton
Fine aggregate	$10/ton
Steel reinforcement	$700/ton
Admixtures - water reducer	$15/gal
Admixture - silica flume	$100/ton

To compute the cost of a reinforced concrete beam using the information in Table 1, use the following estimates (there is no cost associated with shear reinforcement).

The cost of the steel, C_{steel}, may be estimated as:

$$C_{steel} = A_s L \left(490 \frac{\text{lb.}}{\text{ft.}^3} \right) \left(\frac{\$700}{\text{ton}} \right) \left(\frac{\text{ton}}{2{,}000\ \text{lb.}} \right) \tag{18}$$

The total cost of concrete, $C_{concrete}$, is estimated from the mix design as

$$C_{concrete} = C_{cement} + C_{CA} + C_{FA} \tag{19}$$

where C_{cement} is the cost of the cement, C_{CA} is the cost of the coarse aggregate, and C_{FA} is the cost of the fine aggregate. The cost of each of these components can be computed as

$$C_{cement} = V_{beam} \left(W_{cement} \frac{\text{lb.}}{\text{ft.}^3} \right) \left(\frac{\$116}{\text{ton}} \right) \left(\frac{\text{ton}}{2{,}000\ \text{lb.}} \right) \tag{20}$$

$$C_{CA} = V_{beam} \left(W_{CA} \frac{\text{lb.}}{\text{ft.}^3} \right) \left(\frac{\$18}{\text{ton}} \right) \left(\frac{\text{ton}}{2{,}000\ \text{lb.}} \right) \tag{21}$$

$$C_{FA} = V_{beam} \left(W_{FA} \frac{\text{lb.}}{\text{ft.}^3} \right) \left(\frac{\$10}{\text{ton}} \right) \left(\frac{\text{ton}}{2{,}000\ \text{lb.}} \right) \tag{22}$$

where W_{cement} is the weight of cement, W_{CA} is the weight of coarse aggregate, and W_{FA} is the weight of fine aggregate (each weight is per ft.^{3} of concrete).

Estimation of Cost-Adjusted SWR

The predicted value of the cost-adjusted SWR (*ASWR*) may be computed as:

$$\begin{aligned} &\text{If } C_{Beam} < \$2 \quad ASWR = \frac{S}{W} \\ &\qquad\qquad > \$2 \quad ASWR = \frac{S}{W} \left(\frac{\$2}{C_{Beam}} \right) \end{aligned} \tag{23}$$

References

American Concrete Institute (2011). *Building Code Requirements for Structural Concrete (ACI 318-11) and Commentary.*

Whitney, C.S. (1937) "Design of Reinforced Concrete Members Under Flexure and Combined Flexure and Direct Compression." *ACI Journal*, March-April; 33, 483-498.